Ex. 5.3

32

$$\begin{aligned}
y' =& \frac{(e^u + e^{-u})(e^u - e^{-u})' - (e^u - e^{-u})(e^u + e^{-u})'}{(e^u + e^{-u})^2} \\
=& \frac{(e^u + e^{-u})(e^u + e^{-u}) - (e^u - e^{-u})(e^u - e^{-u})}{(e^u + e^{-u})^2} \\
=& \frac{(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u + e^{-u})^2} \\
=& \frac{(e^{2u} + 2e^u e^{-u} + e^{-2u}) - (e^{2u} - 2e^u e^{-u} + e^{-2u})}{(e^u + e^{-u})^2} \quad (\because (e^u)^2 = e^{2u}, \left(e^{-u}\right)^2 = e^{-2u}) \\
=& \frac{(e^{2u} + 2 + e^{-2u}) - (e^{2u} - 2 + e^{-2u})}{(e^u + e^{-u})^2} \qquad (e^u \cdot e^{-u} = 1) \\
=& \frac{4}{(e^u + e^{-u})^2}.
\end{aligned}$$

68 Let $u = e^{-x} + 1$, so $du = -e^{-x}dx = -\frac{1}{e^x}dx$. So

$$\begin{aligned}
\int_0^1 \frac{\sqrt{e^{-x}+1}}{e^x}dx =& -\int_{e^0+1}^{e^{-1}+1} \sqrt{u}du \\
=& -\int_2^{e^{-1}+1} u^{\frac{1}{2}}du \\
=& -\frac{2}{3}\left[u^{\frac{3}{2}}\right]_2^{e^{-1}+1} \\
=& -\frac{2}{3}\left((e^{-1}+1)^{\frac{3}{2}} - 2\sqrt{2}\right).
\end{aligned}$$

Ex. 5.4

32 Let $y = x^{\cos x}$, then $\ln y = \cos x \ln x$. Differentiating w.r.t. x,

$$\begin{aligned}
\frac{y'}{y} =& -\sin x \ln x + \frac{\cos x}{x} \\
y' =& y(-\sin x \ln x + \frac{\cos x}{x}) \\
=& x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right).
\end{aligned}$$

46 Let $u = 2^x + 1$, then $du = 2^x \ln 2\ dx$, i.e. $2^x dx = \frac{1}{\ln 2}du$. So

$$\begin{aligned}
\int \frac{2^x}{2^x+1}dx =& \frac{1}{\ln 2}\int \frac{1}{u}du \\
=& \frac{1}{\ln 2}\ln|u| + C \\
=& \frac{1}{\ln 2}\ln|2^x+1| + C \\
=& \frac{1}{\ln 2}\ln(2^x+1) + C \quad (\because 2^x + 1 \geq 1)
\end{aligned}$$

Ex 5.3 2(a) 15 (b) $\ln e^{-1} = -1$

3(a) $(e^{\ln 5})^{-2} = 5^{-2} = \frac{1}{25}$

(b) $\ln(\ln e^{e^{30}})$

$= \ln e^{30}$

$= 30.$

4(a) $\sin x$ (b) $e^x \cdot e^{\ln x} = e^x x$

5(a) $e^{7-4x} = 6$

Take ln.

$7 - 4x = \ln 6$

$4x = 7 - \ln 6$

$x = \frac{1}{4}(7 - \ln 6)$

(b) Take exp.

$3x - 10 = e^2$

$x = \frac{1}{3}(e^2 + 10)$

6(a) $x^2 - 1 = e^3$

$x^2 = e^3 + 1$

$x = \pm\sqrt{e^3+1}$

(b) $(e^x)^2 - 3e^x + 2 = 0$

$(e^x - 1)(e^x - 2) = 0$

$e^x = 1$ or 2

$x = 0$ or $\ln 2$.

7 (a) Take ln.

$3x+1 = \ln k$

$x = \frac{1}{3}(\ln k - 1)$

(b) $\ln(x^2 - x) = 1$

Take exp.

$x^2 - x = e$ i.e. $x^2 - x - e = 0$

Use quadratic formula:

$$x = \frac{1 \pm \sqrt{1^2 + 4e}}{2}$$

$$= \frac{1 \pm \sqrt{4e+1}}{2}$$

The original equation is undefined if $x<1$

($\because \ln(x-1)$ undefined)

$\therefore x = \frac{1+\sqrt{4e+1}}{2}$ (As $e > 2$,

$\sqrt{4e+1} > 3$)

8 (a) Take exp. 2 times:

$x = e^e$.

(b) Take ln 2 times:

$x = \ln \ln 10$.

23) $f'(x) = (3x^2+2)e^x + (x^3+2x)e^x$

$= (x^3+3x^2+2x+2)e^x$

24) $y' = \dfrac{(1-e^x)e^x - (-e^x)e^x}{(1-e^x)^2}$

$= \dfrac{1}{(1-e^x)^2}$

25) $y' = e^{ax^3}(ax^3)'$

$= 3ax^2 e^{ax^3}$

26) $y' = -2e^{-2t}\cos 4t - 4e^{-2t}\sin 4t$

27) $f'(u) = e^{\frac{1}{u}}(-u^{-2}) = -u^{-2}e^{\frac{1}{u}}$

28) $y' = 2xe^{-1/x} + x^2 e^{-\frac{1}{x}}(x^{-2})$

$= 2xe^{-1/x} + e^{-\frac{1}{x}}$

29) $F'(t) = e^{t\sin 2t}(\sin 2t + 2t\cos 2t)$

30) $y' = e^{k\tan\sqrt{x}}(k\sec^2\sqrt{x} \cdot \frac{1}{2}x^{-\frac{1}{2}})$

$= \frac{k}{2}\sec^2\sqrt{x} \cdot x^{-\frac{1}{2}} \cdot e^{k\tan\sqrt{x}}$

33) $y' = e^{e^x} \cdot e^x$

34) $y' = \frac{1}{2}(1+xe^{-2x})^{-\frac{1}{2}}(e^{-2x} + xe^{-2x}(-2))$

$= \frac{1}{2}(1+xe^{-2x})^{-\frac{1}{2}}(e^{-2x} - 2xe^{-2x})$.

35) $\ln y = \ln(ae^x+b) - \ln(ce^x+d)$

$\therefore \frac{y'}{y} = \frac{ae^x}{ae^x+b} - \frac{ce^x}{ce^x+d}$

$y' = \left(\frac{ae^x+b}{ce^x+d}\right)\left(\frac{ae^x}{ae^x+b} - \frac{ce^x}{ce^x+d}\right)$

$= \frac{ae^x}{ce^x+d} - \frac{ce^x(ae^x+b)}{(ce^x+d)^2}$

36) $f'(t) = 2\sin(e^{\sin^2 t})\cos(e^{\sin^2 t}) \cdot e^{\sin^2 t}(\sin^2 t)'$

$= 2\sin(e^{\sin^2 t})\cos(e^{\sin^2 t}) \cdot e^{\sin^2 t}(2\sin t\cos t)$

$= 4\sin t\cos t \cdot \sin(e^{\sin^2 t})\cos(e^{\sin^2 t}) \cdot e^{\sin^2 t}$.

Ex 5.3 (61) $\int_0^2 e^{-\pi x}\,dx$

Let $u = -\pi x$ $\therefore du = -\pi\,dx$

$\int_0^2 e^{-\pi x}dx = -\frac{1}{\pi}\int_0^{-2\pi} e^u\,du = -\frac{1}{\pi}\left[e^u\right]_0^{-2\pi} = -\frac{1}{\pi}(e^{-2\pi}-1)$

$= \frac{1}{\pi}(1-e^{-2\pi})$

62) Let $u = x^3$ $\therefore du = 3x^2 dx$

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int e^u du$$
$$= \frac{1}{3}e^u + C$$
$$= \frac{1}{3}e^{x^3} + C.$$

63) Let $u = e^x + 1$ $\therefore du = e^x dx$

$$\int e^x \sqrt{1+e^x}\, dx = \int \sqrt{u}\, du$$
$$= \frac{2}{3}u^{\frac{3}{2}} + C$$
$$= \frac{2}{3}(e^x+1)^{\frac{3}{2}} + C.$$

64) $$\int \left(\frac{1+e^x}{e^x}\right)^2 dx = \int \frac{1+2e^x+e^{2x}}{e^x} dx$$
$$= \int (e^{-x} + 2 + e^x) dx$$

Consider $\int e^{-x} dx = -\int e^u du$ $(u = -x)$
$$= -e^u + C$$
$$= -e^{-x} + C$$

$$\int (2 + e^x) dx = 2x + e^x + C$$
$$\therefore \int (e^{-x} + 2 + e^x)\, dx = -e^{-x} + 2x + e^x + C.$$

Remark:

Useful formula: $\int e^{ax}dx = \frac{e^{ax}}{a} + C$.

(Exercise)

65) Let $u = \tan x$ $\therefore du = \sec^2 x\,dx$

$$\int e^{\tan x}\sec^2 x\,dx = \int e^u\,du$$
$$= e^u + C$$
$$= e^{\tan x} + C.$$

66) Let $u = e^x$ $\therefore du = e^x dx$

$$\therefore \int e^x \cos(e^x)\,dx = \int \cos u\,du$$
$$= \sin u + C$$
$$= \sin(e^x) + C.$$

67) Let $u = 1/x$ $\therefore du = -x^{-2}dx$

$$\therefore \int_1^2 \frac{e^{1/x}}{x^2}\,dx = -\int_{1/1}^{1/2} e^u\,du$$
$$= -\int_1^{1/2} e^u\,du$$
$$= \int_{1/2}^{1} e^u\,du$$
$$= \left[e^u\right]_{1/2}^{1} = e - e^{1/2}$$

§5.4) (3) $4^{-\pi} = e^{-\pi \ln 4}$

4) $x^{\sqrt{5}} = e^{\sqrt{5} \ln x}$

5) $10^{x^2} = e^{x^2 \ln 10}$

6) $(\tan x)^{\sec x} = e^{\sec x \ln(\tan x)}$

23) $f'(x) = 5x^4 + 5^x \ln 5$.

24) $g'(x) = \sin 2^x + x\cos(2^x)\cdot(2^x)'$

$= \sin 2^x + x\cos(2^x)\cdot 2^x \ln 2$

$= \sin 2^x + \ln 2 \cdot x\, 2^x \cos(2^x)$.

25) $f'(t) = 10^{\sqrt{t}} \ln 10 \cdot (\sqrt{t})'$

$= 10^{\sqrt{t}} \ln 10 \cdot \frac{1}{2} t^{-\frac{1}{2}}$

$= \frac{\ln 10}{2} \cdot t^{-\frac{1}{2}} \cdot 10^{\sqrt{t}}$.

26) $F'(t) = \ln 3 \cdot 3^{\cos 2t} (-\sin 2t \cdot 2)$

$= -2\ln 3 \cdot 3^{\cos(2t)} \sin 2t$.

27) $L'(v) = \sec^2(4^{v^2}) \cdot \ln 4 \cdot 4^{v^2} \cdot 2v$

$= 2(\ln 4) \cdot v \sec^2(4^{v^2}) \cdot 4^{v^2}$.

28) $G'(u) = 6(1+10^{\ln u})^5 (1+10^{\ln u})'$

$= 6(1+10^{\ln u})^5 10^{\ln u} \cdot \ln 10 \cdot \frac{1}{u}$

29) $y' = \left(2x \log_{10} \sqrt{x}\right)'$

$= \left(2x \cdot \frac{1}{2} \log_{10} x\right)'$

$= \left(x \log_{10} x\right)'$

$= \left(x \frac{\ln x}{\ln 10}\right)'$

$= \frac{1}{\ln 10}\left(\ln x + \frac{x}{x}\right)$

$= \frac{1}{\ln 10}\left(\ln x + 1\right).$

30) $y' = \left[\log_2 (e^{-x} \cos \pi x)\right]'$

$= \frac{1}{\ln 2}\left[\ln (e^{-x} \cos \pi x)\right]'$

$= \frac{1}{\ln 2}\left(\ln(e^{-x}) + \ln(\cos \pi x)\right)'$

$= \frac{1}{\ln 2}\left(-x + \ln(\cos \pi x)\right)'$

$= \frac{1}{\ln 2}\left(-1 + \frac{\pi \sin \pi x}{\cos \pi x}\right)$

$= \frac{1}{\ln 2}\left(-1 + \pi \tan \pi x\right).$

31) $\ln y = x \ln x$

$\Rightarrow \dfrac{y'}{y} = \ln x + \dfrac{x}{x} = \ln x + 1$

$\therefore y' = y(\ln x + 1)$

$= x^x (\ln x + 1)$

33) $\ln y = \sin x \ln x$

$\Rightarrow \dfrac{y'}{y} = \cos x \ln x + \dfrac{\sin x}{x}$

$y' = x^{\sin x}\left(\cos x \cdot \ln x + \dfrac{\sin x}{x}\right)$

34) $\ln y = x \ln \sqrt{x} = \frac{1}{2} x \ln x$

$\therefore \dfrac{y'}{y} = \frac{1}{2}\ln x + \frac{1}{2}$

$y' = \sqrt{x}^{\,x}\left(\frac{1}{2}\ln x + \frac{1}{2}\right)$.

35) $\ln y = x \ln \cos x$

$\Rightarrow \dfrac{y'}{y} = \ln(\cos x) + \dfrac{x}{\cos x}(-\sin x)$

$= \ln(\cos x) - x\tan x$

$\therefore y' = (\cos x)^x \left[\ln(\cos x) - x \tan x\right]$.

36) $\ln y = \ln x \ln(\sin x)$

$$\therefore \frac{y'}{y} = \frac{1}{x}\ln(\sin x) + \ln x \cdot \frac{\cos x}{\sin x}$$

$$y' = (\sin x)^{\ln x}\left[\frac{\ln \sin x}{x} + \frac{\ln x}{\tan x}\right]$$

37) $\ln y = \frac{1}{x}\ln \tan x$

$$\frac{y'}{y} = \frac{-1}{x^2}\ln \tan x + \frac{1}{x}\cdot\frac{\sec^2 x}{\tan x}$$

$$y' = (\tan x)^{\frac{1}{x}}\left[-\frac{\ln \tan x}{x^2} + \frac{\sec^2 x}{x\tan x}\right].$$

38) $\ln y = \cos x \ \ln(\ln x)$

$$\therefore \frac{y'}{y} = -\sin x \ln(\ln x) + \cos x\,\frac{(\ln x)'}{\ln x}$$

$$= -\sin x \ln(\ln x) + \frac{\cos x}{x \ln x}$$

$$y' = (\ln x)^{\cos x}\left(-\sin x \ln(\ln x) + \frac{\cos x}{x\ln x}\right)$$

41) $\int_1^2 10^t\,dt = \left[\frac{10^t}{\ln 10}\right]_1^2 = \frac{100-1}{\ln 10} = \frac{99}{\ln 10}.$

42) $\int (x^5 + 5^x)\,dx = \frac{x^6}{6} + \frac{5^x}{\ln 5} + C.$

43) $\int \frac{\log_{10} x}{x} dx$

$= \frac{1}{\ln 10} \int \frac{\ln x}{x} dx$

Let $u = \ln x$ $\therefore du = \frac{1}{x} dx$

$\therefore \frac{1}{\ln 10} \int \frac{\ln x}{x} dx$

$= \frac{1}{\ln 10} \int u \, du$

$= \frac{1}{\ln 10} \cdot \frac{u^2}{2} + C$

$= \frac{(\ln x)^2}{2 \ln 10} + C$

44) Let $u = x^2$ $\therefore du = 2x \, dx$

$\therefore \int x 2^{x^2} dx = \frac{1}{2} \int 2^u du$

$= \frac{1}{2} \cdot \frac{2^u}{\ln 2} + C$

$= \frac{2^{x^2}}{2 \ln 2} + C$

(Alternatively, let $u = 2^{x^2}$).

45) Let $u = \sin\theta$ $\therefore du = \cos\theta\, d\theta$

$$\therefore \int 3^{\sin\theta}\cos\theta\, d\theta = \int 3^u\, du$$
$$= \frac{3^u}{\ln 3} + C$$
$$= \frac{3^{\sin\theta}}{\ln 3} + C.$$

Harder problems (only if you're interested!)

72) Let $f(x) = e^{\sin x}$

$$\therefore f(\pi) = e^{\sin\pi} = e^0 = 1.$$

$$\therefore \lim_{x\to\pi} \frac{e^{\sin x} - 1}{x - \pi} = f'(\pi).$$

On the other hand, $f'(x) = e^{\sin x}\cos x$

$$\therefore f'(\pi) = e^{\sin\pi}\cos\pi = e^0(-1) = -1$$

$$\therefore \lim_{x\to\pi} \frac{e^{\sin x} - 1}{x - \pi} = -1.$$

75) (a) Let $f(x) = e^x - (1+x)$

$$\therefore f'(x) = e^x - 1$$
> 0 if $x > 0$. (as e^x is increasing)

$\therefore f$ is increasing for $x \geq 0$.

As $f(0) = 0$, $f(x) \geq 0$ i.e. $e^x \geq 1+x$.

(b) From (a),

$$e^{x^2} \geq 1 + x^2 \quad \text{for } 0 \leq x \leq 1$$

On the other hand, as $x^2 \leq 1$ for $0 \leq x \leq 1$,

$$e^{x^2} \leq e^1 = e \quad \text{on } [0,1]$$

$$\therefore \int_0^1 (1+x^2)\,dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e\,dx$$

$$\left[x + \frac{x^3}{3}\right]_0^1 \leq \int_0^1 e^{x^2} dx \leq \left[ex\right]_0^1$$

$$1 + \frac{1}{3} \leq \int_0^1 e^{x^2} dx \leq e$$

$$\frac{4}{3} \leq \int_0^1 e^{x^2} dx \leq e.$$

76(a) By 75(a), $e^x \geq 1 + x$ if $x \geq 0$

Integrating this inequality, for $x \geq 0$

$$\int_0^x e^t\,dt \geq \int_0^x (1+t)\,dt$$

$$e^x - 1 \geq x + \frac{x^2}{2}$$

i.e. $e^x \geq 1 + x + \dfrac{x^2}{2}$.

(b) is left as an exercise.

Ex 5.4

49) Let $y = x^{-\ln x}$

$\therefore \ln y = -\ln x \cdot \ln x$
$= -(\ln x)^2.$

As $x \to \infty$, $\ln x \to \infty$

$\therefore \lim_{x\to\infty} \ln y = -\infty.$

$$\therefore \lim_{x\to\infty} x^{-\ln x} = \lim_{x\to\infty} y$$
$$= \lim_{x\to\infty} e^{\ln y}$$
$$= \lim_{t\to-\infty} e^{t} \qquad (t = \ln y \to -\infty \Leftrightarrow x \to \infty)$$
$$= 0.$$

54)
$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$
$$= \lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right]^x$$
$$= \left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right]^x \qquad \left(y = \frac{x}{n} \to 0 \Leftrightarrow n \to \infty\right)$$
$$= \left[\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right]^x$$
$$= e^x$$