Ex 5.5 (20)

(a) Let A_0 = initial amount, t = time

$\therefore A(t) = A_0 e^{0.06t}$

Solve for $2A_0 = A_0 e^{0.06t}$

$$2 = e^{0.06t}$$

$$0.06t = \ln 2$$

$$t = \frac{\ln 2}{0.06}$$

$\therefore$ Time $= \dfrac{\ln 2}{0.06} \approx 11.55$ years.

(b) $A(1) = A_0 e^{0.06}$

$\therefore$ Equiv. annual interest rate

$$= \frac{A(1) - A_0}{A_0}$$

$$= \frac{A_0 e^{0.06} - A_0}{A_0}$$

$$= e^{0.06} - 1$$

$$\approx 6.18\ \%$$

Ex. 5.6 (1) (a) $\frac{\pi}{3}$ (b) π

2 (a) $\frac{\pi}{6}$ (b) $x = \sec^{-1} 2$

$\Leftrightarrow \sec x = 2 \Leftrightarrow \cos x = \frac{1}{2}$

$\therefore \sec^{-1} 2 = \frac{\pi}{3}$

3 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4}$

16) $y' = \frac{1}{1+(x^2)^2}(x^2)'$ (chain rule)

$= \frac{1}{1+x^4} \cdot 2x$

$= \frac{2x}{1+x^4}$

19) $y' = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot (2x+1)'$

$= \frac{1}{\sqrt{1-(4x^2+4x+1)}} \cdot 2$

$= \frac{2}{\sqrt{-4x^2-4x}}$

$= \frac{2}{\sqrt{2^2(-x^2-x)}}$

$= \frac{1}{\sqrt{-x^2-x}}$ ($-1 < x < 0$ for y to be well-defined)

Recommended

4 (a) $x = \cot^{-1}(-\sqrt{3}) \Leftrightarrow \cot x = -\sqrt{3}$

$\Leftrightarrow \tan x = \frac{-1}{\sqrt{3}}$

$\Leftrightarrow x = -\frac{\pi}{6}$ $\therefore \cot^{-1}(-\sqrt{3}) = -\frac{\pi}{6}$.

(b) As $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos(\pi-\theta) = -\cos\theta$,

$\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

5(a) $\tan(\tan^{-1} 10) = 10$.

(b) Note that $\frac{7\pi}{3} = 2\pi + \frac{\pi}{3}$ and

$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\therefore \sin^{-1}\left(\sin \frac{7\pi}{3}\right) = \sin^{-1}\left(\sin\left(2\pi + \frac{\pi}{3}\right)\right)$$

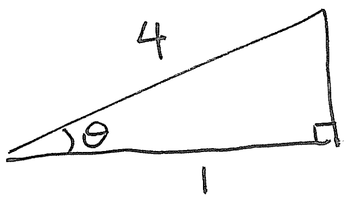
$$= \sin^{-1}\left(\sin \frac{\pi}{3}\right) \qquad (\sin \text{ has period } 2\pi)$$

$$= \frac{\pi}{3}.$$

(Note: $\sin^{-1}\left(\sin \frac{7\pi}{3}\right) \neq \frac{7\pi}{3}$ as we define the range of $\sin^{-1}$ to be $[-\frac{\pi}{2}, \frac{\pi}{2}]$)

6(a) (Triangle trick)

Let $\theta = \sec^{-1} 4$, $\therefore \sec\theta = 4$, or $\cos\theta = \frac{1}{4}$

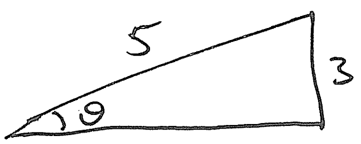


By Pythagoras theorem, $\tan\theta = \frac{\sqrt{4^2 - 1^2}}{1} = \sqrt{15}$

$\therefore \tan(\sec^{-1} 4) = \sqrt{15}$.

(Alternatively, use $1 + \tan^2\theta = \sec^2\theta = 4^2$. (Ex))

(b) Let $\theta = \sin^{-1}\left(\frac{3}{5}\right)$ $\therefore \sin\theta = \frac{3}{5}$



$\cos\theta = \frac{\sqrt{5^2 - 3^2}}{5} = \frac{4}{5}$ (Pythagoras)

By the formula $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin\left(2\sin^{-1}\frac{3}{5}\right) = \sin 2\theta$$

$$= 2\sin\theta\cos\theta$$

$$= 2\cdot\frac{3}{5}\cdot\frac{4}{5}$$

$$= \frac{24}{25}.$$

17) $y' = \dfrac{2\tan^{-1}x}{1+x^2}$.

18) $g'(x) = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}\cdot 2x\cdot\sec^{-1}x + \sqrt{x^2-1}\cdot\dfrac{1}{x\sqrt{x^2-1}}$

$$= \frac{x\sec^{-1}x}{\sqrt{x^2-1}} + \frac{1}{x}.$$

$\left(\text{Ex}: (\sec^{-1}x)' = \dfrac{1}{x\sqrt{x^2-1}}\right)$

20) Omitted.

21) $G'(x) = \dfrac{-2x}{2\sqrt{1-x^2}}\cos^{-1}x - \dfrac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$

$$= -\frac{x\cos^{-1}x}{\sqrt{1-x^2}} - 1$$

22) $F'(\theta) = \dfrac{1}{\sqrt{1-\sqrt{\sin\theta}^{\,2}}}\cdot\dfrac{1}{2\sqrt{\sin\theta}}\cdot\cos\theta$

$$= \frac{1}{\sqrt{1-\sin\theta}}\cdot\frac{1}{2\sqrt{\sin\theta}}\cdot\cos\theta$$

$$= \frac{\cos\theta}{2\sqrt{\sin\theta-\sin^2\theta}}.$$

23) Omitted

24) $$y' = \frac{-1}{\sqrt{1-(\sin^{-1}t)^2}} \cdot \frac{1}{\sqrt{1-t^2}}$$

$$= \frac{-1}{\sqrt{(1-(\sin^{-1}t)^2)(1-t^2)}}$$

(DON'T write $(\sin^{-1}t)^2$ as $\sin^{-2}t$)

25) $$y' = \frac{1}{1+\cos^2\theta} \cdot (-\sin\theta) = \frac{-\sin\theta}{1+\cos^2\theta}$$

26) $$f'(x) = \ln(\tan^{-1}x) + \frac{x}{\tan^{-1}x} \cdot \frac{1}{1+x^2}$$

$$= \ln(\tan^{-1}x) + \frac{x}{(1+x^2)\tan^{-1}x}.$$

27) Omitted.

28) $$y' = \frac{1}{1+\frac{1-x}{1+x}} \cdot \frac{1}{2} \frac{1}{\sqrt{\frac{1-x}{1+x}}} \cdot \left(\frac{1-x}{1+x}\right)'$$

$$= \frac{1+x}{(1+x)+(1-x)} \cdot \frac{1}{2} \cdot \sqrt{\frac{1+x}{1-x}} \cdot \left(\frac{(-1-x)+2}{1+x}\right)'$$

$$= \frac{1}{4} \cdot \frac{(1+x)^{\frac{3}{2}}}{(1-x)^{\frac{1}{2}}} \cdot \left(-1+\frac{2}{1+x}\right)'$$

$$= \frac{(1+x)^{\frac{3}{2}}}{4(1-x)^{\frac{1}{2}}} \cdot \frac{-2}{(1+x)^2}$$

$$\left(\left(\frac{2}{1+x}\right)' = \left(2(1+x)^{-1}\right)' = -2(1+x)^{-2}\right)$$

$$= -\frac{1}{2} \cdot \frac{1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}} = -\frac{1}{2(1-x^2)^{\frac{1}{2}}}.$$

29) $$y' = \frac{-1}{\left[1-\left(\frac{b+a\cos x}{a+b\cos x}\right)^2\right]^{\frac{1}{2}}} \cdot \left(\frac{b+a\cos x}{a+b\cos x}\right)'$$

Consider $$\frac{1}{\left[1-\left(\frac{c}{d}\right)^2\right]^{\frac{1}{2}}} = \frac{1}{\left(\frac{d^2-c^2}{d^2}\right)^{\frac{1}{2}}}$$

$$= \frac{d}{(d^2-c^2)^{\frac{1}{2}}} \quad \text{if } d>0$$

For $c = b+a\cos x$, $d = a+b\cos x \geq a+b(-1) = a-b > 0$,

$$d^2-c^2 = (a+b\cos x)^2 - (b+a\cos x)^2$$

$$= (a^2+2ab\cos x + b^2\cos^2 x)$$
$$-(b^2+2ab\cos x + a^2\cos^2 x)$$

$$= a^2-b^2 + (b^2-a^2)\cos^2 x$$

$$= (a^2-b)^2 - (a^2-b^2)\cos^2 x$$

$$= (a^2-b^2)(1-\cos^2 x)$$

$$= (a^2-b^2)\sin^2 x$$

$$\therefore \frac{-1}{\left[1-\left(\frac{b+a\cos x}{a+b\cos x}\right)^2\right]^{\frac{1}{2}}} = -\frac{a+b\cos x}{\sqrt{a^2-b^2}\,\sin x} \quad (0<x<\pi)$$

Now consider

$$\left(\frac{b+a\cos x}{a+b\cos x}\right)' = \frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{-a^2\sin x - ab\sin x\cos x + b^2\sin x + ab\sin x\cos x}{(a+b\cos x)^2}$$

$$= -\frac{(a^2-b^2)\sin x}{(a+b\cos x)^2}$$

$$\therefore y' = -\frac{a+b\cos x}{\sqrt{a^2-b^2}\,\sin x}\cdot\left[-\frac{(a^2-b^2)\sin x}{(a+b\cos x)^2}\right]$$

$$= \frac{\sqrt{a^2-b^2}}{a+b\cos x}.$$

Challenging

8) Let $\theta = \sin^{-1} x \quad \therefore \sin\theta = x$

$$1+\tan^2\theta = \frac{1}{\cos^2\theta} = \frac{1}{1-\sin^2\theta} = \frac{1}{1-x^2}$$

$$\therefore \tan^2\theta = \frac{1}{1-x^2} - 1 = \frac{x^2}{1-x^2}$$

If $x \geq 0$, then $\theta \geq 0$ and $\tan\theta \geq 0 \quad \therefore \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$.

If $x < 0$, then $\theta < 0$ and $\tan\theta < 0 \quad \therefore \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$.

$$\therefore \tan(\sin^{-1}x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}.$$

9) As in (8),

$$\theta = \tan^{-1} x \Leftrightarrow \tan\theta = x$$

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

$$\Leftrightarrow 1 + x^2 = \frac{1}{\cos^2\theta} = \frac{1}{1-\sin^2\theta}$$

$$\Leftrightarrow 1 - \sin^2\theta = \frac{1}{1+x^2}$$

$$\sin^2\theta = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$$

Similar to (8), $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

$$\therefore \sin(\tan^{-1}x) = \sin\theta = \frac{x}{\sqrt{1+x^2}}.$$

10) Let $\theta = \tan^{-1} x$ $\quad \therefore \tan\theta = x$

$$\therefore \cos(2\tan^{-1}x) = \cos(2\theta) \qquad \text{(Double } \angle \text{ formula)}$$

$$= 2\cos^2\theta - 1 \quad \text{— ①}$$

$$1 + \tan^2\theta = \sec^2\theta = \frac{1}{\cos^2\theta}$$

$\therefore 1 + x^2 = \dfrac{1}{\cos^2\theta}$ ie. $\cos^2\theta = \dfrac{1}{1+x^2}$

$\therefore$ ① becomes $\cos(2\tan^{-1}x) = \dfrac{2}{1+x^2} - 1$

$$= \frac{1-x^2}{1+x^2}.$$

12 a) Let $\theta = \sin^{-1} x$ $\therefore \sin\theta = x$.

Note that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. —— (1)

On the other hand, we have

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$= x \quad \text{—— (2)}$$

It's easy to see that (1) implies

$$0 \le \frac{\pi}{2} - \theta \le \pi$$

$\therefore$ (2) implies $\frac{\pi}{2} - \theta = \cos^{-1} x$

i.e. $\frac{\pi}{2} - \sin^{-1} x = \cos^{-1} x$

Therefore $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) $\frac{d}{dx} \cos^{-1} x = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right)$

$$= -\frac{d}{dx}\left(\sin^{-1} x\right)$$

$$= -\frac{1}{\sqrt{1-x^2}}.$$