Compulsory

39

$$\begin{aligned}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}dx =& 8\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+x^2}dx\\ =& 8\left[\tan^{-1}x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\\ =& 8\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\\ =& \frac{4\pi}{3}.\end{aligned}$$

40

$$\begin{aligned}\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{4}{\sqrt{1-x^2}}dx =& 4\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}dx\\ =& 4\left[\sin^{-1}x\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\\ =& 4\left[\frac{\pi}{4}-\frac{\pi}{6}\right]\\ =& \frac{\pi}{3}.\end{aligned}$$

42 Let $u = 4x$, then $du = 4dx$, so

$$\begin{aligned}\int_0^{\frac{\sqrt{3}}{4}} \frac{dx}{1+16x^2} =& \frac{1}{4}\int_{4\cdot 0}^{4\cdot\frac{\sqrt{3}}{4}} \frac{du}{1+u^2}\\ =& \frac{1}{4}\int_0^{\sqrt{3}} \frac{du}{1+u^2}\\ =& \frac{1}{4}\left[\tan^{-1}u\right]_0^{\sqrt{3}}\\ =& \frac{1}{4}\left[\frac{\pi}{3}-0\right]\\ =& \frac{\pi}{12}.\end{aligned}$$

45 Let $u = t^3$, then $du = 3t^2dt$, so

$$\begin{aligned}\int \frac{t^2}{\sqrt{1-t^6}}dt =& \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}\\ =& \frac{1}{3}\sin^{-1}u + C\\ =& \frac{1}{3}\sin^{-1}(t^3) + C.\end{aligned}$$

Check: Exercise.

Recommended

41 Let $u = \sin^{-1} x$, so $du = \frac{1}{\sqrt{1-x^2}}dx$, so

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx &= \int_{\sin^{-1} 0}^{\sin^{-1}\left(\frac{1}{2}\right)} u\ du \\
&= \int_0^{\frac{\pi}{6}} u\ du \\
&= \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{6}} \\
&= \frac{\pi^2}{72}.
\end{aligned}$$

43

$$\begin{aligned}
\int \frac{1+x}{1+x^2}dx &= \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right) dx \\
&= \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx \\
&= \tan^{-1} x + \int \frac{x}{1+x^2}dx \\
&= \tan^{-1} x + \frac{1}{2}\int \frac{du}{u} \qquad (u = 1+x^2) \\
&= \tan^{-1} x + \frac{1}{2}\ln|u| + C \\
&= \tan^{-1} x + \frac{1}{2}\ln|1+x^2| + C \\
&= \tan^{-1} x + \frac{1}{2}\ln(1+x^2) + C \qquad (\because 1+x^2 > 0)
\end{aligned}$$

Check: Exercise.

44 Let $u = \cos x$, then $du = -\sin x dx$, so

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx &= -\int_{\cos 0}^{\cos\frac{\pi}{2}} \frac{du}{1+u^2} \\
&= -\int_1^0 \frac{du}{1+u^2} \\
&= -\left[\tan^{-1} u\right]_1^0 \\
&= -\left[0 - \frac{\pi}{4}\right] \\
&= \frac{\pi}{4}.
\end{aligned}$$

46 We use the formula $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$, which you can check.

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-4}}dx =& \int \frac{1}{x(2\sqrt{\frac{x^2}{4}}-1)}dx \\ =& \frac{1}{2}\int \frac{1}{x\sqrt{\left(\frac{x}{2}\right)^2-1}}dx\end{aligned}$$

Let $u = \frac{x}{2}$. Then $du = \frac{1}{2}dx$, so

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-4}}dx =& \int \frac{du}{2u\sqrt{u^2-1}} \\ =& \frac{1}{2}\sec^{-1}u + C \\ =& \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C.\end{aligned}$$

Check: Exercise.

47 Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$. So

$$\begin{aligned}\int \frac{dx}{\sqrt{x}(1+x)} =& 2\int \frac{du}{1+u^2} \\ =& 2\tan^{-1}u + C \\ =& 2\tan^{-1}(\sqrt{x}) + C.\end{aligned}$$

48 Let $u = e^{2x}$. Then $du = 2e^{2x}dx$, so

$$\begin{aligned}\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx =& \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} \\ =& \frac{1}{2}\sin^{-1}u + C \\ =& \frac{1}{2}\sin^{-1}(e^{2x}) + C\end{aligned}$$