Ex 9.2 (6)

$$\frac{dy}{dx} = \frac{d}{d\theta}(\cos^3\theta) \Big/ \frac{d}{d\theta}(\sin^3\theta)$$

$$= -3\cos^2\theta\sin\theta \Big/ 3\sin^2\theta\cos\theta$$

$$= -\frac{\cos\theta}{\sin\theta}.$$

$\therefore$ At $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}$.

At $\theta = \frac{\pi}{6}$, $(x, y) = \left(\sin^3\frac{\pi}{6}, \cos^3\frac{\pi}{6}\right)$

$$= \left(\frac{1}{2^3}, \left(\frac{\sqrt{3}}{2}\right)^3\right)$$

$$= \left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$$

$\therefore$ Tangent:

$$y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right).$$

29) At the intersection points:

$$0 = y = t - t^2$$

$\therefore$ $t(t-1) = 0$ ie $t = 0$ or 1.

$$\text{Area} = \int_0^1 y\frac{dx}{dt}\,dt = \int_0^1 (t-t^2)e^t\,dt$$

$$= \left[(t-t^2)e^t\right]_0^1 - \int_0^1 e^t(1-2t)\,dt$$

$$= -\int_0^1 e^t\,dt + 2\int_0^1 e^t \cdot t\,dt$$

$$= \left[-e^t\right]_0^1 + 2\left[e^t \cdot t\right]_0^1 - 2\int_0^1 e^t \, dt$$

$$= (-e+1) + 2e - 2\left[e^t\right]_0^1$$

$$= e+1 - 2(e-1)$$

$$= 3 - e$$

37) $$\text{Length} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2} \, dt$$

$$= 6\int_0^1 t\sqrt{1+t^2} \, dt$$

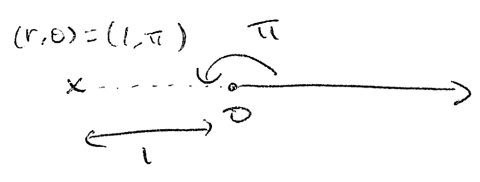
$$= 3\int_1^2 \sqrt{u} \, du \qquad u = 1+t^2$$
$$du = 2t\,dt$$

$$= 2\left[u^{\frac{3}{2}}\right]_1^2$$

$$= 2(2\sqrt{2} - 1)$$

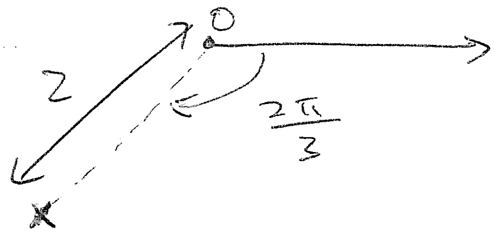
$$= 4\sqrt{2} - 2.$$

<u>Ex 9.3</u> 3(a)



$$(x,y) = (\cos\pi, \sin\pi)$$
$$= (-1, 0)$$

(b)



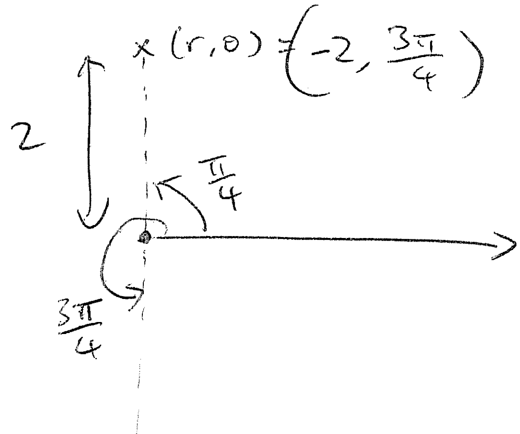
$(r,\theta) = \left(2, -\frac{2\pi}{3}\right)$

$$(x,y) = \left(2\cos\left(\frac{-2\pi}{3}\right),\ 2\sin\left(\frac{-2\pi}{3}\right)\right)$$

$$= \left(2\left(-\frac{1}{2}\right),\ 2\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \left(-1,\ -\sqrt{3}\right)$$

(c)



$(r,\theta) = \left(-2, \frac{3\pi}{4}\right)$

$$(x,y) = \left(-2\cos\frac{3\pi}{4},\ -2\sin\frac{3\pi}{4}\right)$$

$$= \left(-2\cdot 0,\ -2(-1)\right)$$

$$= (0, 2)$$

<u>Recommended</u>

1) $$\frac{dy}{dx} = \frac{(t^2+t)'}{(t\sin t)'} = \frac{2t+1}{\sin t + t\cos t}.$$

2) $$\frac{dy}{dx} = \frac{(\sqrt{t}\,e^{-t})'}{(1/t)'} = \frac{\frac{1}{2\sqrt{t}}e^{-t} - \sqrt{t}\,e^{-t}}{-1/t^2}$$

$$= -\frac{t^{3/2}e^{-t}}{2} + t^{\frac{5}{2}}e^{-t}.$$

3) $\frac{dy}{dx} = \frac{(2-t^3)'}{(1+4t-t^2)'} = \frac{-3t^2}{4-2t}$

At $t=1$, $\left.\frac{dy}{dx}\right|_{t=1} = \frac{-3}{2}$

$(x(1), y(1)) = (4, 1)$

$\therefore$ Tangent: $y - 1 = -\frac{3}{2}(x-4)$.

4) Omitted.

5) $\frac{dy}{dx} = \frac{(t\sin t)'}{(t\cos t)'} = \frac{\sin t + t\cos t}{\cos t - t\sin t}$

$\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{0 + \pi(-1)}{-1 - \pi \cdot 0} = \pi$

$(x(\pi), y(\pi)) = (-\pi, 0)$

$\therefore$ Tangent: $y = \pi(x+\pi)$.

13) $\frac{dy}{dx} = \frac{(t^2-3)'}{(t^3-3t)'} = \frac{2t}{3t^2-3}$

$\frac{dy}{dx} = 0 \iff t = 0$ $\qquad \frac{dy}{dx} = \infty \iff t = \pm 1$.

$\therefore$ Horizontal tangents at $t=0$ i.e. $(x,y) = (0,-3)$

Vertical tangents at $(x,y) = (-2,-2)$ and $(2,-2)$

14, 16) Omitted.

15) $$\frac{dy}{dx} = \frac{(\sin 2\theta)'}{(2\cos\theta)'} = \frac{2\cos 2\theta}{-2\sin\theta}$$

$$\frac{dy}{dx} = 0 \Leftrightarrow \cos 2\theta = 0$$

$$\Leftrightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

(for these θ, $\sin\theta \neq 0$)

$$\frac{dy}{dx} = \infty \Leftrightarrow \sin\theta = 0$$

$$\Leftrightarrow \theta = 0 \text{ or } \pi$$

$\therefore$ Horizontal tangents: $(x,y) = (2\cos\theta, \sin 2\theta)$ where $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ ie. $(x,y) = (\sqrt{2}, 1)$, $(-\sqrt{2}, -1)$, $(-\sqrt{2}, 1)$, $(\sqrt{2}, -1)$

Vertical tangents: $(x,y) = (2,0), (-2,0)$

38, 40) Omitted.

39) Length $= \int_0^1 \sqrt{(t\sin t)'^2 + (t\cos t)'^2}\, dt$

$$= \int_0^1 \left[(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2\right]^{\frac{1}{2}} dt$$

$$= \int_0^1 \sqrt{1+t^2}\, dt$$

Let $t = \tan\theta$. $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dt = \sec^2\theta \, d\theta$

$$\therefore \int_0^1 \sqrt{1+t^2}\, dt$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta} \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec\theta \, (\tan\theta)' \, d\theta$$

$$= \Big[\sec\theta \tan\theta\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2\theta \cdot \sec\theta \, d\theta$$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\sec\theta \, d\theta$$

$$= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta + \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$\Rightarrow 2\int_0^1 \sqrt{1+t^2}\, dt = \sqrt{2} + \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

(note: $\int_0^1 \sqrt{1+t^2}\, dt = \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$)

$$= \sqrt{2} + \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2} + \ln(1+\sqrt{2})$$

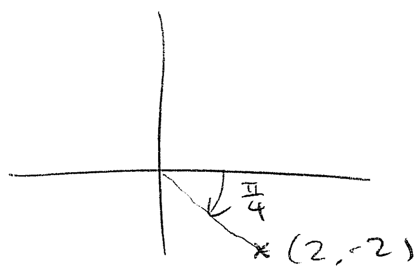
$$\therefore \text{Length} = \int_0^1 \sqrt{1+t^2}\, dt = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln(1+\sqrt{2}) .$$

Ex 9.3) (3), (4) omitted.

5(a) (i) $r = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\tan\theta = \frac{-2}{2} = -1 \Rightarrow \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

(x, y) lies on 4th quad.

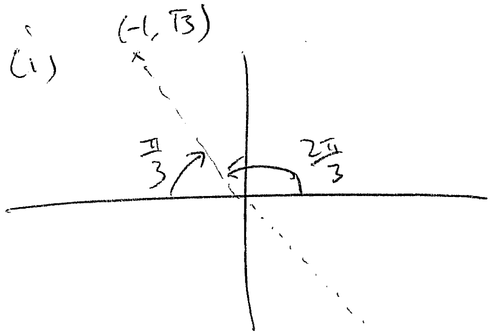


$\therefore \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$\therefore (r, \theta) = \left(2\sqrt{2}, \frac{7\pi}{4}\right)$

(ii) $(r, \theta) = \left(-2\sqrt{2}, \frac{3\pi}{4}\right)$

(b) (i)



$r = \sqrt{(-1)^2 + \sqrt{3}^2}$

$= \sqrt{4} = 2$,

$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

$\therefore \theta = \pi - \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3}$

ie. $\frac{2\pi}{3}$ or $\frac{5\pi}{3}$

(x, y) lies on 2nd quad. $\therefore \theta = \frac{2\pi}{3}$

$\therefore (r, \theta) = \left(2, \frac{2\pi}{3}\right)$,

(ii) $(r, \theta) = \left(-2, \frac{5\pi}{3}\right)$. (6) Omitted