Ex. 9.3

13

$$\begin{aligned} r = 2\cos\theta &= \frac{2x}{r} \\ r^2 &= 2x \\ x^2 + y^2 &= 2x \\ (x^2 - 2x + 1) + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1^2. \end{aligned}$$

The curve is a circle centered at $(1, 0)$ with radius 1.

17

$$\begin{aligned} y &= 1 + 3x \\ r\sin\theta &= 1 + 3r\cos\theta \\ r(\sin\theta - 3\cos\theta) &= 1. \end{aligned}$$

28

$$\begin{aligned} \text{Slope} = \frac{dy}{dx} &= \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} \\ &= \frac{\frac{d}{d\theta}((2-\sin\theta)\sin\theta)}{\frac{d}{d\theta}((2-\sin\theta)\cos\theta)} \\ &= \frac{2\cos\theta - 2\sin\theta\cos\theta}{-2\sin\theta - \cos^2\theta + \sin^2\theta} \end{aligned}$$

So at $\theta = \frac{\pi}{3}$,

$$\begin{aligned} \text{Slope} &= \frac{2\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3}\cos\frac{\pi}{3}}{-2\sin\frac{\pi}{3} - \cos^2\frac{\pi}{3} + \sin^2\frac{\pi}{3}} \\ &= \frac{2\cdot\frac{1}{2} - 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}{-2\cdot\frac{\sqrt{3}}{2} - \frac{1}{4} + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{2-\sqrt{3}}{1-2\sqrt{3}} \\ &= \frac{4-3\sqrt{3}}{11} \quad \text{(by rationalization)} \end{aligned}$$

Ex. 9.4

5

$$\begin{aligned} \text{Area} &= \frac{1}{2}\int_0^{2\pi} r^2 d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \theta d\theta \\ &= \frac{1}{2}\left[\frac{\theta^2}{2}\right]_0^{2\pi} \\ &= \pi^2. \end{aligned}$$

33

$$
\begin{aligned}
\text{Length} &= \int_0^{\frac{\pi}{3}} \sqrt{r^2 + \frac{dr}{d\theta}^2} d\theta \\
&= \int_0^{\frac{\pi}{3}} \sqrt{(3\sin\theta)^2 + (3\cos\theta)^2} d\theta \\
&= \int_0^{\frac{\pi}{3}} \sqrt{9(\sin^2\theta + \cos^2\theta)} d\theta \\
&= \int_0^{\frac{\pi}{3}} 3 d\theta \\
&= \pi.
\end{aligned}
$$

Recommended

14) $\theta = \frac{\pi}{3}$ $\therefore$ $\tan\theta = \tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow$ $\frac{y}{x} = \sqrt{3}$ or

$y = \sqrt{3}\,x$. (line)

15) $r^2 \cos 2\theta = 1$

$\Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1$ $(\cos 2\theta = \cos^2\theta - \sin^2\theta)$

$(r\cos\theta)^2 - (r\sin\theta)^2 = 1$

$x^2 - y^2 = 1$ (hyperbola)

16) $r = \tan\theta \sec\theta$

$= \frac{\tan\theta}{\cos\theta}$

$\therefore r\cos\theta = \tan\theta$

$\Rightarrow x = \frac{y}{x}$ $(r\cos\theta = x,\ \frac{y}{x} = \tan\theta)$

$\Rightarrow y = x^2$ (parabola)

18) $4y^2 = x$

$4r^2\sin^2\theta = r\cos\theta$

$\Rightarrow \quad 4r\sin^2\theta = \cos\theta$ (which also includes the origin)

19) $x^2 + y^2 = 2cx$

$\Rightarrow \quad r^2 = 2c \cdot r\cos\theta$

$r = 2c\cos\theta$.

20) $xy = 4$

$r^2\cos\theta\sin\theta = 4$

47) Slope $= \frac{d}{d\theta}(r\sin\theta) \Big/ \frac{d}{d\theta}(r\cos\theta)$

$= \frac{d}{d\theta}\left(2\sin^2\theta\right) \Big/ \frac{d}{d\theta}(2\sin\theta\cos\theta)$

$= 4\sin\theta\cos\theta \Big/ \left(2\cos^2\theta - 2\sin^2\theta\right)$

$= \dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$

At $\theta = \frac{\pi}{6}$, slope $= \dfrac{2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2^2}} = \sqrt{3}$

49)
$$\frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\left(\frac{\sin\theta}{\theta}\right)'}{\left(\frac{\cos\theta}{\theta}\right)'}$$

$$= \frac{\frac{\cos\theta}{\theta} - \frac{\sin\theta}{\theta^2}}{\frac{-\sin\theta}{\theta} - \frac{\cos\theta}{\theta^2}}$$

At $\theta = \pi$, slope $= \dfrac{\frac{-1}{\pi} - \frac{0}{\pi^2}}{\frac{0}{\pi} - \frac{(-1)}{\pi^2}}$

$$= -\pi.$$

50)
$$\frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\left(\cos\frac{\theta}{3}\sin\theta\right)'}{\left(\cos\frac{\theta}{3}\cos\theta\right)'}$$

$$= \frac{-\frac{1}{3}\sin\frac{\theta}{3}\sin\theta + \cos\frac{\theta}{3}\cos\theta}{-\frac{1}{3}\sin\frac{\theta}{3}\cos\theta - \cos\frac{\theta}{3}\sin\theta}$$

At $\theta = \pi$, slope $= \dfrac{-\frac{1}{3}\sin\frac{\pi}{3}\cdot 0 + \cos\frac{\pi}{3}\cdot(-1)}{-\frac{1}{3}\sin\frac{\pi}{3}\cdot(-1) - \cos\frac{\pi}{3}\cdot 0}$

$$= \frac{-1/2}{\frac{1}{3}\cdot\frac{\sqrt{3}}{2}}$$

$$= -\sqrt{3}.$$

6) $\text{Area} = \frac{1}{2}\int_0^{\pi} (1+\cos\theta)^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} (1+2\cos\theta + \cos^2\theta) \, d\theta$

$= \frac{1}{2}\int_0^{\pi} \left[1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta$

$= \frac{1}{2}\int_0^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi}$

$= \frac{3\pi}{4}$.

7) $\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4+3\sin\theta)^2 \, d\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (16 + 24\sin\theta + 9\sin^2\theta) \, d\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(16 + 24\sin\theta + \frac{9}{2}(1-\cos 2\theta)\right) d\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{41}{2} + 24\sin\theta - \frac{9}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\left[\frac{41}{2}\theta - 24\cos\theta - \frac{9}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{41\pi}{4}$.

8) Area $= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos 4\theta) \, d\theta$

$= \frac{1}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{8}$.

15) Area $= \frac{1}{2} \int_0^{2\pi} 16 \cos^2 3\theta \, d\theta$

$= 8 \int_0^{2\pi} \cos^2 3\theta \, d\theta$

$= 4 \int_0^{2\pi} (1 + \cos 6\theta) \, d\theta$

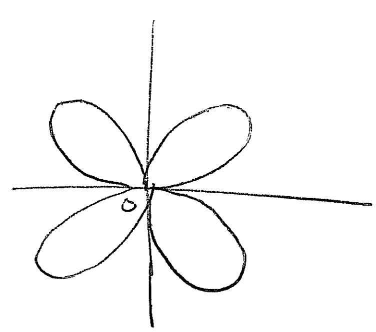
$= 4 \left[\theta + \frac{\sin 6\theta}{6} \right]_0^{2\pi}$

$= 8\pi$.

16) Area $= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$

$= \frac{1}{2} \left[-\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}$.



17, 18 : Omitted.

19) At the intersection,

$$2\cos\theta = r = 1$$

$$\therefore \cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \pm\frac{\pi}{3}$$

$\therefore$ By symmetry,

$$\text{Area} = \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{3}} 4\cos^2\theta\, d\theta - \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{3}} 1^2\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} (4\cos^2\theta - 1)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} (2(1+\cos 2\theta) - 1)\, d\theta$$

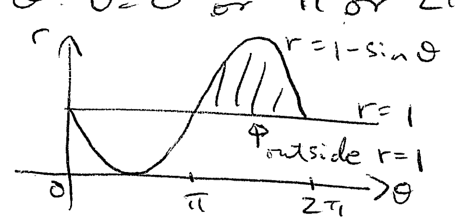
$$= \int_0^{\frac{\pi}{3}} (1 + 2\cos 2\theta)\, d\theta$$

$$= \Big[\theta + \sin 2\theta\Big]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

20) $1 - \sin\theta = r = 1 \Leftrightarrow \sin\theta = 0$ i.e. $\theta = 0$ or π or 2π

$$\text{Area} = \frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\pi}^{2\pi} 1^2\, d\theta$$



$$= \frac{1}{2}\int_{\pi}^{2\pi} (1 - 2\sin\theta + \sin^2\theta - 1)\, d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi} \left[-2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right] d\theta$$

$$= \frac{1}{2}\left[\frac{\theta}{2} + 2\cos\theta - \frac{\sin 2\theta}{4}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[(\pi + 2) - \left(\frac{\pi}{2} + (-2)\right)\right]$$

$$= \frac{\pi}{4} + 2.$$

21, 22 : omitted.

34) Length $= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{2\pi} \sqrt{e^{4\theta} + (2e^{2\theta})^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{5e^{4\theta}}\, d\theta$$

$$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\, d\theta$$

$$= \sqrt{5}\left[\frac{e^{2\theta}}{2}\right]_0^{2\pi}$$

$$= \frac{\sqrt{5}}{2}(e^{4\pi} - 1).$$

35) Length $= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^4 + (2\theta)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{2\pi} \theta\sqrt{\theta^2 + 1}\, d\theta$

$= \frac{1}{2}\int_1^{4\pi^2+1} \sqrt{u}\, du$ $\quad (u = \theta^2 + 1$

$\therefore du = 2\theta\, d\theta)$

$= \frac{1}{2} \cdot \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^{4\pi^2+1}$

$= \frac{1}{3}\left[(4\pi^2+1)^{3/2} - 1\right].$

36) Length $= \int_0^{2\pi} \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{\frac{1}{2}} d\theta$

$= \int_0^{2\pi} \left(4(1+\cos\theta)^2 + 4\sin^2\theta\right)^{\frac{1}{2}} d\theta$

$= \int_0^{2\pi} (4 + 8\cos\theta + 4)^{\frac{1}{2}} d\theta$

$= \int_0^{2\pi} (8 + 8\cos\theta)^{\frac{1}{2}} d\theta$

$= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta$

$= 4\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta$

Consider $\int \sqrt{2+2\cos\theta}\, d\theta$ $\quad (\theta \in (0, \pi))$

$$= \int \frac{\sqrt{2+2\cos\theta} \cdot \sqrt{2-2\cos\theta}}{\sqrt{2-2\cos\theta}}\, d\theta$$

$$= \int \frac{\sqrt{4(1-\cos^2\theta)}}{\sqrt{2-2\cos\theta}}$$

$$= \int \frac{2\sin\theta}{\sqrt{2-2\cos\theta}}\, d\theta$$

$$= \sqrt{2}\int \frac{\sin\theta}{\sqrt{1-\cos\theta}}\, d\theta$$

$$= \sqrt{2}\int \frac{du}{u^{1/2}} \quad \left(\begin{array}{l} u = 1-\cos\theta \\ du = \sin\theta\, d\theta \end{array}\right)$$

$$= 2\sqrt{2}(1-\cos\theta)^{\frac{1}{2}} + C.$$

$\therefore$ Length $= 4\int_0^{\pi} \sqrt{2+2\cos\theta}\, d\theta$

$$= 4\left[2\sqrt{2}(1-\cos\theta)^{\frac{1}{2}}\right]_0^{\pi}$$

$$= 16.$$