Ex 8.6 (6)

$$f(x) = \frac{1}{x+10} = \frac{1}{10} \cdot \frac{1}{1+\frac{x}{10}}$$

$$= \frac{1}{10} \cdot \frac{1}{1-(-\frac{x}{10})}$$

$$= \frac{1}{10}\left(1 + \left(-\frac{x}{10}\right) + \left(\frac{-x}{10}\right)^2 + \left(\frac{-x}{10}\right)^3 + \dots\right)$$

$$= \frac{1}{10} - \frac{x}{10^2} + \frac{x^2}{10^3} - \frac{x^3}{10^4} + \dots$$

Ex 8.7 (9) $f(x) = \sinh x \qquad f(0) = 0$

$$f'(x) = \cosh x \quad \Rightarrow \quad f'(0) = 1$$

$$f''(x) = \sinh x \qquad f''(0) = 0$$

$$f^{(3)}(x) = \cosh x \qquad f^{(3)}(0) = 1$$

$$\vdots \qquad \vdots$$

Maclaurin Series of $\sinh x$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

17) $f(x) = \cos x \qquad \therefore f(\pi) = -1$

$$f'(x) = -\sin x \qquad f'(\pi) = 0$$

$$f''(x) = -\cos x \qquad f''(\pi) = 1$$

$$f^{(3)}(x) = \sin x \qquad f^{(3)}(\pi) = 0$$

$$f^{(4)}(x) = \cos x = f^{(0)}(x) \dots \qquad f^{(4)}(\pi) = -1 \text{ etc.}$$

$\therefore$ Taylor series at π

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$$

$$= -\frac{1}{0!} + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots$$

44) $$e^x - 1 = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - 1$$

$$= \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\therefore \frac{e^x-1}{x} = \frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

$$\Rightarrow \int \frac{e^x-1}{x}\,dx = \int\left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right)dx$$

$$= C + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \frac{x^5}{5 \cdot 5!} + \cdots$$

Recommended

4) For $|4x^2| < 1$ ie. $-\frac{1}{2} < x < \frac{1}{2}$,

$$\frac{5}{1-4x^2} = 5\left(1 + 4x^2 + (4x^2)^2 + (4x^2)^3 + \dots\right)$$

$$= 5 + 5 \cdot 4x^2 + 5 \cdot 4^2 x^4 + 5 \cdot 4^3 x^6$$
$$+ 5 \cdot 4^4 x^8 + \dots$$

3) $$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$= 1 + (-x) + (-x)^2 + (-x)^3 + \dots$$

$$= 1 - x + x^2 - x^3 + x^4 \dots$$

for $-1 < x < 1$.

5) Omitted.

7) $$\frac{x}{9+x^2} = \frac{1}{9} \cdot \frac{x}{1+\frac{x^2}{9}}$$

$\therefore$ for $\left|\frac{x^2}{9}\right| < 1$ ie. $-3 < x < 3$,

$$\frac{x}{9+x^2} = \frac{x}{9}\left(1 - \frac{x^2}{9} + \left(\frac{x^2}{9}\right)^2 - \left(\frac{x^2}{9}\right)^3 + \dots\right)$$

$$= \frac{x}{9} - \frac{x^3}{9^2} + \frac{x^5}{9^3} - \frac{x^7}{9^4} + \frac{x^9}{9^5} - \dots$$

8) Omitted

9) For $-1 < x < 1$,

$$\frac{1+x}{1-x} = (1+x)\cdot\frac{1}{1-x}$$
$$= (1+x)(1+x+x^2+x^3+\dots)$$
$$= (1+x+x^2+x^3+x^4+\dots)$$
$$+ (x+x^2+x^3+x^4+\dots)$$
$$= 1+2x+2x^2+2x^3+2x^4+\dots$$

10) For $\left|\frac{x^3}{a^3}\right| < 1$ ie. $-|a| < x < |a|$,

$$\frac{x^2}{a^3-x^3} = \frac{x^2}{a^3}\cdot\frac{1}{1-\frac{x^3}{a^3}}$$
$$= \frac{x^2}{a^3}\left(1+\frac{x^3}{a^3}+\left(\frac{x^3}{a^3}\right)^2+\left(\frac{x^3}{a^3}\right)^3+\dots\right)$$
$$= \frac{x^2}{a^3}\left(1+\frac{x^3}{a^3}+\frac{x^6}{a^6}+\frac{x^9}{a^9}+\dots\right)$$
$$= \frac{x^2}{a^3}+\frac{x^5}{a^6}+\frac{x^8}{a^9}+\frac{x^{11}}{a^{12}}+\dots$$

15) $\ln(5-x)$

$$= \ln 5 + \ln\left(1-\frac{x}{5}\right)$$
$$= \ln 5 + \ln\left(1+\left(-\frac{x}{5}\right)\right)$$
$$= \ln 5 + \left[\left(\frac{-x}{5}\right) - \frac{1}{2}\left(\frac{-x}{5}\right)^2 + \frac{1}{3}\left(\frac{-x}{5}\right)^3 - \frac{1}{4}\left(\frac{-x}{5}\right)^4 + \dots\right]$$
(if $\left|\frac{x}{5}\right| < 1$ ie $-5 < x < 5$)

$$= \ln 5 - \frac{x}{5} - \frac{x^2}{2\cdot 5^2} - \frac{x^3}{3\cdot 5^3} - \frac{x^4}{4\cdot 5^4} - \cdots$$

for $-5 < x < 5$,

(We've used $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$)

16) $x^2 \tan^{-1}(x^3)$

$$= x^2\left(x^3 - \frac{(x^3)^3}{3} + \frac{(x^3)^5}{5} - \frac{(x^3)^7}{7} + \cdots\right)$$

$$= x^5 - \frac{x^{11}}{3} + \frac{x^{17}}{5} - \frac{x^{23}}{7} + \cdots$$ (if $|x^3|<1$, ie $-1<x<1$)

17) We first find the series of $\frac{1}{(1+4x)^2}$.

Consider $-\frac{1}{4}\left(\frac{1}{1+4x}\right)' = \frac{1}{(1+4x)^2}$ and for $-\frac{1}{4}<x<1$,

$$\frac{1}{1+4x} = \frac{1}{1-(-4x)} = 1 - 4x + 4^2x^2 - 4^3x^3 + 4^4x^4 + \cdots$$

$$\therefore \frac{1}{(1+4x)^2} = -\frac{1}{4}\left[1 - 4x + 4^2x^2 - 4^3x^3 + 4^4x^4 + \cdots\right]'$$

$$= -\frac{1}{4}\left[-4 + 2\cdot 4^2 x - 3\cdot 4^3 x^2 + 4\cdot 4^4 x^3 + \cdots\right]$$

$$= 1 - 2\cdot 4x + 3\cdot 4^2x^2 - 4\cdot 4^3\cdot x^3 + \cdots$$

$$\therefore \frac{x}{(1+4x)^2} = x - 2\cdot 4x^2 + 3\cdot 4^2x^3 - 4\cdot 4^3x^4 + 5\cdot 4^4x^5 - \cdots$$

18 and 19 are similar to (17).

20) Consider $\left(\frac{1}{1-x}\right)'' = \frac{2}{(1-x)^3}$

$$\therefore \frac{1}{(1-x)^3} = \frac{1}{2}\left(\frac{1}{1-x}\right)''$$

$$= \frac{1}{2}\left(1+x+x^2+x^3+\cdots\right)''$$

(if $-1<x<1$.)

$$= \frac{1}{2}\left(1+2x+3x^2+4x^3+\cdots\right)'$$

$$= \frac{1}{2}\left(2+3\cdot 2x+4\cdot 3x^2+5\cdot 4x^3+\cdots\right)$$

$$\therefore \frac{x^2+x}{(1-x)^3} = \frac{1}{2}(x^2+x)\left(2+3\cdot 2x+4\cdot 3x^2+5\cdot 4x^3+\cdots\right)$$

$$= \frac{1}{2}\Big[\left(2x^2+3\cdot 2x^3+4\cdot 3x^4+5\cdot 4x^5+\cdots\right) + \left(2x+3\cdot 2x^2+4\cdot 3x^3+5\cdot 4x^4+\cdots\right)\Big]$$

$$= \frac{1}{2}\Big[2x+(2\cdot 1+3\cdot 2)x^2+(3\cdot 2+4\cdot 3)x^3 + (4\cdot 3+5\cdot 4)x^4+\cdots\Big]$$

$$= x+4x^2+9x^3+16x^4+\cdots$$

(note: $\frac{n(n-1)+(n+1)\cdot n}{2} = n^2$)

We now ignore the radius of convergence.

25) $\int \frac{t}{1-t^8}\,dt$

$= \int t(1+t^8+t^{16}+t^{24}+\dots)\,dt$

$= \int (t+t^9+t^{17}+t^{25}+\dots)\,dt$

$= C + \frac{t^2}{2} + \frac{t^{10}}{10} + \frac{t^{18}}{18} + \frac{t^{26}}{26} + \dots$

26) similar to (25).

27) $\int x^2 \ln(1+x)\,dx$

$= \int x^2 (x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots)\,dx$

$= \int (x^3 + \frac{x^4}{2} + \frac{x^5}{3} + \frac{x^6}{4} + \dots)\,dx$

$= C + \frac{x^4}{1\cdot 4} + \frac{x^5}{2\cdot 5} + \frac{x^6}{3\cdot 6} + \frac{x^7}{4\cdot 7} + \dots$

28) $\int \frac{\tan^{-1} x}{x}\,dx = \int \frac{1}{x}(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots)\,dx$

$= \int (1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots)\,dx$

$= C + x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \dots$

Ex 8.7 (5) $f(x) = (1-x)^{-2}$

$$f'(x) = 2(1-x)^{-3}$$

$$f''(x) = 3!\,(1-x)^{-4}$$

$$f^{(3)}(x) = 4!\,(1-x)^{-5} \quad \text{etc.}$$

$\therefore f(0) = 1, \quad f'(0) = 2! \quad f''(0) = 3!$ etc.

$$\therefore (1-x)^{-2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (\text{for } |x|<1)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

(This also follows by diff. $\frac{1}{1-x} = 1 + x + x^2 + \ldots$)

6) $f(x) = e^{-2x}$ $\qquad \therefore f(0) = 1$

$$f'(x) = -2e^{-2x} \qquad f'(0) = -2$$

$$f''(x) = 2^2 e^{-2x} \qquad f''(0) = 2^2$$

$$f^{(3)}(x) = -2^3 e^{-2x} \qquad f'''(0) = -2^3 \quad \text{etc.}$$

$$\vdots$$

$$\therefore f(x) = 1 - 2x + \frac{2^2x^2}{2!} - \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} - \ldots$$

for $-\infty < x < \infty$.

(Also follows by applying $e^x = 1 + x + \frac{x^2}{2!} + \ldots$)

7) Using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\sin \pi x = \pi x - \frac{\pi^3 x^3}{3!} + \frac{\pi^5 x^5}{5!} - \frac{\pi^7 x^7}{7!} + \dots$$

(for $-\infty < x < \infty$)

8) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ for $-\infty < x < \infty$

$$\therefore x \cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots$$

10) $f^{(2n)}(0) = 1$ and $f^{(2n+1)}(0) = 0$ for all n.

$$\therefore \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

for $-\infty < x < \infty$.

11) By long division, or by letting $x = y+1$ (ie. $y = x-1$),

$$\begin{aligned} x^4 - 3x^2 + 1 &= (y+1)^4 - 3(y+1)^2 + 1 \\ &= y^4 + 4y^3 + 3y^2 - 2y - 1 \\ &= -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 \\ &\quad + (x-1)^4. \end{aligned}$$

13) $\ln x = \ln\left(2+(x-2)\right)$

$$= \ln 2 + \ln\left(1+\frac{x-2}{2}\right)$$

$$= \ln 2 + \left(\frac{x-2}{2} - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 - \frac{1}{4}\left(\frac{x-2}{2}\right)^4 + \dots\right)$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{2\cdot 2^2}(x-2)^2 + \frac{1}{3\cdot 2^3}(x-2)^3 - \frac{1}{4\cdot 2^4}(x-2)^4 + \dots$$

$\left(\text{for } \left|\frac{x-2}{2}\right| < 1 \text{ ie. } -2 < x-2 < 2\right)$

14) $f(x) = \frac{1}{x} = \frac{1}{x-(-3)-3}$

$$= -\frac{1}{3-(x+3)}$$

$$= -\frac{1}{3}\cdot\frac{1}{1-\frac{x+3}{3}}$$

$$= -\frac{1}{3}\left(1+\frac{x+3}{3}+\left(\frac{x+3}{3}\right)^2+\left(\frac{x+3}{3}\right)^3+\dots\right)$$

$$= -\frac{1}{3} - \frac{x-(-3)}{3^2} - \frac{(x-(-3))^2}{3^3} - \frac{(x-(-3))^3}{3^4} + \dots$$

if $|x-(-3)| < 3$.

15) $$e^{2x} = e^{2(x-3)+6}$$

$$= e^6 \cdot e^{2(x-3)}$$

$$= e^6\left[1 + 2(x-3) + \frac{2^2(x-3)^2}{2!} + \frac{2^3(x-3)^3}{3!} + \dots\right]$$

$$= e^6 + 2e^6(x-3) + \frac{2^2e^6(x-3)^2}{2!}$$

$$+ \frac{2^3e^6}{3!}(x-3)^3 + \dots$$

for $-\infty < x < \infty$.

16) $$f(x) = \sin x$$

$$= \sin\left[\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= \cos\left(x - \frac{\pi}{2}\right)$$

$$= 1 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6 + \dots$$

for $-\infty < x < \infty$.

17) Similar to (16).

18) $$\sqrt{x} = x^{\frac{1}{2}} = (16 + (x-16))^{\frac{1}{2}}$$

$$= 4\left(1 + \frac{x-16}{16}\right)^{\frac{1}{2}}$$

$$= 4\left[1 + \frac{1}{2}\cdot\frac{x-16}{16} + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}\left(\frac{x-16}{16}\right)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}\left(\frac{x-16}{16}\right)^3 + \dots\right]$$

(Binomial thm.)

$$= 4 + \frac{x-16}{8} - \frac{1}{2!\,16^2}(x-16)^2$$

$$+ \frac{\frac{3}{2}}{3!\,16^3}\cdot(x-16)^3 - \frac{\frac{3}{2}\cdot\frac{5}{2}}{4!\,16^4}(x-16)^4 + \dots$$

for $|x-16| < 16$.

(Binomial thm. not included in exam!).

27, 28 similar to (7).

29 similar to:

30) $e^x + 2e^{-x}$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + 2\left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \left(2 - 2x + \frac{2x^2}{2!} - \frac{2x^3}{3!} + \dots\right)$$

$$= 3 - x + \frac{3x^2}{2!} - \frac{x^3}{3!} + \frac{3x^4}{4!} - \dots$$

31) similar to (8).

32) $x^2 \ln(1+x^3)$

$$= x^2\left(x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \dots\right)$$

$$= x^5 - \frac{x^8}{2} + \frac{x^{11}}{3} - \frac{x^{14}}{4} + \dots \qquad (|x| < 1)$$

33). (Require Binomial thm.)

$$\frac{1}{\sqrt{4+x^2}} = (4+x^2)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\cdot\frac{x^2}{4} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(\frac{x^2}{4}\right)^3 + \cdots\right]$$

$$= \frac{1}{2} - \frac{1}{2^2\cdot 4}x^2 + \frac{3\cdot 1}{2^3\cdot 2!\,4^2}x^4 - \frac{5\cdot 3\cdot 1}{2^4\cdot 3!\,4^3}x^6 + \cdots \quad (|x|<2)$$

$$\frac{x}{\sqrt{4+x^2}} = \frac{1}{2}x - \frac{1}{2^2\cdot 4}x^3 + \frac{3\cdot 1}{2^3\cdot 2!\,4^2}x^5 - \frac{5\cdot 3\cdot 1}{2^4\cdot 3!\,4^3}x^7 + \cdots \quad (|x|<2).$$

34) Similar to (33).

35) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$= \frac{1}{2} - \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$

$$= \frac{2x^2}{2!} - \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \cdots$$

for $-\infty < x < \infty$.

36) Omitted.

43) $\int x\cos(x^3)\,dx$

$$= \int x\left(1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \dots\right)dx$$

$$= \int \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \dots\right)dx$$

$$= C + \frac{x^2}{2!} - \frac{x^8}{8\cdot 2!} + \frac{x^{14}}{14\cdot 4!} - \frac{x^{20}}{20\cdot 6!} + \dots$$

45) $\int \frac{\cos x - 1}{x}\,dx$

$$= \int \frac{1}{x}\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - 1\right]dx$$

$$= \int \frac{1}{x}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)dx$$

$$= \int \left(-\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \dots\right)dx$$

$$= C - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \frac{x^8}{8\cdot 8!} - \dots$$

46) $\int \tan^{-1}(x^2)\,dx = \int \left(x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \dots\right)dx$

$$= \int \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots\right)dx$$

$$= C + \frac{x^3}{3} - \frac{x^7}{7\cdot 3} + \frac{x^{11}}{11\cdot 5} - \frac{x^{15}}{15\cdot 7} + \dots$$

47-50: We ignore the error estimates and just compute a few terms.

47) $\int_0^1 x \cos x^3 \, dx$

$$= \int_0^1 x \left(1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \dots \right) dx$$

$$= \int_0^1 \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \dots \right) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} + \dots \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} + \frac{1}{20 \cdot 6!} + \dots$$

$\approx 0.440.$

48, 49 omitted.

50) $\int_0^{0.5} x^2 e^{-x^2} dx$

$$= \int_0^{0.5} x^2 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots \right) dx$$

$$= \int_0^{0.5} \left(x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \frac{x^{10}}{4!} - \dots \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \dots \right]_0^{0.5}$$

$$= \frac{0.5^3}{3} - \frac{0.5^5}{5} + \frac{0.5^7}{7 \cdot 2!} - \frac{0.5^9}{9 \cdot 3!} + \dots \approx 0.036$$

51) $\lim_{x\to 0} \dfrac{x - \ln(1+x)}{x^2}$

$= \lim_{x\to 0} \dfrac{x - (x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots)}{x^2}$

$= \lim_{x\to 0} \dfrac{\frac{x^2}{2} - \frac{x^3}{3} + \cdots}{x^2}$

$= \lim_{x\to 0} \left(\frac{1}{2} - \frac{x}{3} + \cdots\right)$

$= \frac{1}{2}$.

52) $\lim_{x\to 0} \dfrac{1 - \cos x}{1 + x - e^x}$

$= \lim_{x\to 0} \dfrac{1 - (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)}{1 + x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)}$

$= \lim_{x\to 0} \dfrac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{-\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}$

$= \lim_{x\to 0} \dfrac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots}$

$= \frac{1}{2!} \Big/ -\frac{1}{2!}$

$= -1$.

53) $\displaystyle \lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) - x + \frac{x^3}{6}}{x^5}$$

$$= \lim_{x \to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5}$$

$$= \lim_{x \to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \dots \right)$$

$$= \frac{1}{5!} .$$