Math 531 - Topology I - Problem Set
Euler Characteristic & Surfaces

1. Euler characteristics

Recall that for a finite graph Γ, its *Euler characteristic* is the count $\chi(\Gamma) = \#V - \#E$ where V is the set of vertices of Γ and E is the set of edges. If Σ is a compact surface (possibly with boundary) and Γ is a finite graph embedded in Σ such that $F = \Sigma - \Gamma$ is a disjoint union of open disks, then we define $\chi(\Sigma, \Gamma) = \chi(\Gamma) + \#F = \#V - \#E + \#F$. (Here we take the perspective that Γ is realized as a simplicial complex where vertices are points and an edge joining two vertices is identified with a line segment joining the corresponding points.)

In the next problems, assume Γ is a finite graph embedded in a compact surface Σ such that $\Sigma - \Gamma$ is a disjoint union of open disks.

Problem 1. Show that if Γ' is a graph in Σ obtained from Γ by either

(A) subdividing an edge of Γ (adding a vertex to the interior of an edge, splitting that edge into two edges),
(B) placing a vertex v in $\Sigma - \Gamma$ and joining it to an existing vertex of Γ by an edge contained in $\Sigma - (\Gamma \cup \{v\})$, or
(C) joining two vertices of Γ by an edge in $\Sigma - \Gamma$,

then $\chi(\Sigma, \Gamma) = \chi(\Sigma, \Gamma')$.

Problem 2. If $h\colon \Sigma \to \Sigma$ is a homeomorphism, then show $\chi(\Sigma, \Gamma) = \chi(\Sigma, h(\Gamma))$.

Problem 3. Assume Γ_0 and Γ_1 are two finite graphs in Σ whose complements are disjoint unions of open disks. Prove that there exists a homeomorphism $h\colon \Sigma \to \Sigma$ such that there exists a finite graph Γ in Σ for which

(1) $\Sigma - \Gamma$ is a disjoint union of open disks and
(2) for each $i = 0, 1$, the graph Γ may be obtained from Γ_i by a sequence of the operations (A), (B), (C) in Problem 1.

Problem 4. Conclude that $\chi(\Sigma, \Gamma)$ does not depend on the choice of finite graph Γ or its embedding into Σ, only that $\Sigma - \Gamma$ is a collection of open disks.

Thus the Euler characteristic of a compact surface Σ is $\chi(\Sigma) = \chi(\Sigma, \Gamma) = \#V - \#E + \#F$ for any finite graph Γ embedded in Σ such that $\Sigma - \Gamma$ is a collection of open disks.

Problem 5. How might you generalize the Euler characteristic formula for when components of $\Sigma - \Gamma$ are not open disks? (You might begin by considering the Euler characteristic of a closed disk, a circle, and an open disk...)

2. Surfaces

Problem 6. Calculate the Euler characteristic of (a) a sphere, (b) a sphere with one handle, (c) a sphere with two handles, (d) a sphere with one crosscap, and (e) a sphere with two crosscaps.

Let Σ_0 and Σ_1 both be connected compact surfaces. The *connected sum* of Σ_0 and Σ_1 is the surface $\Sigma_0 \# \Sigma_1$ obtained by deleting an open disk from the interior of each Σ_0 and Σ_1 and joining the two surfaces along the two new boundary circles. For example, the connected sum of two tori is a genus 2 surface.

Problem 7. Show that for any connected compact surface Σ, $S^2 \# \Sigma \cong \Sigma$.

Problem 8. Determine $\chi(\Sigma_0 \# \Sigma_1)$ in terms of $\chi(\Sigma_0)$ and $\chi(\Sigma_1)$.

Problem 9. Determine a formula for the Euler characteristic of:

(1) a sphere with n perforations and g handles, $n, g \in \mathbb{Z}_+$
(2) a sphere with n perforations and c crosscaps, $n, c \in \mathbb{Z}_+$

Problem 10. Prove that the torus T and the Klein bottle K are not homeomorphic even though $\chi(T) = \chi(K)$.