Small Seifert Fibred Spaces (SSFs) are formed from 3 solid tori glued together along annuli.

They have Heegaard genus 2 and are non-Haken or fibered.
Some S5FS's may be obtained by Dehn surgery on torus knots.

Very non hyperbolic.
For all known examples of hyperbolic knots with $SSFS$ surgeries, the surgery is integral.

The $S^3$ meridian $\mu$ intersects the surgery slope $\lambda$ (the $SFS$ meridian) just once.

The distance of this surgery is $\Delta = \Delta(\mu, \lambda) = 1$.

Question: Must the distance be 1?
What hyperbolic knots in $S^3$ admit $SSFS$ surgeries?

Dean & Edidin-Muñoz describe many by generalizing Berge.

- all TN1 and Strongly Invertible.

Mattman, Miyazaki, Motejai describe an interesting family.

- all TN2 ad not Strongly Invertible.

$P(-3,3,5) \Rightarrow$

Question: Must the tunnel number be $\leq 2$?
Theorem (B–Gordon–Luecke)

If a distance $d \geq 3$ surgery on a hyperbolic knot in $S^3$ produces an $SSFS$, then $TN \leq 2$.

Actually only use Heegaard genus 2 and non-Haken

(Homology $\Rightarrow$ Can't get fibered SSFS by $d \geq 2$ surgery)
Almost Theorem

If a distance $\Delta \geq 3$ surgery on a hyperbolic knot in $S^3$
produces a non-Haken, Heegaard genus 2 manifold then the core of the surgery is 1-bridge with respect to either
a genus 2 Heegaard Splitting or a one-sided genus 3 Heegaard Splitting.
Set up:

Two Surjects: \( S^3 = X_k(y) \) \( \rightarrow X_k \) \( \rightarrow X_k(y) = M \) non-Haken genus 2

\( \Delta = \Delta(y, \theta) \geq 3 \)

Non-Haken + \( \Delta \geq 2 + S^3 \) sorry

\( \Rightarrow \) thin is bridge

(Thompson)

Put knots in thin position

Thick Sphere

\[ Q = \hat{Q} \wedge X_k \]

\[ F = \hat{F} \wedge X_k \]

Gabai, Rieck \( \Rightarrow \) May take \( \hat{Q}, \hat{F} \) so that no arc of \( Q \cap F \) is trivial.
\( Q, F \) are punctual surfaces.
View punctures as (fat) vertices on \( \hat{Q}, \hat{F} \)
View arcs of \( Q \lor F \) as edges on \( \hat{Q}, \hat{F} \)

Fat vertex: graphs

vertex \( \leftrightarrow \) copies of \( \mu \)

vertex \( \leftrightarrow \) copies of \( \xi \)
Give structure in the other manifold.
Combinatorics of Fat Vertex Graphs imply that we will always see an 6G cycle which forms Mobius bands in M.
Together they form a "Mobius Band".

In one handlebody in the other handlebody, the other two faces form an annulus. The Sicherman cycle makes a Mobius band.

Extended Sicherman Cycle

Sometimes we find...
And sometimes we find

Forked Extend SCHURLEMANN Cycles

\[ a - 1 \times a(1) \]
\[ a + 2 \quad a(2) \quad a \]

Again we have a Möbius band from the Schurlemann cycle.

How do the other two pieces fit?

Look at graph on GF.