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$$\begin{aligned}\int_1^\infty \frac{\ln x}{x}dx &= \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx\\ &= \lim_{t\to\infty}\int_1^t \ln x(\ln x)'dx\\ &= \lim_{t\to\infty}\int_0^{\ln t} udu \quad (u=\ln x)\\ &= \lim_{t\to\infty}\frac{1}{2}(\ln t)^2\\ &=\infty.\end{aligned}$$

So the improper integral $\int_1^\infty \frac{\ln x}{x}dx$ is divergent.

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$$\begin{aligned}\int_0^\infty \frac{e^x}{e^{2x}+3}dx &= \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+3}dx\\ &= \lim_{t\to\infty}\int_1^{e^t}\frac{1}{u^2+3}du \quad (u=e^x)\\ &= \lim_{t\to\infty}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_1^{e^t}\\ &=\frac{1}{\sqrt{3}}\lim_{s\to\infty}\tan^{-1}(s)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \left(\frac{e^t}{\sqrt{3}}\to\infty \text{ as } t\to\infty\right)\\ &=\frac{1}{\sqrt{3}}\cdot\frac{\pi}{2}-\frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}\\ &=\frac{\pi}{3\sqrt{3}}.\end{aligned}$$

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$$\begin{aligned}\int_0^5 \frac{w}{w-2}dw &= \int_0^2 \frac{w}{w-2}dw+\int_2^5\frac{w}{w-2}dw\\ &= \lim_{t\to 2^-}\int_0^t \frac{w}{w-2}dw+\lim_{t\to 2^+}\int_t^5\frac{w}{w-2}dw.\end{aligned}$$

Consider

$$\begin{aligned}\lim_{t\to 2^-}\int_0^t \frac{w}{w-2}dw &= \lim_{t\to 2^-}\int_0^t\left(1+\frac{2}{w-2}\right)dw\\ &= \lim_{t\to 2^-}\left[w+2\ln|w-2|\right]_0^t\\ &=2+\lim_{t\to 2^-}\ln(2-t)\\ &=-\infty.\end{aligned}$$

Therefore the improper integral $\int_0^5 \frac{w}{w-2}dw$ is divergent.

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$$\begin{aligned}\int_0^\infty \frac{x}{x^3+1}dx &= \int_0^1 \frac{x}{x^3+1}dx+\int_1^\infty \frac{x}{x^3+1}dx\\ &\le \int_0^1 \frac{x}{x^3+1}dx+\int_1^\infty \frac{x}{x^3}dx\\ &= \int_0^1 \frac{x}{x^3+1}dx+\int_1^\infty \frac{1}{x^2}dx.\end{aligned}$$

Note that $\int_0^1 \frac{x}{x^3+1}dx$ and $\int_1^\infty \frac{1}{x^2}dx = 1$ are both finite. So by comparison, $\int_0^\infty \frac{x}{x^2+1}dx$ is convergent.

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$$\begin{aligned}
\int_1^\infty \frac{2+e^{-x}}{x}dx \geq& \int_1^\infty \frac{2}{x}dx \\
&= \lim_{t\to\infty} \int_1^t \frac{2}{x}dx \\
&= \lim_{t\to\infty} \left[2\ln|x|\right]_1^t \\
&= \infty.
\end{aligned}$$

So by comparison, $\int_1^\infty \frac{2+e^{-x}}{x}dx$ is divergent.

5) $\int_3^\infty \frac{1}{(x-2)^{3/2}}dx = \lim_{t\to\infty}\int_3^t \frac{1}{(x-2)^{3/2}}dx$

$= \lim_{t\to\infty}\int_1^{t-2} \frac{1}{u^{3/2}}du \qquad (u = x-2)$

$= \lim_{s\to\infty}\int_1^{s} \frac{1}{u^{3/2}}du \qquad (s = t-2 \to \infty)$

$= \lim_{s\to\infty}\left[-2u^{-\frac{1}{2}}\right]_1^s$

$= 2$. i.e. convergent.

6) $\int_0^\infty \frac{1}{(x+1)^{1/4}}dx = \int_0^1 \frac{1}{(x+1)^{1/4}}dx + \int_1^\infty \frac{1}{(x+1)^{1/4}}dx$

$\geq \int_0^1 \frac{1}{(x+1)^{1/4}}dx + \int_1^\infty \frac{1}{(2x)^{1/4}}dx \qquad (x+1 \leq 2x \text{ for } x \geq 1)$

$= \int_0^1 \frac{1}{(x+1)^{1/4}}dx + \frac{1}{2^{\frac{1}{4}}}\int_1^\infty \frac{1}{x^{1/4}}dx$

As $\int_1^\infty \frac{1}{x^p}dx = \infty$ ~~for~~ $p \leq 1$ (Ex.),

$\therefore \int_0^\infty \frac{1}{(x+1)^{1/4}}dx = \infty$ i.e. divergent.

7) $\int_{\infty}^{0} \frac{1}{3-4x} dx = -\int_{\infty}^{0} \frac{1}{3+4u} du \qquad (u=-x)$

$= \int_{0}^{\infty} \frac{1}{4u+3} du$

$= \int_{0}^{3} \frac{1}{4u+3} du + \int_{3}^{\infty} \frac{1}{4u+3} du$

$\geq \int_{0}^{3} \frac{1}{4u+3} du + \int_{3}^{\infty} \frac{1}{5u} du \qquad (4u+3 \leq 5u$ if $u \geq 3)$

$= \infty \qquad$ as $\int_{3}^{\infty} \frac{1}{u} du = \infty$.

$\therefore$ divergent.

8) Similar to (5).

9) $\int_{2}^{\infty} e^{-5p} dp = \lim_{t\to\infty} \int_{2}^{t} e^{-5p} dp$

$= \lim_{t\to\infty} \left[-\frac{1}{5} e^{-5p}\right]_{2}^{t}$

$= \lim_{t\to\infty} \left(-\frac{1}{5} e^{-5t} + \frac{1}{5} e^{-10}\right)$

$= \frac{1}{5} e^{-10}$. $\quad \therefore$ Convergent.

10) $\int_{-\infty}^{0} 2^{-r} dr = -\int_{\infty}^{0} 2^{u} du \qquad (u=-r)$

$= \int_{0}^{\infty} 2^{u} du$

$= \lim_{t\to\infty} \int_{0}^{t} 2^{u} du$

$= \lim_{t\to\infty} \left[\frac{2^{u}}{\ln 2}\right]_{0}^{t} = \infty$. $\quad \therefore$ divergent.

11) $\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}dx = \int_0^1 \frac{x^2}{(1+x^3)^{\frac{1}{2}}}dx + \int_1^\infty \frac{x^2}{(1+x^3)^{\frac{1}{2}}}dx$

$\geq \int_0^1 \frac{x^2}{(1+x^3)^{\frac{1}{2}}}dx + \int_1^\infty \frac{x^2}{(2x^3)^{\frac{1}{2}}}dx$ ($x^3+1 \leq 2x^3$ for $x \geq 1$)

$= \int_0^1 \frac{x^2}{(1+x^3)^{\frac{1}{2}}}dx + \frac{1}{2^{\frac{1}{2}}}\int_1^\infty x^{\frac{1}{2}}dx$

$= \infty$ as $\int_1^\infty x^p dx = \infty$ for $p \geq -1$.

$\therefore$ divergent.

12) $\int_{-\infty}^\infty (y^3-3y^2)dy = \int_{-\infty}^0 (y^3-3y^2)dy + \int_0^\infty (y^3-3y^2)dy$

Consider $\int_0^\infty (y^3-3y^2)dy$

$= \int_0^6 (y^3-3y^2)dy + \int_6^\infty (y^3-3y^2)dy$

$\geq \int_0^6 (y^3-3y^2)dy + \int_6^\infty (y^3-\frac{1}{2}y^3)dy$ ($3y^2 \leq \frac{1}{2}y^3$ if $y \geq 6$)

$= \int_0^6 (y^3-3y^2)dy + \frac{1}{2}\int_6^\infty y^3 dy$

$= \infty$ as $\int_6^\infty y^3 dy = \infty$. $\therefore$ divergent.

$\therefore \int_{-\infty}^\infty (y^3-3y^2)dy$ is divergent.

13) $\int_{-\infty}^\infty x e^{-x^2}dx = \int_{-\infty}^0 x e^{-x^2}dx + \int_0^\infty x e^{-x^2}dx$

Consider $\int_0^\infty x e^{-x^2} dx$. Let $u = -x^2$ $\therefore du = -2x dx$

$$\int_0^\infty x e^{-x^2} dx = \lim_{t \to \infty} \int_0^t x e^{-x^2} dx$$

$$= \lim_{t \to \infty} \int_0^{-t^2} e^u \cdot (-\tfrac{1}{2} du)$$

$$= \lim_{t \to \infty} \left[-\tfrac{1}{2} e^u \right]_0^{-t^2}$$

$$= \lim_{t \to \infty} \left(-\tfrac{1}{2} e^{-t^2} + \tfrac{1}{2} \right)$$

$$= \tfrac{1}{2}.$$

Consider $\int_{-\infty}^0 x e^{-x^2} dx$. Let $u = -x^2$ $\therefore du = -2x dx$

$$\int_{-\infty}^0 x e^{-x^2} dx = \lim_{t \to -\infty} \int_t^0 x e^{-x^2} dx$$

$$= \lim_{t \to -\infty} \int_{-t^2}^0 e^u \cdot (\tfrac{-1}{2} du)$$

$$= \lim_{t \to -\infty} \left[\tfrac{-1}{2} e^u \right]_{-t^2}^0$$

$$= -\tfrac{1}{2}.$$

$\therefore \int_{-\infty}^\infty x e^{-x^2} dx = -\tfrac{1}{2} + \tfrac{1}{2} = 0$. ie. convergent.

14) $\int_0^{\infty} x^2 e^{-x^3} dx$

$= \lim_{t\to\infty} \int_0^t x^2 e^{-x^3} dx$

$= \lim_{t\to\infty} \int_0^{-t^3} e^u (-\frac{1}{3} du)$ $\quad (u=-x^3, \quad du=-3x^2dx)$

$= \lim_{t\to\infty} \left[-\frac{1}{3} e^u\right]_0^{-t^3}$

$= \frac{1}{3}$.

$\int_{-\infty}^0 x^2 e^{-x^3} dx = \lim_{t\to-\infty} \int_t^0 x^2 e^{-x^3} dx$

$= \lim_{t\to-\infty} \int_{-t^3}^0 e^u (-\frac{1}{3} du)$ $\quad (u=-x^3, du=-3x^2dx)$

$= \lim_{t\to-\infty} \left[-\frac{1}{3} e^u\right]_{-t^3}^0$

$= \lim_{t\to-\infty} \left(-\frac{1}{3} + \frac{1}{3} e^{-t^3}\right)$

$= \infty$.

$\therefore \int_{-\infty}^{\infty} x^2 e^{-x^3} dx = \int_{-\infty}^0 x^2 e^{-x^3} dx + \int_0^{\infty} x^2 e^{-x^3} dx$

$= \infty$ ie. divergent.

15) $\int_{-\infty}^0 z e^{2z} dz = \lim_{t\to-\infty} \int_t^0 z\left(\frac{e^{2z}}{2}\right)' dz$

$= \lim_{t\to-\infty} \left(\left[\frac{z e^{2z}}{2}\right]_t^0 - \frac{1}{2} \int_t^0 e^{2z} dz\right)$

$= \lim_{t\to-\infty} -\frac{1}{2} \int_t^0 e^{2z} dz$ $\quad (\lim_{t\to-\infty} t e^{2t} = 0)$

$$= \lim_{t\to-\infty} \left[\frac{-1}{4} e^{2z} \right]_t^0$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{4} + \frac{1}{4} e^{2t} \right)$$

$$= -\frac{1}{4}. \quad \therefore \text{Convergent.}$$

16) $\int_0^\infty \cos \pi t \, dt = \lim_{s\to\infty} \int_0^s \cos\pi t \, dt$

$$= \lim_{s\to\infty} \left[\frac{\sin \pi t}{\pi} \right]_0^s$$

$$= \lim_{s\to\infty} \left(\frac{\sin \pi s}{\pi} \right)$$

which doesn't exist. (why?) ie. divergent.

$\therefore \int_{-\infty}^\infty \cos\pi t \, dt$ is also divergent.

18) Similar to (15).

19) $\int_1^\infty \frac{1}{x^2+x} dx \leq \int_1^\infty \frac{1}{x^2} dx < \infty \quad (x^2+x \geq x^2 \text{ for } x \geq 1)$

$\therefore$ convergent.

To find its value, by partial fraction decomp.

$$\int_1^\infty \frac{1}{x^2+x} dx = \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \lim_{t\to\infty} \left[\ln \left| \frac{x}{x+1} \right| \right]_1^t$$

$$= -\ln \frac{1}{2} \quad \left(\lim_{t\to\infty} \frac{t}{t+1} = 1 \right)$$

$$= \ln 2.$$

20) $\int_1^\infty \frac{\ln x}{x^3}\,dx = \int_1^\infty \ln x \left(\frac{-x^{-2}}{2}\right)' dx$

$= \lim_{t\to\infty}\left[-\frac{x^{-2}\ln x}{2}\right]_1^t + \int_1^\infty \frac{1}{2}\cdot x^{-3}\,dx$ (I.B.P.)

$= \frac{1}{2}\int_1^\infty x^{-3}\,dx$ $\left(\lim_{t\to\infty} t^{-2}\ln t \overset{L'H}{=} 0\right)$

$= \frac{1}{2}\lim_{t\to\infty}\left[\frac{x^{-2}}{-2}\right]_1^t$

$= \frac{1}{4}$.

21) $\int_0^\infty \frac{x^2}{9+x^6}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{x^6+9}\,dx$

$= \lim_{t\to\infty}\frac{1}{3}\int_0^{t^3}\frac{du}{u^2+9}$ $u = x^3,\ du = 3x^2dx$

$= \lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\frac{u}{3}\right]_0^{t^3}$

$= \frac{1}{9}\cdot\frac{\pi}{2} = \frac{\pi}{18}$

By symmetry. $\int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx = \int_0^\infty \frac{x^2}{9+x^6}\,dx$ $\left(\frac{x^2}{9+x^6}\text{ even}\right)$

$= \frac{\pi}{18}$

$\therefore \int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx = \frac{\pi}{18} + \frac{\pi}{18} = \frac{\pi}{9}$ is convergent,

23) $\int_0^1 \frac{3}{x^5}dx = \lim_{t\to 0^+} \int_t^1 \frac{3}{x^5}dx$

$= \lim_{t\to 0^+} \left[\frac{-3}{4}x^{-4}\right]_t^1$

$= \lim_{t\to 0^+}\left(-\frac{3}{4} + \frac{3}{4t^4}\right)$

$= \infty$. $\therefore$ divergent.

24) $\int_2^3 \frac{1}{\sqrt{3-x}}dx = \lim_{t\to 3^-} \int_2^t \frac{1}{\sqrt{3-x}}dx$

$= \lim_{t\to 3^-} -\int_1^{3-t} \frac{1}{\sqrt{u}}du$ $\quad (u=3-x \;\therefore du=-dx)$

$= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{u}}du$ $\quad$ (why?)

$= \lim_{t\to 0^+}\left[2\sqrt{u}\right]_t^1$

$= 2$. $\therefore$ convergent.

25) Similar to (24)

26) $\int_6^8 \frac{4}{(x-6)^3}dx = \lim_{t\to 6^+} \int_t^8 \frac{4}{(x-6)^3}dx$

$= \lim_{t\to 6^+} \int_{t-6}^2 \frac{4}{u^3}du$ $\quad (u=x-6)$

$= \lim_{t\to 0^+} \int_t^2 \frac{4}{u^3}du$

$= \lim_{t\to 0^+}\left[-2u^{-2}\right]_t^2$

$= \infty$. $\therefore$ divergent.

27) $$\int_0^9 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx = \int_0^1 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx + \int_1^9 \frac{1}{(x-1)^{\frac{1}{3}}}\,dx$$

Consider $$\int_0^1 \frac{1}{(x-1)^{1/3}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^{1/3}}\,dx$$

$$= \lim_{t\to 1^-} \int_{-1}^{t-1} \frac{du}{u^{\frac{1}{3}}} \qquad (u = x-1)$$

$$= \lim_{t\to 0^-} \int_{-1}^{t} \frac{du}{u^{1/3}}$$

$$= \lim_{t\to 0^-} \left[\frac{3}{2} u^{2/3}\right]_{-1}^{t}$$

$$= -\frac{3}{2}.$$

$$\int_1^9 \frac{1}{(x-1)^{1/3}}\,dx = \lim_{t\to 1^+} \int_t^9 \frac{1}{(x-1)^{1/3}}\,dx$$

$$= \lim_{t\to 1^+} \int_{t-1}^{8} \frac{du}{u^{\frac{1}{3}}} \qquad (u = x-1)$$

$$= \lim_{t\to 0^+} \int_t^8 \frac{du}{u^{1/3}}$$

$$= \lim_{t\to 0^+} \left[\frac{3}{2} u^{2/3}\right]_t^8$$

$$= \frac{3}{2}\cdot 8^{2/3} = 6.$$

$\therefore$ $$\int_0^9 \frac{1}{(x-1)^{1/3}}\,dx = -\frac{3}{2} + 6 = \frac{9}{2}$$ is convergent.

29) $$\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx = \int_{-1}^{0} \frac{e^x}{e^x-1}\,dx + \int_{0}^{1} \frac{e^x}{e^x-1}\,dx$$

$$\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^-} \int_{-1}^{t} \frac{e^x}{e^x-1}\,dx$$

$$= \lim_{t\to0^-} \int_{e^{-1}}^{e^t} \frac{du}{u-1} \qquad u = e^x$$

$$= \lim_{t\to0^-} \Big[\ln|u-1|\Big]_{e^{-1}}^{e^t}$$

$$= \lim_{t\to0^-} \left[\ln|e^t-1| - \ln\left(1-\frac{1}{e}\right)\right]$$

$$= \lim_{t\to0^-} \left[\ln(1-e^t) - \ln\left(1-\frac{1}{e}\right)\right]$$

$$= -\infty.$$

$1-e^t \to 0^+$ as $t\to0^-$

$\therefore \int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$ is divergent.

30) $$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to1} \int_0^t \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{t\to1} \left(\sin^{-1} t\right) \qquad \left(\sin^{-1} 0 = 0\right)$$

$= \frac{\pi}{2}$ is convergent.

31) $$\int_0^2 z^2 \ln z\,dz = \lim_{t\to0^+} \int_t^2 z^2 \ln z\,dz$$

For $t > 0$, consider

$$\int_t^2 z^2 \ln z \, dz = \frac{z^3}{3}\ln z - \frac{1}{3}\int z^2 \, dz \quad \text{(I.b.p.)}$$

$$= \frac{z^3 \ln z}{3} - \frac{z^3}{9} + C$$

$$\therefore \int_0^2 z^2 \ln z \, dz = \lim_{t \to 0^+} \int_t^2 z^2 \ln z \, dz$$

$$= \lim_{t \to 0^+} \left[\frac{z^3 \ln z}{3} - \frac{z^3}{9} \right]_t^2$$

$$= \frac{8 \ln 2}{3} - \frac{8}{9}. \quad \left(\lim_{t \to 0^+} t^3 \ln t \overset{L'H}{=} 0 \right)$$

i.e. convergent.

32) Let $t > 0$. consider

$$\int_t^1 \frac{\ln x}{\sqrt{x}} \, dx = \int_t^1 \ln x \, (2\sqrt{x})' \, dx$$

$$= \left[2\sqrt{x} \ln x \right]_t^1 - 2\int_t^1 \frac{1}{\sqrt{x}} \, dx$$

$$= -2\sqrt{t} \ln t - 4\left[\sqrt{x} \right]_t^1$$

$$\to -4 \text{ as } t \to 0^+ \quad (\sqrt{t}\ln t \to 0 \text{ as } t \to 0^+)$$

$$\therefore \int_0^1 \frac{\ln x}{\sqrt{x}} \, dx = \lim_{t \to 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}} \, dx = -4. \quad \text{(convergent.)}$$

43) $$\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx \geq \int_1^\infty \frac{x}{\sqrt{x^4-x}}\,dx$$

$$\geq \int_1^\infty \frac{x}{\sqrt{x^4}}\,dx$$

$$= \int_1^\infty \frac{1}{x}\,dx$$

$$= \infty$$

$\therefore$ divergent.

44) $$\int_0^\infty \frac{\tan^{-1}x}{2+e^x}\,dx \leq \int_0^\infty \frac{\pi/2}{2+e^x}\,dx \qquad \left(\tan^{-1}x < \frac{\pi}{2}\right)$$

$$\leq \int_0^\infty \frac{\pi/2}{e^x}\,dx$$

$$= \frac{\pi}{2}\int_0^\infty e^{-x}\,dx$$

$$= \frac{\pi}{2}\lim_{t\to\infty}\left[e^{-x}\right]_0^t$$

$$= \frac{\pi e}{2}$$

$\therefore \int_0^\infty \frac{\tan^{-1}x}{2+e^x}\,dx$ is convergent.

45) $$\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx \geq \int_0^1 \frac{1}{x\sqrt{x}}\,dx \qquad (\sec^2 x \geq 1 \text{ as } \cos^2 x \leq 1)$$

$$= \int_0^1 \frac{1}{x^{3/2}}\,dx = \infty \qquad \left(\int_0^1 \frac{1}{x^p}\,dx = \infty \text{ if } p \geq 1\right)$$

$\therefore$ divergent.

46) $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx$

$\leq \int_0^{\pi} \frac{1}{\sqrt{x}} dx$

$= \lim_{t \to 0^+} \int_t^{\pi} x^{-\frac{1}{2}} dx$

$= \lim_{t \to 0^+} \left[2x^{\frac{1}{2}} \right]_t^{\pi}$

$= 2\sqrt{\pi}$

$\therefore \int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx < \infty$ is convergent.