(2) Solve the each of the systems of linear differential equations and sketch their phase portraits.

- $\begin{cases} x' &= 5x + 6y \\ y' &= -2x - 2y \end{cases}$
- $\begin{cases} x' &= 3x + y \\ y' &= 3y \end{cases}$
- $\begin{cases} x' &= 3x + y \\ y' &= 5x - y \end{cases}$
- $\begin{cases} x' &= -2y \\ y' &= 5x - 2y \end{cases}$

$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ has a double eigenvalue $\lambda = 3$

We did not discuss what to do in this situation. Such systems will not be on the exams.

(Nevertheless, you can solve it by transforming it into a 2nd order linear equation ...)

$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$ $\quad \det\begin{pmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{pmatrix} = (5-\lambda)(-2-\lambda) + 12$

$= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

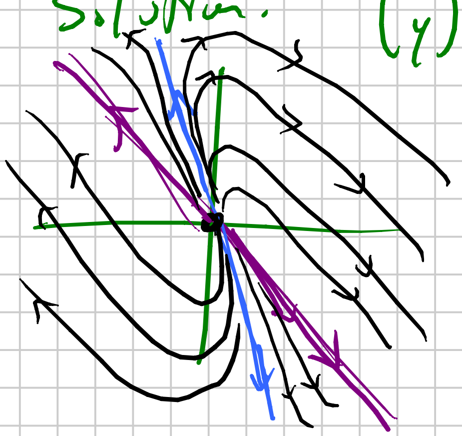
eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$

$\underline{\lambda_1 = 1}$ $\begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix}$ so $\bar{v}_1 = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 1$

$\underline{\lambda_2 = 2}$ $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$ so $\bar{v}_2 = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ is an eigenvector for $\lambda_2 = 2$.

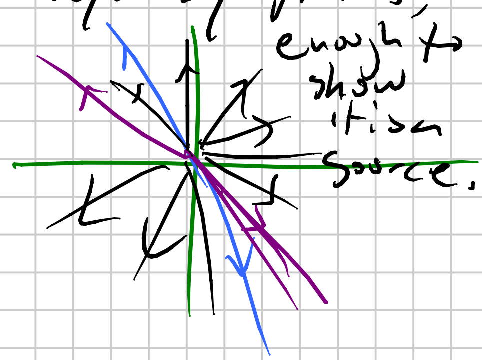
Genl solution: $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^t \begin{pmatrix} 6 \\ -4 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 6 \\ -3 \end{pmatrix}$

Quick Sketch.



this has the asymptotic information

w/o asymptotics, enough to show it is a source.



$x' = 3x + y$
$y' = 5x - y$ $\rightsquigarrow A = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}$ $\det\begin{pmatrix} 3-\lambda & 1 \\ 5 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) - 5$

$= \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$

eigenvalues $\lambda_1 = 4 \quad \lambda_2 = -2$

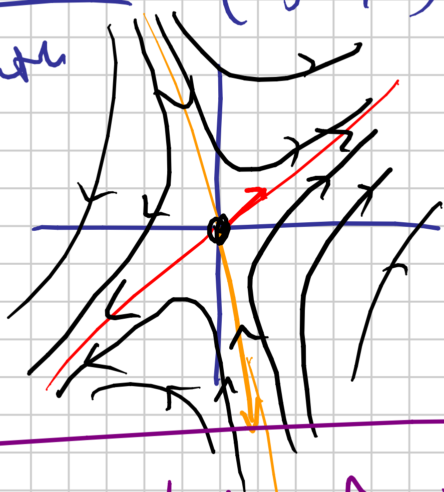
$\underline{\lambda_1 = 4}$ $\begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix}$ $\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Genl soln

$\underline{\lambda_2 = -2}$ $\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix}$ $\bar{v}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

$\bar{x} = C_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -5 \end{pmatrix}$

sketch



Saddle

$x' = 0x + -2y$
$y' = 5x - 2y$ $A = \begin{pmatrix} 0 & -2 \\ 5 & -2 \end{pmatrix}$

$\det\begin{pmatrix} -\lambda & -2 \\ 5 & -2-\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 10$

$\lambda = \frac{-2 \pm \sqrt{4 - 40}}{2}$ (-36)

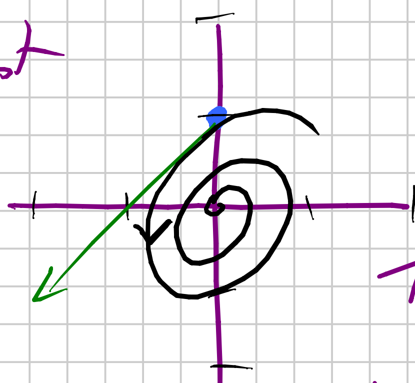
$= -1 \pm 3i$

spiral sink

Let's Sketch first

Test pt $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 & -2 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$



Soln through $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
should be tangent to $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$
So spiral counterclockwise

To get genl solution.

Need complex eigenvector
Using $\lambda = -1 + 3i$

$A - \lambda I = \begin{pmatrix} 1-3i & -2 \\ 5 & -1-3i \end{pmatrix}$

So $\bar{v} = \begin{pmatrix} -2 \\ -1+3i \end{pmatrix}$ works. $\underline{(-2)}(1-3i) + \underline{-(1-3i)}(-2) = 0$

$\left(\underline{Q} \cdot P + \underline{-P} \cdot Q = 0 \right)$

next page

With $\lambda = -1+3i$ and $\bar{v} = \begin{pmatrix} -2 \\ -1+3i \end{pmatrix}$

the gen'l soln is

$$\bar{x} = C_1 \cdot \mathrm{Re}\left(e^{\lambda t}\bar{v}\right) + C_2\, \mathrm{Im}\left(e^{\lambda t}\bar{v}\right)$$

$$e^{\lambda t} = e^{(-1+3i)t} = e^{-t+3t\,i} = e^{-t}\cdot e^{3t\,i}$$

$$= e^{-t}\cdot\left(\cos 3t + i\sin 3t\right)$$

So $e^{\lambda t}\bar{v} = e^{-t}\left(\cos 3t + i\sin 3t\right)\begin{pmatrix} -2 \\ -1+3i \end{pmatrix}$

$$= e^{-t}\begin{pmatrix} -2\cos 3t + (-2\sin 3t)\,i \\ -\cos 3t - (\sin 3t)\,i + (3\cos 3t)\,i - (3\sin 3t) \end{pmatrix}$$

Re

Im

this is multiplied through.

Gen'l Soln is

$$\bar{x} = C_1\, e^{-t}\begin{pmatrix} -2\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + C_2\, e^{-t}\begin{pmatrix} -2\sin 3t \\ -\sin 3t + 3\cos 3t \end{pmatrix}$$

(4) The non-linear system $\begin{cases} x' &= x^2 - 2x - xy \\ y' &= y^2 - 4y + xy \end{cases}$ models the population of a predator-prey relationship.

(a) Which variable describes the predator population? the prey population?
y x
(b) Find the equilibrium points.
(c) Sketch the phase portrait of the linearization of the system at each equilibrium point.
(d) Sketch the phase portrait of the non-linear system. Use the previous info and nullclines.
(e) Discuss the possible the long term behaviors of these populations.

b) Equilibrium pts. solve $\begin{cases} 0 = x^2 - 2x - xy \\ 0 = y^2 - 4y + xy \end{cases}$

$\begin{cases} 0 = x(x-2-y) \\ 0 = y(y-4-x) \end{cases}$ either $x=0$ or $x = y+2$

$x=0$: $0 = y(y-4)$ so $y=0$ or $y=4$

$x=y+2$: $0 = y(\cancel{y}-4-\cancel{y}-2)$ so $y=0$ ⇒ $x=6$

Eq pts are $(0,0)$, $(0,4)$, $(6,0)$

c) Linearization $A(x,y) = \begin{pmatrix} 2x-2-y & -x \\ y & 2y-\cancel{4}+x \end{pmatrix}$

@$(0,0)$ $A(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$ $\lambda_1 = -2$ $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = -4$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

w/o eigenvectors.

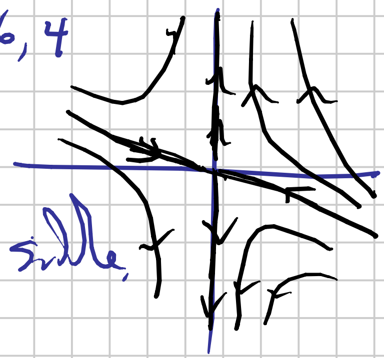
sink

@$(0,4)$ $A(0,4) = \begin{pmatrix} -6 & 0 \\ 4 & 4 \end{pmatrix}$ ⇒ $\det\begin{pmatrix} -6-\lambda & 0 \\ 4 & 4-\lambda \end{pmatrix} = (-6-\lambda)(4-\lambda) = 0$

$\lambda = -6, 4$

$\lambda=-6$ $\begin{pmatrix} 0 & 0 \\ 4 & 10 \end{pmatrix}$ so $\vec{v} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$ or $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\lambda=4$ $\begin{pmatrix} -10 & 0 \\ 4 & 0 \end{pmatrix}$ so $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



saddle

Center

(a) $(6,0)$ $A_{(6,0)} = \begin{pmatrix} 10 & -6 \\ 0 & 6 \end{pmatrix}$

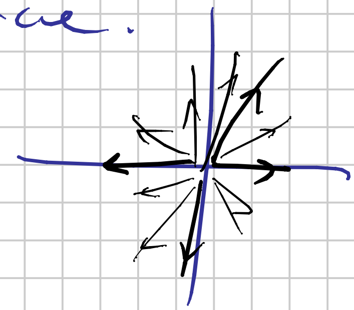
$\det\begin{pmatrix} 10-\lambda & -6 \\ 0 & 6-\lambda \end{pmatrix} = (10-\lambda)(6-\lambda)$ $\quad \lambda = 10, 6$

source.

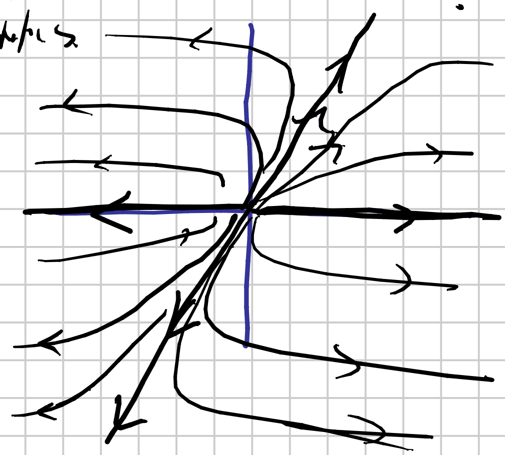
$\lambda_1 = 10 \quad \begin{pmatrix} 0 & -6 \\ 0 & -4 \end{pmatrix} \quad \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 6 \quad \begin{pmatrix} 4 & -6 \\ 0 & 0 \end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

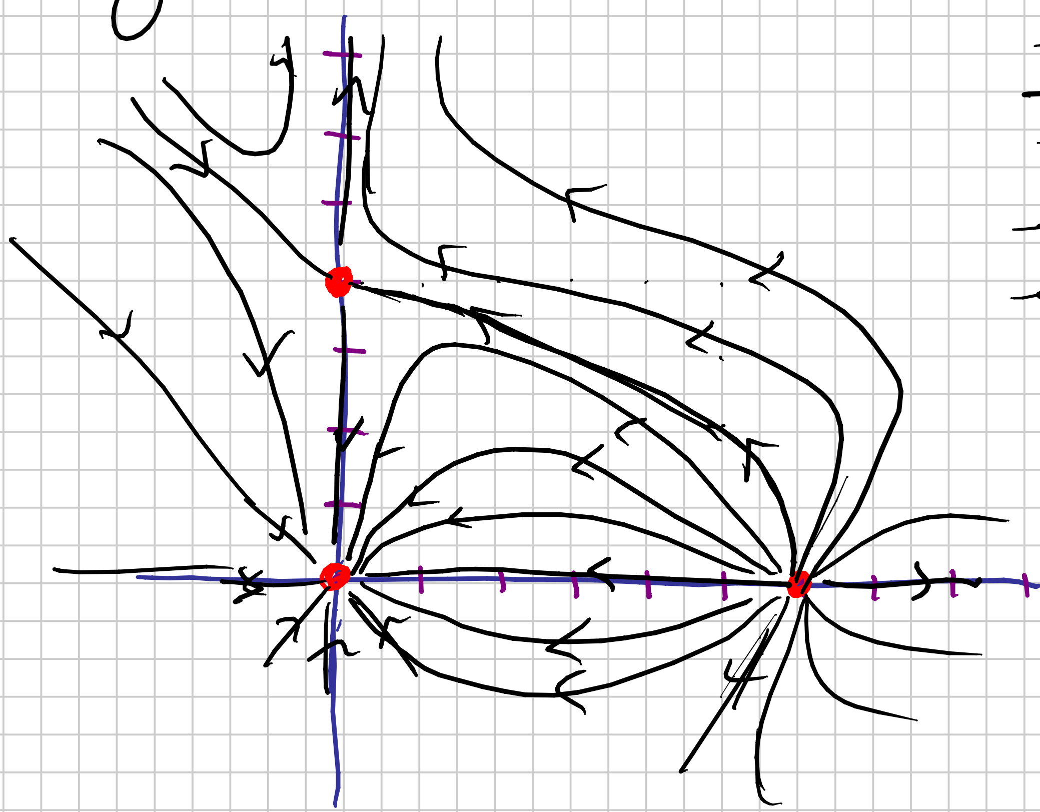
w/o asymptotics



with asymptotics



All together



7) Solve the following initial value problems.

(a) $x'' + 9x = 0; \quad x(0) = 3, x'(0) = 4$

(b) $x'' - 4x = 3t; \quad x(0) = x'(0) = 0$

(c) $x'' + 4x' + 5x = 5e^{-2t}; \quad x(0) = x'(0) = 0$

(d) $x^{(4)} + x = 0; \quad x(0) = x'(0) = x''(0) = 0, x^{(3)}(0) = 1$

(e) $x'' + 4x = E(t); \quad x(0) = 10, x'(0) = 0$, and $E(t) = \begin{cases} 1 \text{ if } t \leq 1 \\ 0 \text{ if } t > 1 \end{cases}$

(f) $x'' + 9x = \delta_4(t); \quad x(0) = x'(0) = 1$

(g) $\begin{cases} 0 &= x' + 2y' + x \\ 0 &= x' - y' + y \end{cases}; \quad x(0) = 0, y(0) = 1$

we didn't do systems w/ Laplace, but you can.

$$\begin{cases} 0 = x' + 2y' + x \\ 0 = x' - y' + y \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} 0 = (sX - 0) + 2(sY - 1) + X \\ 0 = (sX - 0) - (sY - 1) + Y \end{cases}$$

now solve this system for X + Y

then do $\mathcal{L}^{-1}$!

a) $x'' + 9x = 0, \quad x(0) = 3, \quad x'(0) = 4$

$\mathcal{L}$

$$(s^2X - s3 - 4) + 9X = 0 \qquad (s^2+9)X = 3s + 4$$

$$X = \frac{3s}{s^2+9} + \frac{4}{s^2+9} = 3 \cdot \boxed{\frac{s}{s^2+3^2}} + \frac{4}{3}\boxed{\frac{3}{s^2+3^2}}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad x = 3\cos(3t) + \frac{4}{3}\sin(3t)$$

f) $x'' + 9x = \delta_4(t) \qquad x(0) = 1, \quad x'(0) = 1$

$$\Big\}\mathcal{L}$$

$$(s^2X - s - 1) + 9X = e^{-4s}$$

$$(s^2+9)X = e^{-4s} + 1 + s$$

$$X = e^{-4s}\cdot\frac{1}{s^2+9} + \frac{1}{s^2+9} + \frac{s}{s^2+9}$$

$$= e^{-4s}\boxed{\frac{1}{3}\cdot\frac{3}{s^2+3^2}} + \boxed{\frac{1}{3}\,\frac{3}{s^2+3^2}} + \frac{s}{s^2+3^2}$$

$\mathcal{L}^{-1}$: $\quad \frac{1}{3}\sin(3t) \qquad \cos(3t)$

use

$$e^{-as}\boxed{F(s)}$$

$$\Big\}$$

$$u_a(t)f(t-a)$$

So $\mathcal{L}^{-1}\left(e^{-4s}\cdot\frac{1}{3}\,\frac{3}{s^2+3^2}\right) = u_4(t)\,\frac{1}{3}\sin\big(3(t-4)\big)$

Hence

$$x(t) = u_4(t)\cdot\frac{1}{3}\sin(3t-12) + \frac{1}{3}\sin(3t) + \cos(3t)$$

$$x'' + 4x' + 5x = 5e^{-2t}, \qquad x(0) = x'(0) = 0$$

$$\Big\downarrow$$

$$s^2 X + 4sX + 5X = 5\frac{1}{s+2}$$

$$(s^2+4s+5)X = 5\frac{1}{s+2}$$

29. $e^{ct}f(t)$ $F(\underline{s-c})$

(Thinly $s^2+4s+5 = (\underline{s+2})^2 + 1^2$)

$$X = 5\frac{1}{(s^2+4s+5)(s+2)} = 5\left(\frac{As+B}{s^2+4s+5} + \frac{C}{s+2}\right)$$

$$1 = As^2 + (A+2B)s + 2B$$
$$+ Cs^2 + 4Cs + 5C$$

$A + C = 0 \qquad A = -C$

$A + 2B + 4C = 0 \rightsquigarrow 2B + 3C = 0$

$2B + 5C = 1 \rightsquigarrow 2B + 5C = 1$

$\Rightarrow 2C = 1 \quad C = 1/2, \ A = -1/2$

$B = -3/4$

$$X = 5\left(\frac{-\frac{1}{2}s - \frac{3}{4}}{(s+2)^2+1^2} + \frac{1/2}{s+2}\right)$$

$$= -\frac{5}{2}\left(\frac{s + 3/2}{(s+2)^2+1^2} - \frac{1}{s+2}\right)$$

$$= \frac{-5}{2}\left(\frac{(s+2) + (-2+3/2)}{(s+2)^2+1^2} - \frac{1}{s+2}\right)$$

$$= -\frac{5}{2}\left(\frac{s+2}{(s+2)^2+1^2} + \frac{-1}{2}\frac{1}{(s+2)^2+1^2} - \frac{1}{s+2}\right)$$

$$\frac{s}{s^2+1} \overset{\mathcal{L}^{-1}}{\rightsquigarrow} \sin(t) \qquad \frac{1}{s} \overset{\mathcal{L}^{-1}}{\rightsquigarrow} 1$$

$$\overset{\mathcal{L}^{-1}}{\rightsquigarrow} \quad x = \frac{-5}{2}\left(e^{-2t}\sin(t) + \frac{-1}{2}e^{-2t}\cos(t) - e^{-2t}\right)$$

(6) Let $f(t) = t - 2$ when $2 \le t \le 3$ and let $f(t) = 0$ otherwise.

(a) Compute the Laplace transform of $f(t)$ directly from the definition of the Laplace transform.

(b) Compute the Laplace transform by first expressing $f(t)$ in terms of fundamental functions, step functions, and translations and then using the properties of the Laplace transform.

$$f(t) = \begin{cases} 0 & t \le 2 \\ t-2 & 2 < t \le 3 \\ 0 & 3 < t \end{cases}$$

$$\mathcal{L}\{f\} \underset{\text{from defn}}{=} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} \underset{\overset{\shortparallel}{0}}{f(t)}\,dt + \int_2^3 e^{-st} \underset{\overset{\shortparallel}{t-2}}{f(t)}\,dt + \int_3^\infty e^{-st} \underset{\overset{\shortparallel}{0}}{f(t)}\,dt$$

$$= 0 + \int_2^3 e^{-st}(t-2)\,dt + 0$$

$$= \int_2^3 \underbrace{t}_{u} \underbrace{e^{-st}\,dt}_{dv} - 2\int_2^3 e^{-st}\,dt$$

$du = dt \quad v = \frac{-1}{s} e^{-st}$

$$= \left(-\frac{1}{s} t e^{-st}\right)\Big|_2^3 - \int_2^3 \left(\frac{-1}{s}\right) e^{-st}\,dt - 2\int_2^3 e^{-st}\,dt$$

$$= \left(-\frac{3e^{-3s}}{s} + \frac{2e^{-2s}}{s}\right) - \left(\frac{-1}{s} + 2\right)\left(\frac{-1}{s} e^{-st}\right)\Big|_2^3$$

$$= -\frac{3e^{-3s}}{s} + \frac{2e^{-2s}}{s} - \left(\frac{-1}{s} + 2\right)\left(\frac{-1}{s} e^{-3s} + \frac{1}{s} e^{-2s}\right)$$

Well that was a mess!