## 1.5 Problems

*Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.*

**1.** $y' + y = 2,\ y(0) = 0$
**2.** $y' - 2y = 3e^{2x},\ y(0) = 0$
**3.** $y' + 3y = 2xe^{-3x}$
**4.** $y' - 2xy = e^{x^2}$
**5.** $xy' + 2y = 3x,\ y(1) = 5$
**6.** $xy' + 5y = 7x^2,\ y(2) = 5$
**7.** $2xy' + y = 10\sqrt{x}$
**8.** $3xy' + y = 12x$
**9.** $xy' - y = x,\ y(1) = 7$
**10.** $2xy' - 3y = 9x^3$
**11.** $xy' + y = 3xy,\ y(1) = 0$
**12.** $xy' + 3y = 2x^5,\ y(2) = 1$
**13.** $y' + y = e^x,\ y(0) = 1$
**14.** $xy' - 3y = x^3,\ y(1) = 10$
**15.** $y' + 2xy = x,\ y(0) = -2$
**16.** $y' = (1 - y)\cos x,\ y(\pi) = 2$
**17.** $(1 + x)y' + y = \cos x,\ y(0) = 1$
**18.** $xy' = 2y + x^3 \cos x$
**19.** $y' + y \cot x = \cos x$
**20.** $y' = 1 + x + y + xy,\ y(0) = 0$
**21.** $xy' = 3y + x^4 \cos x,\ y(2\pi) = 0$
**22.** $y' = 2xy + 3x^2 \exp(x^2),\ y(0) = 5$
**23.** $xy' + (2x - 3)y = 4x^4$
**24.** $(x^2 + 4)y' + 3xy = x,\ y(0) = 1$
**25.** $(x^2 + 1)\dfrac{dy}{dx} + 3x^3 y = 6x \exp\left(-\frac{3}{2}x^2\right),\ y(0) = 1$

*Solve the differential equations in Problems 26 through 28 by regarding y as the independent variable rather than x.*

**26.** $(1 - 4xy^2)\dfrac{dy}{dx} = y^3$
**27.** $(x + ye^y)\dfrac{dy}{dx} = 1$
**28.** $(1 + 2xy)\dfrac{dy}{dx} = 1 + y^2$

**29.** Express the general solution of $dy/dx = 1 + 2xy$ in terms of the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

**30.** Express the solution of the initial value problem

$$2x\frac{dy}{dx} = y + 2x\cos x, \quad y(1) = 0$$

as an integral as in Example 3 of this section.

*Problems 31 and 32 illustrate—for the special case of first-order linear equations—techniques that will be important when we study higher-order linear equations in Chapter 3.*

**31.** **(a)** Show that

$$y_c(x) = Ce^{-\int P(x)\,dx}$$

is a general solution of $dy/dx + P(x)y = 0$. **(b)** Show that

$$y_p(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)e^{\int P(x)\,dx}\right) dx\right]$$

is a particular solution of $dy/dx + P(x)y = Q(x)$. **(c)** Suppose that $y_c(x)$ is any general solution of $dy/dx + P(x)y = 0$ and that $y_p(x)$ is any particular solution of $dy/dx + P(x)y = Q(x)$. Show that $y(x) = y_c(x) + y_p(x)$ is a general solution of $dy/dx + P(x)y = Q(x)$.

**32.** **(a)** Find constants $A$ and $B$ such that $y_p(x) = A\sin x + B\cos x$ is a solution of $dy/dx + y = 2\sin x$. **(b)** Use the result of part (a) and the method of Problem 31 to find the general solution of $dy/dx + y = 2\sin x$. **(c)** Solve the initial value problem $dy/dx + y = 2\sin x,\ y(0) = 1$.

**33.** A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture—kept uniform by stirring—is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

**34.** Consider a reservoir with a volume of 8 billion cubic feet ($\text{ft}^3$) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million $\text{ft}^3$ of water with a pollutant concentration of 0.05% and an equal daily outflow of the well-mixed water in the reservoir. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?

**35.** Rework Example 4 for the case of Lake Ontario, which empties into the St. Lawrence River and receives inflow from Lake Erie (via the Niagara River). The only differences are that this lake has a volume of 1640 $\text{km}^3$ and an inflow-outflow rate of 410 $\text{km}^3$/year.

**36.** A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 h. **(a)** Find the amount of salt in the tank after $t$ minutes. **(b)** What is the maximum amount of salt ever in the tank?

**37.** A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

**38.** Consider the *cascade* of two tanks shown in Fig. 1.5.5, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1. **(a)** Find the amount $x(t)$ of salt in tank 1 at time $t$. **(b)** Suppose that $y(t)$ is the amount of salt in tank 2 a time $t$. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200},$$

and then solve for $y(t)$, using the function $x(t)$ found part (a). **(c)** Finally, find the maximum amount of s ever in tank 2.

## 1.6 Problems

*Find general solutions of the differential equations in Problems 1 through 30. Primes denote derivatives with respect to x throughout.*

**1.** $(x+y)y' = x - y$
**2.** $2xyy' = x^2 + 2y^2$
**3.** $xy' = y + 2\sqrt{xy}$
**4.** $(x-y)y' = x + y$
**5.** $x(x+y)y' = y(x-y)$
**6.** $(x+2y)y' = y$
**7.** $xy^2y' = x^3 + y^3$
**8.** $x^2y' = xy + x^2e^{y/x}$
**9.** $x^2y' = xy + y^2$
**10.** $xyy' = x^2 + 3y^2$
**11.** $(x^2 - y^2)y' = 2xy$
**12.** $xyy' = y^2 + x\sqrt{4x^2 + y^2}$
**13.** $xy' = y + \sqrt{x^2 + y^2}$
**14.** $yy' + x = \sqrt{x^2 + y^2}$
**15.** $x(x+y)y' + y(3x+y) = 0$
**16.** $y' = \sqrt{x+y+1}$
**17.** $y' = (4x+y)^2$
**18.** $(x+y)y' = 1$
**19.** $x^2y' + 2xy = 5y^3$
**20.** $y^2y' + 2xy^3 = 6x$
**21.** $y' = y + y^3$
**22.** $x^2y' + 2xy = 5y^4$
**23.** $xy' + 6y = 3xy^{4/3}$
**24.** $2xy' + y^3e^{-2x} = 2xy$
**25.** $y^2(xy' + y)(1 + x^4)^{1/2} = x$
**26.** $3y^2y' + y^3 = e^{-x}$
**27.** $3xy^2y' = 3x^4 + y^3$
**28.** $xe^yy' = 2(e^y + x^3e^{2x})$
**29.** $(2x\sin y\cos y)y' = 4x^2 + \sin^2 y$
**30.** $(x + e^y)y' = xe^{-y} - 1$

*In Problems 31 through 42, verify that the given differential equation is exact; then solve it.*

**31.** $(2x + 3y)\,dx + (3x + 2y)\,dy = 0$
**32.** $(4x - y)\,dx + (6y - x)\,dy = 0$
**33.** $(3x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0$
**34.** $(2xy^2 + 3x^2)\,dx + (2x^2y + 4y^3)\,dy = 0$
**35.** $\left(x^3 + \dfrac{y}{x}\right)dx + (y^2 + \ln x)\,dy = 0$
**36.** $(1 + ye^{xy})\,dx + (2y + xe^{xy})\,dy = 0$
**37.** $(\cos x + \ln y)\,dx + \left(\dfrac{x}{y} + e^y\right)dy = 0$
**38.** $(x + \tan^{-1} y)\,dx + \dfrac{x+y}{1+y^2}\,dy = 0$
**39.** $(3x^2y^3 + y^4)\,dx + (3x^3y^2 + y^4 + 4xy^3)\,dy = 0$
**40.** $(e^x\sin y + \tan y)\,dx + (e^x\cos y + x\sec^2 y)\,dy = 0$
**41.** $\left(\dfrac{2x}{y} - \dfrac{3y^2}{x^4}\right)dx + \left(\dfrac{2y}{x^3} - \dfrac{x^2}{y^2} + \dfrac{1}{\sqrt{y}}\right)dy = 0$
**42.** $\dfrac{2x^{5/2} - 3y^{5/3}}{2x^{5/2}y^{2/3}}\,dx + \dfrac{3y^{5/3} - 2x^{5/2}}{3x^{3/2}y^{5/3}}\,dy = 0$

*Find a general solution of each reducible second-order differential equation in Problems 43–54. Assume x, y and/or y′ positive where helpful (as in Example 11).*

**43.** $xy'' = y'$
**44.** $yy'' + (y')^2 = 0$
**45.** $y'' + 4y = 0$
**46.** $xy'' + y' = 4x$
**47.** $y'' = (y')^2$
**48.** $x^2y'' + 3xy' = 2$
**49.** $yy'' + (y')^2 = yy'$
**50.** $y'' = (x + y')^2$
**51.** $y'' = 2y(y')^3$
**52.** $y^3y'' = 1$
**53.** $y'' = 2yy'$
**54.** $yy'' = 3(y')^2$

**55.** Show that the substitution $v = ax + by + c$ transforms the differential equation $dy/dx = F(ax + by + c)$ into a separable equation.

**56.** Suppose that $n \neq 0$ and $n \neq 1$. Show that the substitution $v = y^{1-n}$ transforms the Bernoulli equation $dy/dx + P(x)y = Q(x)y^n$ into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v(x) = (1-n)Q(x).$$

**57.** Show that the substitution $v = \ln y$ transforms the differential equation $dy/dx + P(x)y = Q(x)(y\ln y)$ into the linear equation $dv/dx + P(x) = Q(x)v(x)$.

**58.** Use the idea in Problem 57 to solve the equation

$$x\frac{dy}{dx} - 4x^2y + 2y\ln y = 0.$$

**59.** Solve the differential equation

$$\frac{dy}{dx} = \frac{x - y - 1}{x + y + 3}$$

by finding $h$ and $k$ so that the substitutions $x = u + h$, $y = v + k$ transform it into the homogeneous equation

$$\frac{dv}{du} = \frac{u - v}{u + v}.$$

**60.** Use the method in Problem 59 to solve the differential equation

$$\frac{dy}{dx} = \frac{2y - x + 7}{4x - 3y - 18}.$$

**61.** Make an appropriate substitution to find a solution of the equation $dy/dx = \sin(x - y)$. Does this general solution contain the linear solution $y(x) = x - \pi/2$ that is readily verified by substitution in the differential equation?

**62.** Show that the solution curves of the differential equation

$$\frac{dy}{dx} = -\frac{y(2x^3 - y^3)}{x(2y^3 - x^3)}$$

are of the form $x^3 + y^3 = 3Cxy$.

**63.** The equation $dy/dx = A(x)y^2 + B(x)y + C(x)$ is called a **Riccati equation.** Suppose that one particular solution $y_1(x)$ of this equation is known. Show that the substitution

$$y = y_1 + \frac{1}{v}$$

transforms the Riccati equation into the *linear* equation

$$\frac{dv}{dx} + (B + 2Ay_1)v = -A.$$

*Use the method of Problem 63 to solve the equations in Problems 64 and 65, given that $y_1(x) = x$ is a solution of each.*

**64.** $\dfrac{dy}{dx} + y^2 = 1 + x^2$

**65.** $\dfrac{dy}{dx} + 2xy = 1 + x^2 + y^2$

**66.** An equation of the form

$$y = xy' + g(y') \qquad (37)$$

is called a **Clairaut equation.** Show that the one-parameter family of straight lines described by

$$y(x) = Cx + g(C) \qquad (38)$$

is a general solution of Eq. (37).

**67.** Consider the Clairaut equation

$$y = xy' - \tfrac{1}{4}(y')^2$$

for which $g(y') = -\frac{1}{4}(y')^2$ in Eq. (37). Show that the line

$$y = Cx - \tfrac{1}{4}C^2$$

is tangent to the parabola $y = x^2$ at the point $\left(\frac{1}{2}C, \frac{1}{4}C^2\right)$. Explain why this implies that $y = x^2$ is a singular solution of the given Clairaut equation. This singular solution and the one-parameter family of straight line solutions are illustrated in Fig. 1.6.10.

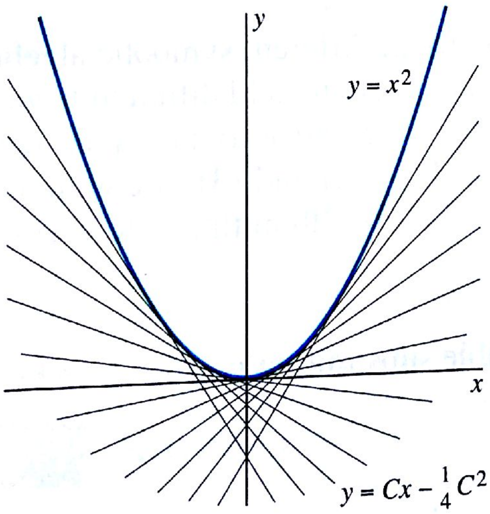


**FIGURE 1.6.10.** Solutions of the Clairaut equation of Problem 67. The "typical" straight line with equation $y = Cx - \frac{1}{4}C^2$ is tangent to the parabola at the point $\left(\frac{1}{2}C, \frac{1}{4}C^2\right)$.

**68.** Derive Eq. (18) in this section from Eqs. (16) a

**69.** In the situation of Example 7, suppose that $a$ $v_0 = 400$ mi/h, and $w = 40$ mi/h. Now how ward does the wind blow the airplane?

**70.** As in the text discussion, suppose that an airp tains a heading toward an airport at the origin. mi/h and $w = 50$ mi/h (with the wind blowing and the plane begins at the point (200, 150), s trajectory is described by

$$y + \sqrt{x^2 + y^2} = 2(200x^9)^{1/10}.$$

**71.** A river 100 ft wide is flowing north at $w$ fee A dog starts at (100, 0) and swims at $v_0 = 4$ heading toward a tree at (0, 0) on the west across from the dog's starting point. **(a)** If show that the dog reaches the tree. **(b)** If show that the dog reaches instead the poin bank 50 ft north of the tree. **(c)** If $w = 6$ f the dog never reaches the west bank.

**72.** In the calculus of plane curves, one learns t *ture* $\kappa$ of the curve $y = y(x)$ at the point by

$$\kappa = \frac{|y''(x)|}{[1 + y'(x)^2]^{3/2}},$$

and that the curvature of a circle of radius [See Example 3 in Section 11.6 of Edwar *Calculus: Early Transcendentals*, 7th editi dle River, NJ: Prentice Hall, 2008).] Co tute $p = y'$ to derive a general solution of t differential equation

$$ry'' = [1 + (y')^2]^{3/2}$$

(with $r$ constant) in the form

$$(x - a)^2 + (y - b)^2 = r$$

Thus a circle of radius $r$ (or a part thereo curve with constant curvature $1/r$.

## Computer Algebra Solutions