Practice Exam 2 Solutions

(1) Give an example three vectors in $\mathbb{R}^3$ for which no two are parallel and yet all three span a plane. Do they form a basis for that plane?

For example, in $\mathbb{R}^2$ $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\1\end{pmatrix}$ all span $\mathbb{R}^2$ but no two are parallel.

But $\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$ so they aren't lin. indep. & can't be a basis.

Put these into $\mathbb{R}^3$ by adding an extra 0.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\\0\end{pmatrix}$$

They span the plane $\left\{\begin{pmatrix}x\\y\\0\end{pmatrix}\right\}$ but are not a basis b/c not lin. indep.

(2) If $A\mathbf{x} = \mathbf{0}$ has just one solution, then what's the most you can say about the matrix A? What if A were a square matrix?

First, it means $\vec{x}=\vec{0}$ is the only solution. Hence the columns of A are linearly indep. So if A is $m \times n$, then rank $r = n$. This also tells us $n \leq m$.

The nullspace of A is the set of solutions to $A\vec{x}=\vec{0}$ so $\dim N(A) = 0$ and $\dim C(A^T) = n$.

If A is square, then A is also invertible.

(3) If the row vectors of an $n \times n$ matrix A are each perpendicular to a non-zero vector $\mathbf{v}$, then A cannot be invertible. Why not?

If each row of A is $\perp$ to $\vec{v}$

then $A\vec{v} = \vec{0}$

(because $\vec{a} \perp \vec{v}$ means $\vec{a}^T \vec{v} = 0$.)

Since $\vec{v} \neq \vec{0}$, then $A\vec{x} = \vec{0}$ has a non zero solution.

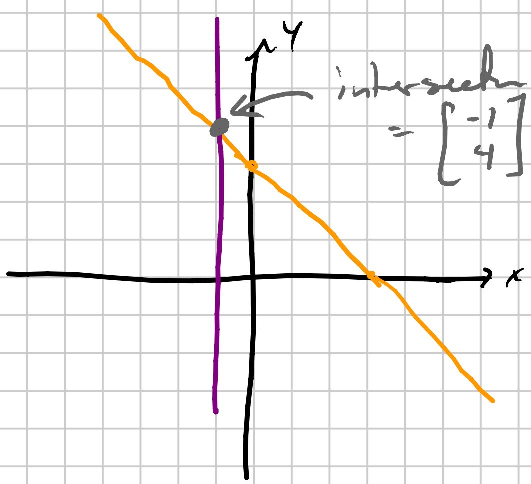
But if A were invertible,

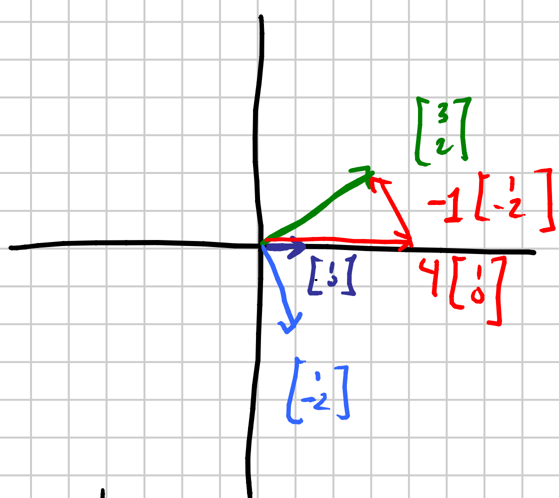
then $\vec{x} = A^{-1}\vec{0}$ would be the only soln.

$= \vec{0}$

Hence A is not invertible.

(4) Draw the Row Picture and Column Picture that illustrate the equation $\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and its solution.





row $1x + 1y = 3$

$-2x + 0y = 2$

Column:

$x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(5) (a) Draw both the schematic and "literal" pictures of the four fundamental subspaces associated to the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$.

(b) On the left side (the row picture) of both the schematic and literal pictures, what corresponds to the set of solutions to $A\mathbf{x} = \mathbf{0}$?

(c) Find the complete solution $\mathbf{x} = \mathbf{x_p} + \mathbf{x_n}$ to the equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. Draw the set of solutions in the left side (the row picture) of the literal picture. Is this set of solutions a subspace?

(d) Repeat this problem using the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

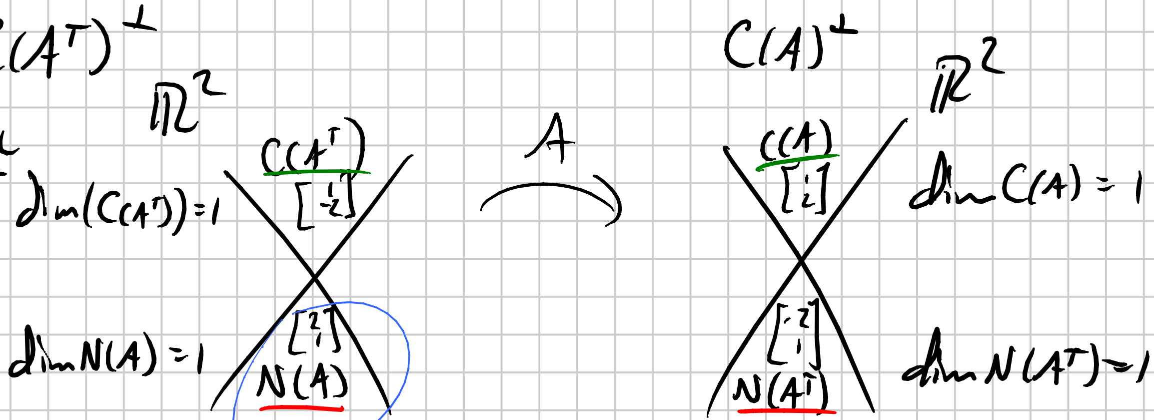
a) $C(A^T)$ has basis $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ row sp

$C(A)$ has basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ col sp.

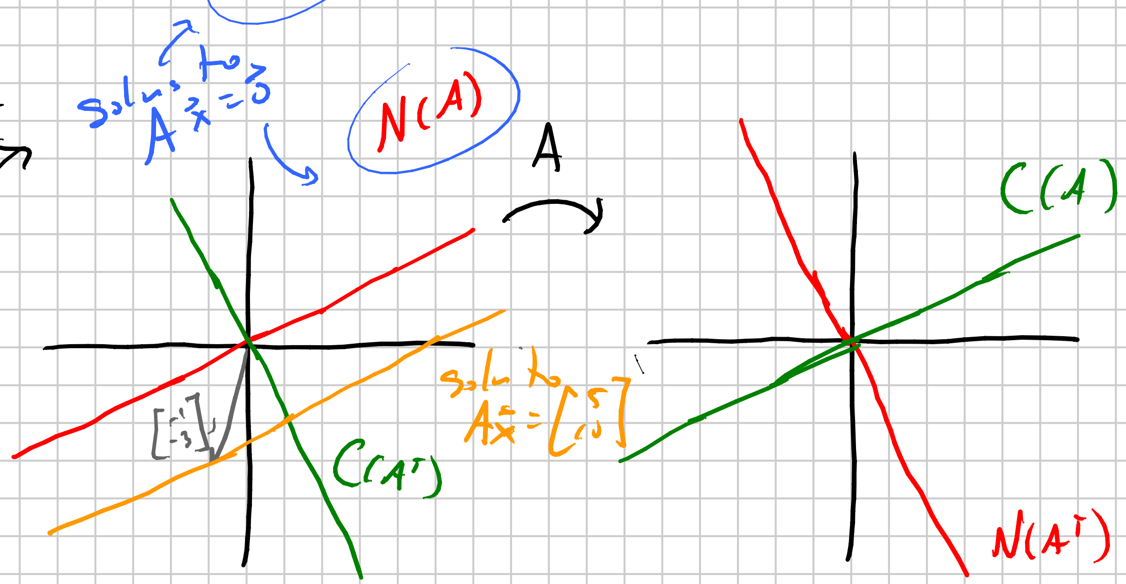
$N(A)$ has basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. $= C(A^T)^\perp$

$N(A^T)$ has basis $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ $= C(A)^\perp$

Schematic



Literal



b)

c) $\vec{x}_n = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (the null space)

just need to find a particular soln $\vec{x}_p$.

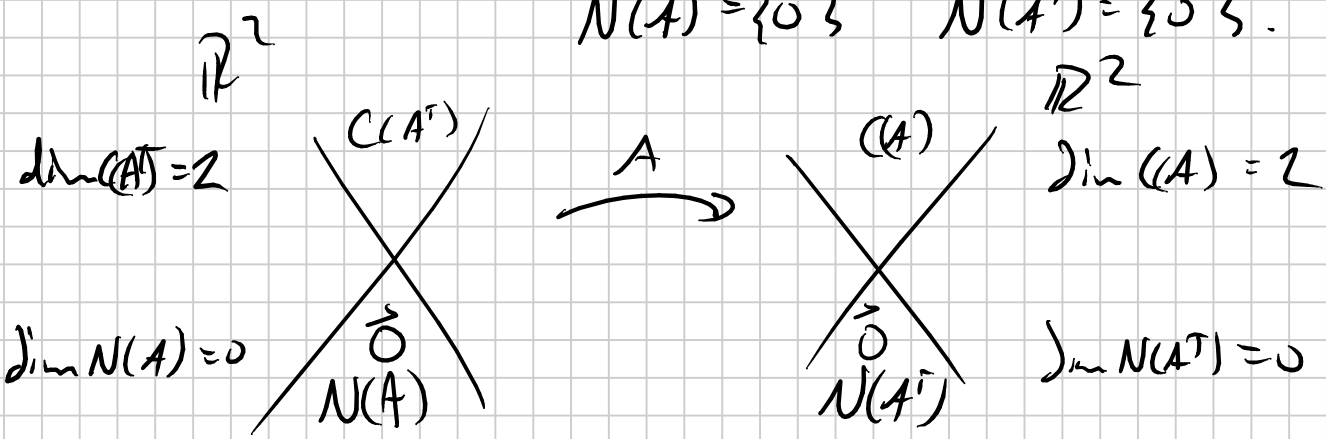
Eyeballing it, $\begin{bmatrix} 5 \\ 10 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + -3\begin{bmatrix} -2 \\ -4 \end{bmatrix}$ so $\vec{x}_p = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ works.

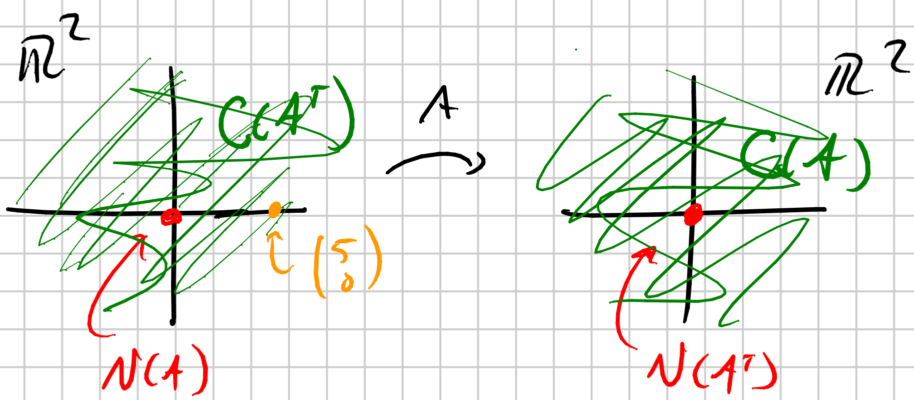
Complete soln: $\vec{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

A is invertible, so $C(A^T) = \mathbb{R}^2$, $C(A) = \mathbb{R}^2$

$N(A) = \{\vec{0}\}$ $N(A^T) = \{\vec{0}\}$.





" solns to $A\vec{x} = 0$.

Solve $\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$x - 2y = 5$

$2x = 10$

$x = 5, \; y = 0$

Soln $\vec{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

(6) (a) Project the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ into the line W spanned by $\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

(b) Determine the matrix P that projects vectors into W.

(c) If a vector $\mathbf{x}$ is in $W^{\perp}$, why must $P\mathbf{x} = \mathbf{0}$?

(d) Use this to find a basis for $W^{\perp}$.

a) $\vec{P} = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \dfrac{\vec{w}\cdot\vec{v}}{\vec{w}\cdot\vec{w}}\vec{w} = \dfrac{0+6+-1}{0+9+1}\begin{bmatrix}0\\3\\1\end{bmatrix} = -\dfrac{6}{10}\begin{bmatrix}0\\3\\1\end{bmatrix}$

$= -\dfrac{3}{5}\begin{bmatrix}0\\3\\1\end{bmatrix}$

b) Projection matrix $P = A(A^TA)^{-1}A^T$

$= \begin{bmatrix}0\\3\\1\end{bmatrix}\left([0\,3\,1]\begin{bmatrix}0\\3\\1\end{bmatrix}\right)^{-1}[0\,3\,1]$

$= \begin{bmatrix}0\\3\\1\end{bmatrix}\left[\frac{1}{10}\right][0\,3\,1] = \frac{1}{10}\begin{bmatrix}0&0&0\\0&9&3\\0&3&1\end{bmatrix}$

c) With $\vec{x} = \vec{p} + \vec{e}$ = "projection + error"

since $\vec{p} \in W \Rightarrow \vec{e} \in W^{\perp}$, $\vec{x} = \vec{e} \Rightarrow \vec{p} = \vec{0}$

Hence $P\vec{x} = \vec{0}$.

d) Part c) says that $W^{\perp} = N(P)$

So we just find basis for $N(P)$.

Scalars don't change $N(P)$

So use $\begin{bmatrix}0&0&0\\0&\boxed{9}&3\\0&3&1\end{bmatrix} \rightsquigarrow \begin{bmatrix}0&\boxed{9}&3\\0&0&0\\0&3&1\end{bmatrix} \rightsquigarrow \begin{bmatrix}0&\boxed{9}&3\\0&0&0\\0&0&0\end{bmatrix} \rightsquigarrow \begin{bmatrix}0&\boxed{1}&1/3\\0&0&0\\0&0&0\end{bmatrix}$

special solns $\vec{s}_1 = \begin{bmatrix}1\\\boxed{0}\\0\end{bmatrix}$ $\vec{s}_3 = \begin{bmatrix}0\\\boxed{-1/3}\\1\end{bmatrix}$

A basis for $W^{\perp}$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1/3\\1\end{bmatrix}\right\}$.

(7) (a) Find the complete solution $\mathbf{x} = \mathbf{x_p} + \mathbf{x_n}$ to the equation $A\mathbf{x} = \mathbf{b}$

where $A = \begin{bmatrix} 1 & -2 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 3 & -6 \\ 2 & -4 & 2 & 0 & -6 \end{bmatrix}$ and $\mathbf{b} \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$.

(b) Find bases for each $C(A)$, $C(A^T)$, and $N(A)$.

a) Put in Reduced row eschelon form.

$$\left[\begin{array}{ccccc|c} \boxed{1} & -2 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 2 & 0 & 0 & 3 & -6 & 0 \\ 2 & -4 & 2 & 0 & -6 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccc|c} \boxed{1} & -2 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & \boxed{4} & -2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{ROW SWAP}} \left[\begin{array}{ccccc|c} \boxed{1} & -2 & 1 & 0 & -3 & -1 \\ 0 & \boxed{4} & -2 & 3 & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} \boxed{1} & -2 & 0 & -1 & -5 & -2 \\ 0 & \boxed{4} & 0 & 5 & 4 & 4 \\ 0 & 0 & \boxed{1} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccc|c} \boxed{1} & -2 & 0 & -1 & -5 & -2 \\ 0 & \boxed{1} & 0 & 5/4 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 3/2 & -3 & 0 \\ 0 & \boxed{1} & 0 & 5/4 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} p & p & p & f & f & \end{array} \end{array}$$

particular soln

$$\vec{x}_p = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

special solns.

$$\vec{s}_4 = \begin{bmatrix} -3/2 \\ -5/4 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{s}_5 = \begin{bmatrix} 3 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Complete Solution

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ -5/4 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

b) <u>Bases</u> $\underset{\text{row sp}}{C(A^T)}$ & $\underset{\text{null sp}}{N(A)}$ are same as for reduced matrix R.

So $C(A^T)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3/2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5/4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

& $N(A)$ has basis $\left\{ \vec{s}_4 = \begin{bmatrix} -3/2 \\ -5/4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \vec{s}_5 = \begin{bmatrix} 3 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for $\underset{\text{col sp.}}{C(A)}$ is the pivot cols of A

So basis for $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}$

(8) These questions are about the vector space **M** of 2×2 matrices (where the "vectors" are 2×2 matrices).

(a) Give a basis for this vector space **M** and explain why it is a basis.
(b) What is the dimension of **M**?
(c) Do the 2×2 matrices A such that $A^{\mathrm{T}} =$ drag to resize or space? If so, give a basis and state its dimension. If not, give a reason why not.
(d) Do the invertible 2×2 matrices form a vector space? If so, give a basis and state its dimension. If not, give a reason why not.

a) Since $\mathcal{M} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$

a basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Clearly they are lin indep..

They also span since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Hence this is a basis.

b) So the dimension of $\mathcal{M}$ is 4.

c) $\{A \mid A^T = -A\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{matrix} a = -a \\ c = -b \\ b = -c \\ d = -d \end{matrix} \right\} = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \;\middle|\; b \in \mathbb{R} \right\}$

A basis is $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. Dimension is 1.

d) No, the set of invertible matrices is not a vector space.

For instance, the zero vector $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.