(1) (a) Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 3 & 1 & 0 & -1 \end{bmatrix}$.

(b) Is A invertible? If so, what is $\det(A^{-1})$? If not, why not?

Add multiples of one row to other rows:

$$A \rightsquigarrow B = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\det A = \det B = 3 \cdot \det \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$= 3 \cdot -1 \cdot \det \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$
$$= 3 \cdot -1 \cdot (0 \cdot 1 - 2 \cdot 1)$$
$$= 6.$$

Since $\det A = 6 \neq 0$, A is invertible.

$$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{6}$$

(2) Quickly, what is the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -\frac{7}{8} & 6 \\ 3 & 1 & 0 & 42 & \frac{238}{19938} & \pi & 0 \\ 0 & -1 & 28 & 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & 1 & -2 & 2 & 9 \\ 0 & 0 & 0 & 1 & 10 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 93 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$?

4 Row swaps makes it upper triangular.

$$\det A = (-1)^4 \cdot 3 \cdot -1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot -1 = 6.$$

(3) What numbers arise as determinants of orthogonal matrices? Justify your answer.

An orthogonal matrix is a square matrix s.t. $Q^{-1} = Q^T$.

since $\det(Q^T) = \det Q$ and $\det(Q^{-1}) = \frac{1}{\det Q}$

we must have $\det Q = \frac{1}{\det Q}$.

Hence $(\det Q)^2 = 1$. Therefore $\det Q = \pm 1$.

Both ± 1 occur, e.g. with permutation matrices.

(3) What numbers arise as determinants of orthogonal matrices? Justify your answer.

(4) (a) How can you easily conclude that the matrix $B = \begin{bmatrix} 3 & 1 & 33 & 8 \\ 9 & -2 & 99 & 1 \\ -7 & 5 & -77 & 11 \\ 2 & 1 & 22 & -8 \end{bmatrix}$ is singular?

(b) Why does this tell you that 4 is an eigenvalue of $A = \begin{bmatrix} 7 & 1 & 33 & 8 \\ 9 & 2 & 99 & 1 \\ -7 & 5 & -73 & 11 \\ 2 & 1 & 22 & -4 \end{bmatrix}$?

(c) Use this to find an eigenvector of A with eigenvalue 4. Then find a unit eigenvector of A with eigenvalue 4.

4a) Columns 1 & 3 are multiples of each other.
(Hence the columns are <u>not</u> linearly independent)
Thus the matrix is not invertible.
(singular matrix = non invertible matrix
= zero determinant)

4b) Notice that $A - 4I = B$
Since $\det B = 0$, 4 is a root of $\det(A - \lambda I)$.

4c) We just need a vector in $N(B)$, the nullspace of B.
Since $11 \cdot \text{col}\,1 = \text{col}\,3$,
the vector $\vec{v} = \begin{bmatrix} -11 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in N(B)$. Hence it's an eigenvector of A w/ eigenvalue 4.

To make it a <u>unit</u> eigenvector, scale by its magnitude.

$$\|\vec{v}\| = \sqrt{(-11)^2 + 0^2 + 1^2 + 0^2} = \sqrt{122}$$

So $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{V} = \frac{1}{\sqrt{122}}\begin{bmatrix} -11 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -11/\sqrt{122} \\ 0 \\ 1/\sqrt{122} \\ 0 \end{bmatrix}$ is a <u>unit</u> eigenvector of A w/ eigenvalue 4

(5) (a) What are the eigenvalues of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$?

(b) Find an eigenvector for each eigenvalue; describe the eigenspace for each eigenvalue.

(c) Use this information to diagonalize A.

(d) Repeat this for the matrix $M = \begin{bmatrix} 3 & 2 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

5a) $\det\begin{pmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) - 2\cdot 3$

$= -6 - \lambda + \lambda^2 - 6$

$= \lambda^2 - \lambda - 12 = (\lambda-4)(\lambda+3)$

So the eigenvalues are 4 and -3.

5b) Since the eigenvectors live in $\mathbb{R}^2$ and we have two distinct eigenvalues, each eigenspace should be 1 dimensional.

So to get a basis for each eigenspace, we just need one eigenvector.

For $\lambda_1 = 4$, $A - 4I = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$

b/c $\left(\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$

so $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvect.

The eigenspace is $E_4 = N(A-4I) = \left\{ t\begin{pmatrix} 2 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$

Or: just say $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ is a basis for E_4

For $\lambda_2 = -3$, $A - -3I = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$

so $\vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvect

The eigenspace is $E_{-3} = N(A-3I) = \left\{ t\begin{pmatrix} 1 \\ -3 \end{pmatrix} \mid t \in \mathbb{R} \right\}$

Or: just say $\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$ is a basis for E_{-3}

5c) Now for the diagonalization $A = S\Lambda S^{-1}$

let's use $S = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}2&1\\1&-3\end{bmatrix}$, $\Lambda = \begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix} = \begin{bmatrix}4&0\\0&-3\end{bmatrix}$

then $S^{-1} = \frac{1}{-7}\begin{bmatrix}-3&-1\\-1&2\end{bmatrix}$ (using inverse formula for 2×2)

Thus $\begin{bmatrix}3&2\\3&-2\end{bmatrix} = \begin{bmatrix}2&1\\1&-3\end{bmatrix}\begin{bmatrix}4&0\\0&-3\end{bmatrix}\begin{bmatrix}3/7&1/7\\1/7&-2/7\end{bmatrix}$

5d) $A = \left[\begin{array}{cc|c}3&2&0\\3&-2&0\\\hline 0&0&4\end{array}\right]$ $\quad \det(A-\lambda I) = (\lambda - -3)(\lambda-4)^2$

Eigenvalues are -3 and 4

This time 4 has multiplicity 2, while -3 has multiplicity 1.

So we expect $\dim E_4 = 2$ and $\dim E_{-3} = 1$

$E_4 = N(A-4I) = N\left(\left[\begin{array}{cc|c}-1&2&0\\3&-6&0\\\hline 0&0&0\end{array}\right]\right)$ Can see that $\begin{bmatrix}2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$

are in null sp. & lin indep.

So E_4 has basis $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$E_{-3} = N(A - -3I) = N\left(\left[\begin{array}{cc|c}6&2&\\3&1&\\\hline &&7\end{array}\right]\right)$ see that $\begin{bmatrix}1\\-3\\0\end{bmatrix}$ is in null sp.

So E_{-3} has basis $\left\{\begin{bmatrix}1\\-3\\0\end{bmatrix}\right\}$

To diagonalize, use $S = \begin{bmatrix}2&0&1\\1&0&-3\\0&1&0\end{bmatrix}$ and $\Lambda = \begin{bmatrix}4&&\\&4&\\&&-3\end{bmatrix}$

Still need S^{-1} ..... but you know how.

(6) The matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ is symmetric. Find an *orthonormal basis* for $\mathbb{R}^3$ consisting of eigenvectors of A. Use this to construct a diagonalization $A = Q\Lambda Q^T$.

Do like in 5), but w/ orthonormal eigenvectors.

Since $\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & & 3 \\ & -2-\lambda & \\ 3 & & 1-\lambda \end{pmatrix}$

$= ((1-\lambda)^2 - 9)(-2-\lambda)$

$= (\lambda^2 - 2\lambda - 8)(-2-\lambda) = -(\lambda+2)(\lambda-4)(\lambda+2)$

$= -(\lambda+2)^2(\lambda-4)$

Eigenvalues $-2, 4$

$\lambda = -2$

$A + 2I = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix}$ w/ eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Can see that they're $\perp$

just need to make the first one into a unit vector.

Since $\left\| \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$, use $\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

$\lambda = 4$

$A - 4I = \begin{pmatrix} -3 & 0 & 3 \\ 0 & -6 & 0 \\ 3 & 0 & -3 \end{pmatrix}$ gives eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

make it unit $\rightsquigarrow$ $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

check that it is $\perp$ to others

Then $Q = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$ $\Lambda = \begin{pmatrix} -2 & & \\ & -2 & \\ & & 4 \end{pmatrix}$

$Q^T = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$

(7) The Singular Value Decomposition of $A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 2 & -2 & 0 & 2 \end{bmatrix}$ is:

$$A = U \quad \Sigma \quad V^T$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 2 & -2 & 0 & 2 \end{bmatrix} = \begin{array}{c} \vec{u}_1 \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \sigma_1 \\ \begin{bmatrix} 2\sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \vec{v}_1^{\,T} \\ \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 0 & 1/\sqrt{6} \\ 1/\sqrt{6} & 0 & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \end{array}$$

(a) What are the eigenvalues of A^TA? What are the eigenvalues of AA^T?
(b) What is the closest rank 1 approximation to A?
(c) Using this SVD, give an orthonormal basis for the null space of A.
(d) What is the pseudoinverse A^+ of A?

7a) Since the sing values are $\sigma_1 = 2\sqrt{3}$
$\Rightarrow \sigma_2 = \sqrt{3}$

The nonzero eigenvalues are $\lambda_1 = 12$, $\lambda_2 = 3$

$\underset{4\times 2\ 2\times 4}{A^TA}$ is 4×4 so it has two more eigenvalues 0 in addition to 12 + 3

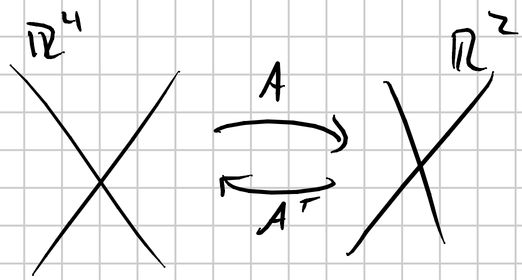
$\underset{2\times 4\ 4\times 2}{AA^T}$ is 2×2 so 12 + 3 are the only eigenvalues.

7b) The best rank 1 approx is

$$\vec{u}_1 \sigma_1 \vec{v}_1^{\,T} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2\sqrt{3} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 2 \end{bmatrix}$$

7c) Recall $N(A) = N(A^TA)$



also, $V = \Bigg[\underbrace{\vec{v}_1 \cdots \vec{v}_r}_{\text{othonormal eigenvectors of } A^TA \text{ w/ nonzero eigenvalues}} \Bigg| \underbrace{\vec{v}_{r+1} \cdots \vec{v}_m}_{\text{orthonormal basis of } N(A^TA)} \Bigg]$

Here $\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}$ is an ON basis for $N(A)$.

d) The pseudo inverse is

$$A^+ = V \Sigma^+ U^T$$

where $\Sigma^+ = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Multiply it out.

(d) What is the pseudoinverse A^+ of A?

(8) Let P_2 be the vector space of polynomials of degree at most 2. That is, $P_2 = \{ax^2+bx+c \mid a, b, c \in \mathbb{R}\}$.

(a) Show that the transformation $T\colon P_2 \to P_2$ that takes a polynomial $p(x)$ to the polynomial $p(x-1)$ is a linear transformation.

(b) Choose a basis for P_2 and find the associated matrix of this linear transformation.

a) Check $T(r \cdot p(x)) = r \cdot p(x-1) = r\,T(p(x))$ ✓

$T(p(x)+q(x)) = p(x-1)+q(x-1) = T(p(x)) + T(q(x))$ ✓

T behaves correctly for scalar mult & vect. addition

So it is a linear transform.

b) Let's use the basis $x^2, x, 1$

then

$$\begin{tikzcd} ax^2+bx+c \in P_2 \arrow[r, "T"] \arrow[d] & P_2 \ni a(x-1)^2+b(x-1)+c \arrow[d] \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \arrow[r, "A"] & \mathbb{R}^3 \ni \begin{bmatrix} a \\ -2a+b \\ a-b+c \end{bmatrix} \end{tikzcd}$$

$$= ax^2 - 2ax + a + bx - b + c$$
$$= ax^2 + (-2a+b)x + a - b + c$$

So
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

hence $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

(this answer depends on your choice of basis.)