Compulsory

Ex. 5.1 23 (We've assumed $2+3x \geq 0$ so that $\sqrt{2+3x}$ makes sense.)

$$\begin{aligned} y =& 1+\sqrt{2+3x} \\ y-1 =& \sqrt{2+3x} \\ (y-1)^2 =& \left(\sqrt{2+3x}\right)^2 = 2+3x \\ 3x =& (y-1)^2-2 \\ x =& \frac{1}{3}\left((y-1)^2-2\right). \end{aligned}$$

So

$$f^{-1}(y) = \frac{1}{3}\left((y-1)^2-2\right).$$

38 By inspection, $f(0) = 0 + 3\sin 0 + 2\cos 0 = 2$, so $f^{-1}(2) = 0$.

$$f'(x) = 3x^2 + 3\cos x - 2\sin x.$$

So

$$f'(0) = 0 + 3\cos 0 - 2\sin 0 = 0 + 3 - 0 = 3.$$

Therefore

$$\left(f^{-1}\right)'(2) = \frac{1}{f'(0)} = \frac{1}{3}.$$

Ex. 5.2 2

$$\begin{aligned} \ln\sqrt[3]{\frac{x-1}{x+1}} =& \ln\left[\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}\right] \\ =& \frac{1}{3}\ln\left(\frac{x-1}{x+1}\right) \\ =& \frac{1}{3}\left(\ln(x-1)-\ln(x+1)\right). \end{aligned}$$

6

$$\begin{aligned} \ln 3 + \frac{1}{3}\ln 8 &= \ln 3 + \ln\left(8^{\frac{1}{3}}\right) \\ &= \ln 3 + \ln 2 \\ &= \ln(3\cdot 2) \\ &= \ln 6. \end{aligned}$$

Recommended
(Check the answers yourself.)

Ex. 5.1 17 The easiest way is by the observation (trial and error) that $h(4) = 4 + \sqrt{4} = 6$. So $h^{-1}(6) = 4$.
(Alternatively you can solve for x by squaring $y - x = \sqrt{x}$ and solve for x, though this will produce false solution because squaring is not an reversible process, and this is more complicated.)

22 Let

$$y = \frac{4x-1}{2x+3}$$

$$\begin{aligned} \therefore 2yx + 3y =& 4x - 1 \\ (4-2y)x =& 3y + 1 \\ x =& \frac{3y+1}{4-2y}. \end{aligned}$$

So $f^{-1}(y) = \frac{3y+1}{4-2y}$.

24 Let

$$y = 2x^3 + 3$$

$$\begin{aligned} \therefore 2x^3 =& y - 3 \\ x^3 =& \frac{1}{2}(y-3) \\ x =& \left(\frac{1}{2}(y-3)\right)^{\frac{1}{3}}. \end{aligned}$$

So $f^{-1}(y) = \left(\frac{1}{2}(y-3)\right)^{\frac{1}{3}}$.

25 Let

$$y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$$

$$\begin{aligned} \therefore y + y\sqrt{x} =& 1 - \sqrt{x} \\ (1+y)\sqrt{x} =& 1 - y \\ \sqrt{x} =& \frac{1-y}{1+y} \\ x =& \left(\frac{1-y}{1+y}\right)^2. \end{aligned}$$

So $f^{-1}(y) = \left(\frac{1-y}{1+y}\right)^2$.

26 Let

$$y = 2x^2 - 8x$$

$$\therefore 2x^2 - 8x - y = 0$$

Regard this as a quadratic equation in x, then

$$x = \frac{8 \pm \sqrt{8^2 - 4 \cdot 2 \cdot y}}{4} = 2 \pm \sqrt{4 + \frac{y}{2}}.$$

As $x \geq 2$, so $x = 2 + \sqrt{4 + \frac{y}{2}}$. So $f^{-1}(y) = 2 + \sqrt{4 + \frac{y}{2}}$.

37 By inspection, $f^{-1}(4) = 0$ ($\because f(0) = 4$).

$$\begin{aligned} f'(x) =& 6x^2 + 6x + 7 \\ \therefore f'(0) =& 7 \end{aligned}$$

So $\left(f^{-1}\right)'(4) = \frac{1}{f'(0)} = \frac{1}{7}$.

39 By inspection, $f^{-1}(3) = 0$ ($\because f(0) = 3$). By chain rule,

$$\begin{aligned} f'(x) =& 2x + \frac{\pi}{2} \sec^2\left(\frac{\pi x}{2}\right) \\ f'(0) =& \frac{\pi}{2} \sec^2(0) = \frac{\pi}{2} \cdot \frac{1}{1^2} = \frac{\pi}{2} \\ \therefore \left(f^{-1}\right)'(3) =& \frac{1}{f'(0)} = \frac{2}{\pi}. \end{aligned}$$

40 By inspection, as $f(1) = \sqrt{4} = 2$, so $f^{-1}(2) = 1$.

$$\begin{aligned} f'(x) =& \frac{1}{2}\left(x^3 + x^2 + x + 1\right)^{-\frac{1}{2}} \cdot \left(3x^2 + 2x + 1\right) \\ \therefore f'(1) =& \frac{1}{2} 4^{-\frac{1}{2}} \cdot 6 = \frac{3}{2} \\ \therefore \left(f^{-1}\right)'(2) =& \frac{2}{3}. \end{aligned}$$

Ex. 5.2

1

$$\ln \sqrt{ab} = \ln(ab)^{\frac{1}{2}} = \frac{1}{2} \ln(ab) = \frac{1}{2}\left(\ln a + \ln b\right).$$

4

$$\ln s^4\sqrt{t\sqrt{u}} = \ln s^4 + \ln\sqrt{t\sqrt{u}} = 4\ln s + \frac{1}{2}\ln(t\sqrt{u})$$
$$= 4\ln s + \frac{1}{2}(\ln t + \frac{1}{2}\ln u)$$
$$= 4\ln s + \frac{1}{2}\ln t + \frac{1}{4}\ln u.$$

5

$$\ln\left(5\cdot 3^5\right)\ (= \ln 1215)$$

7

$$\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}[\ln x - \ln(x^2+3x+2)^2]$$
$$= \ln(x+2)^{\frac{3}{3}} + \frac{1}{2}[\ln\frac{x}{(x^2+3x+2)^2}]$$
$$= \ln(x+2) + \ln\left(\frac{x}{(x^2+3x+2)^2}\right)^{\frac{1}{2}}$$
$$= \ln(x+2) + \ln\left(\frac{\sqrt{x}}{x^2+3x+2}\right)$$
$$= \ln\left(\frac{\sqrt{x}\cdot(x+2)}{x^2+3x+2}\right).$$

8

$$\ln\left[\frac{(a+b)(a-b)}{c^2}\right] = \ln\frac{a^2-b^2}{c^2}$$

Harder problems(Optional)

31 (a) Direct computation gives $f^{-1}(x) = \sqrt{1-x^2}$, which is exactly f itself.

(b) From the equation $y = \sqrt{1-x^2}$, it's easy to see that the graph is (a part of) the circle, which is symmetric about the line $y = x$, so f^{-1} must be the same as f.

43 As $f(3) = \int_3^3 \sqrt{1+t^3}dt = 0$, so $f^{-1}(0) = 3$. By the fundamental theorem of calculus, $f'(3) = \sqrt{1+3^3} = \sqrt{28}$. So $\left(f^{-1}\right)'(0) = \frac{1}{\sqrt{28}}$.

48 (a) We have $f(g(x)) = x$ as $g = f^{-1}$. Differentiating this once,

$$f'(g(x))g'(x) = 1.$$

Differentiate this w.r.t. x again,

$$f''(g(x))\,(g'(x))^2 + f'(g(x))g''(x) = 0.$$

$$\therefore g''(x) = -\frac{f''(g(x))g'(x)^2}{f'(g(x))} = -\frac{f''(g(x))}{f'(g(x))^3}. \quad (\because g'(x) = \frac{1}{f'(g(x))})$$

(b) If f is increasing and concave upward, $f' > 0$ and $f'' > 0$, so by (a), $g''(x) < 0$. i.e. g is concave downward.

Ex. 5.2

69 By definition, $\ln x = \int_1^n \frac{1}{x}dx$. So for a positive integer n, $\ln n$ is the area under the graph $\frac{1}{x}$ for $1 \le x \le n$. In particular, by dividing the interval $[1, n]$ into $n-1$ equal parts and noting that the area under the graph of $\frac{1}{x}$ in the interval $[k, k+1]$ is bounded from above by $\frac{1}{k}$ and from below by $\frac{1}{k+1}$ ($\because \frac{1}{k+1} \le \frac{1}{x} \le \frac{1}{k}$ for $k \le x \le k+1$). By summing these two inequalities from $k = 1$ up to $n-1$, we can get the estimate.

73 By the definition of f', for $f = \ln$

$$f'(1) = \lim_{x\to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0} \frac{\ln(1+x)}{x}.$$

On the other hand, $f'(1) = \frac{1}{1} = 1$, so $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$.