Math 311 — Practice — Test 1

(1) Verify that $y_1 = \cos x - \cos 2x$ is a solution to the differential equation $y'' + y = 3\cos 2x$.

(2) Consider the equation $y' = \sqrt[3]{y} + \sin(2\pi y)$.
   (a) Does there exist a solution to the equation with the initial condition $y(0) = 0$? If so, can you tell if it is unique? Explain.
   (b) What if $y(0) = 1$ instead?
   (c) There is a unique solution to the initial equation that runs through the point $(x, y) = (5, 8)$. What is the slope of the tangent line to that solution at the point $(5, 8)$?

(3) Find general solutions to the following differential equations. (Explicit if convenient, implicit if not.)
   (a) $(x^2 + 1)(\sec^2 y)y' = x$
   (b) $xy' + 2x^2y - x^2 = 0$
   (c) $2xyy' = x^2 + 2y^2$ (This is homogeneous....)
   (d) $(x^3 + 2y^2)dx + (4xy + 6y^2)dy = 0$

(4) Find particular solutions to these initial value problems.
   (a) $\frac{dx}{dt} = 10x - x^2$, $x(0) = 1$
   (b) $\frac{dy}{dx} = 3x^2(y^2 + 1)$, $y(0) = 1$

(5) A $200m^3$ room initially contains fresh air. At time $t = 0$, a faulty heating system causes gas containing 20% carbon monoxide to be pumped into the room at a rate of $3m^3$ per minute. The well-mixed air is vented out at the same rate.
   (a) Write a differential equation with appropriate initial conditions that describes this situation.
   (b) Solve the initial value problem.
   (c) A carbon monoxide detector in the room is triggered when the carbon monoxide reaches 1%. Give an expression (reasonably simplified) for the time at which the alarm will sound.

(6) "Reverse engineer" a substitution problem: (1) Start with a differential equation you can solve, but use v instead of y. (2) Then make a substitution like $v = e^y$ to replace the v's with y's. (3) Differentiate your substitution to replace dv/dx with dy/dx.

   For example, from (3)(b) we get $xv' + 2x^2v - x^2 = 0$. Using the substitution $v = 1/y + 1$, differentiating gives $\frac{dv}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$. (That is, $v' = -y'/y^2$.) Plugging in makes the equation become $\frac{-x}{y^2}y' + 2x^2(\frac{1}{y} + 1) - x^2 = 0$. This simplifies to $y' - 2xy - 2y^2 + xy^2 = 0$ which seems much more difficult to solve. How might this new differential equation be solved?

   Try a few of your own design.

(7) Create some exact differential equations that you can solve.

(8) Create some linear equations that you can solve.

(9) Sketch the slope field to $\frac{dy}{dx} = 1/y - x$ on the domain $x > 0$ and $y > 0$. Also sketch some solution curves.