Ex 5.2

25) $g'(x) = \left[\ln\left(x\sqrt{x^2-1}\right)\right]'$

$$= \left[\ln x + \ln\sqrt{x^2-1}\right]'$$

$$= \left[\ln x + \frac{1}{2}\ln(x^2-1)\right]'$$

$$= \frac{1}{x} + \frac{1}{2}\cdot\frac{(x^2-1)'}{x^2-1}$$

$$= \frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2-1}$$

$$= \frac{1}{x} + \frac{x}{x^2-1} \quad \#$$

29) Using $(\ln|f|)' = \frac{f'}{f}$,

$$y' = \left[\ln|2-x-5x^2|\right]' = \frac{(2-x-5x^2)'}{2-x-5x^2}$$

$$= \frac{-1-10x}{2-x-5x^2} \quad \#$$

53) $y = \sqrt{\frac{x-1}{x^4+1}} = \left(\frac{x-1}{x^4+1}\right)^{\frac{1}{2}}$

$$\therefore \ln y = \frac{1}{2}\ln\left(\frac{x-1}{x^4+1}\right) = \frac{1}{2}\left[\ln(x-1) - \ln(x^4+1)\right]$$

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{(x-1)'}{x-1} - \frac{(x^4+1)'}{x^4+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$\therefore y' = \frac{y}{2}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$= \sqrt{\frac{x-1}{x^4+1}}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$= \frac{1}{\sqrt{(x^4+1)(x-1)}} - \frac{4x^3\sqrt{x-1}}{(x^4+1)^{\frac{3}{2}}} \quad \#$$

57) $$\int_1^e \frac{x^2+x+1}{x}\,dx$$

$$= \int_1^e \left(x+1+\frac{1}{x}\right)dx$$

$$= \left[\frac{x^2}{2} + x + \ln|x|\right]_1^e$$

$$= \left(\frac{e^2}{2} + e + \ln e\right) - \left(\frac{1^2}{2} + 1 + \ln 1\right) \quad (e>0)$$

$$= \left(\frac{e^2}{2} + e + 1\right) - \left(\frac{1}{2} + 1 + 0\right)$$

$$= \frac{e^2}{2} + e - \frac{1}{2} \quad \#$$

Recommended

Ex. 5.2 (15) $f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \ln x + \frac{\sqrt{x}}{x}$

$= \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$.

16) $f'(x) = \ln x + \frac{x}{x} - 1 = \ln x + 1 - 1 = \ln x$.

17) $f'(x) = \cos(\ln x) \cdot \frac{1}{x} = \frac{\cos(\ln x)}{x}$.

18) $f'(x) = \frac{(\sin^2 x)'}{\sin^2 x} = \frac{2\sin x \cos x}{\sin^2 x} = \frac{2\cos x}{\sin x}$ $(= 2\cot x)$

19) $f'(x) = (\ln x^{-1})' = -(\ln x)' = -\frac{1}{x}$.

20) $y' = [(\ln x)^{-1}]' = -(\ln x)^{-2} \cdot (\ln x)'$

$= -\frac{1}{(\ln x)^2} \cdot \frac{1}{x}$

$= -\frac{1}{x(\ln x)^2}$.

21) $g'(x) = [\ln(a-x) - \ln(a+x)]'$

$= \frac{-1}{a-x} - \frac{1}{a+x}$.

22) $h'(x) = \frac{(x + \sqrt{x^2-1})'}{x + \sqrt{x^2-1}}$

$= \frac{1 + \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{x^2-1}}$

$$= \frac{1 + x(x^2-1)^{-\frac{1}{2}}}{x + (x^2-1)^{\frac{1}{2}}}.$$

(You can multiply by $\frac{(x^2-1)^{\frac{1}{2}}}{(x^2-1)^{\frac{1}{2}}}$ to remove $(\cdots)^{-\frac{1}{2}}$, but this doesn't simplify a lot.)

23) $$G'(y) = \left[5\ln(2y+1) - \frac{1}{2}\ln(y^2+1)\right]'$$

$$= 5 \cdot \frac{2}{2y+1} - \frac{1}{2} \cdot \frac{2y}{y^2+1}$$

$$= \frac{10}{2y+1} - \frac{y}{y^2+1}.$$

24) $$f'(u) = \frac{(1+\ln u) - u\left(\frac{1}{u}\right)}{(1+\ln u)^2}$$

$$= \frac{\ln u}{(1+\ln u)^2}.$$

26) $$H'(z) = \frac{1}{2}\left[\ln(a^2-z^2) - \ln(a^2+z^2)\right]'$$

$$= \frac{1}{2}\left[\frac{-2z}{a^2-z^2} - \frac{2z}{a^2+z^2}\right]$$

$$= \frac{-z}{a^2-z^2} - \frac{z}{a^2+z^2}.$$

27) $f'(u) = \dfrac{(1+\ln(2u))\cdot\frac{1}{u} + \ln u\cdot\left(\frac{u}{2u}\right)}{[1+\ln(2u)]^2}$

$= \dfrac{1}{u(1+\ln u)} + \dfrac{\ln u}{2(1+\ln(2u))^2}.$

28) $y' = 2\ln(\tan x)\,\dfrac{\sec^2 x}{\tan x}$

31) $y' = \sec^2(\ln(ax+b))\cdot[\ln(ax+b)]'$

$= \sec^2(\ln(ax+b))\cdot\dfrac{a}{ax+b}.$

32) $y' = \dfrac{(\cos(\ln x))'}{\cos(\ln x)}$

$= \dfrac{-\sin(\ln x)\cdot\frac{1}{x}}{\cos(\ln x)}$

$= -\dfrac{\tan(\ln x)}{x}$

33) $y' = 2x\ln(2x) + x^2\cdot\dfrac{2}{2x} = 2x\ln(2x) + x$

$y'' = 2\ln(2x) + 2x\cdot\dfrac{2}{2x} + 1$

$= 2\ln(2x) + 3.$

34) $$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \sec x$$

$$\therefore y'' = \sec x \tan x$$

38) $$f'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2}.$$

$$f''(x) = -2x^{-3} - \frac{1}{x^3} + \frac{2\ln x}{x^3} = \frac{-3}{x^3} + \frac{2\ln x}{x^3}$$

$$\therefore f''(e) = -3e^{-3} + 2e^{-3} = -e^{-3}.$$

42) $\ln(xy) = y\sin x$

$$\Rightarrow \frac{y + xy'}{xy} = y'\sin x + y\cos x$$

$$y + xy' = xy\sin x \cdot y' + xy^2\cos x$$

$$(x - xy\sin x)\,y' = xy^2\cos x - y$$

$$\therefore y' = \frac{xy^2\cos x - y}{x - xy\sin x}.$$

51) $\ln y = 2\ln(x^2+2) + 4\ln(x^4+4)$

$$\therefore \frac{y'}{y} = 2\cdot\frac{2x}{x^2+2} + 4\cdot\frac{4x^3}{x^4+4}$$

$$= \frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}$$

$$y' = (x^2+2)^2(x^4+4)^4\left[\frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}\right]$$

$$= 4x(x^2+2)(x^4+4)^4 + 16x^3(x^2+2)^2(x^4+4)^3.$$

52) omitted.

54) $$\ln y = 4\ln(x^3+1) + 2\ln\sin x - \frac{1}{3}\ln x$$

$$\therefore \frac{y'}{y} = 4 \cdot \frac{3x^2}{x^3+1} + 2 \cdot \frac{\cos x}{\sin x} - \frac{1}{3x}$$

$$= \frac{12x^2}{x^3+1} + \frac{2\cos x}{\sin x} + \frac{1}{3x}$$

$$\therefore y' = \frac{(x^3+1)^4 \sin^2 x}{x^{\frac{1}{3}}}\left(\frac{12x^2}{x^3+1} + \frac{2\cos x}{\sin x} + \frac{1}{3x}\right)$$

$$= \frac{12(x^3+1)^3 \sin^2 x}{x^{\frac{1}{3}}} + \frac{2(x^3+1)^4 \sin x \cos x}{x^{\frac{1}{3}}} + \frac{(x^3+1)^4 \sin^2 x}{3x^{4/3}}.$$

55) Let $u = 8-3t \quad \therefore du = -3dt$

$$\int_1^2 \frac{dt}{8-3t} = \frac{-1}{3}\int_{8-3\cdot 1}^{8-3\cdot 2} \frac{du}{u}$$

$$= \frac{-1}{3}\int_5^2 \frac{du}{u}$$

$$= \frac{1}{3}\int_2^5 \frac{du}{u}$$

$$= \frac{1}{3}\Big[\ln|u|\Big]_2^5$$

$$= \frac{1}{3}(\ln 5 - \ln 2).$$

56) Omitted.

58) $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

$= \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx$

$= \left[\frac{x^2}{2} + 2x + \ln|x|\right]_4^9$

$= \frac{1}{2}(9^2 - 4^2) + 2(9-4) + (\ln 9 - \ln 4)$

$= \frac{65}{2} + 10 + \ln\frac{9}{4}$

$= \frac{85}{2} + \ln\frac{9}{4}$.

59) Let $u = \ln x$ $\therefore du = \frac{1}{x} dx$

$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$.

60) Let $u = \ln x$ $\therefore du = \frac{1}{x} dx$

$\int_e^6 \frac{dx}{x \ln x} = \int_{\ln e}^{\ln 6} \frac{du}{u} = \int_1^{\ln 6} \frac{du}{u}$

$= \left[\ln|u|\right]_1^{\ln 6}$

$= \ln(\ln 6)$. $(\ln 6 > 0)$

Challenging

43) $f'(x) = \frac{1}{x-1} = (x-1)^{-1}$

$f''(x) = -(x-1)^{-2}$

$f^{(3)}(x) = 2(x-1)^{-3}$

From this, it's not hard to see that

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(x-1)^{-n}.$$

44) Let $f(x) = x^8 \ln x$

$\therefore f'(x) = 8x^7 \ln x + x^7$

$f''(x) = 8 \cdot 7\, x^6 \ln x + 8x^6 + 7x^6$

From this, it's easy to see that

$f^{(8)}(x) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ln x$

$+$ (a polynomial of deg 0, i.e. a constant)

$= 8! \ln x + (\text{a constant})$

$\therefore f^{(9)}(x) = \frac{8!}{x}$. $\quad (8! = 8 \cdot 7 \cdots 2 \cdot 1)$