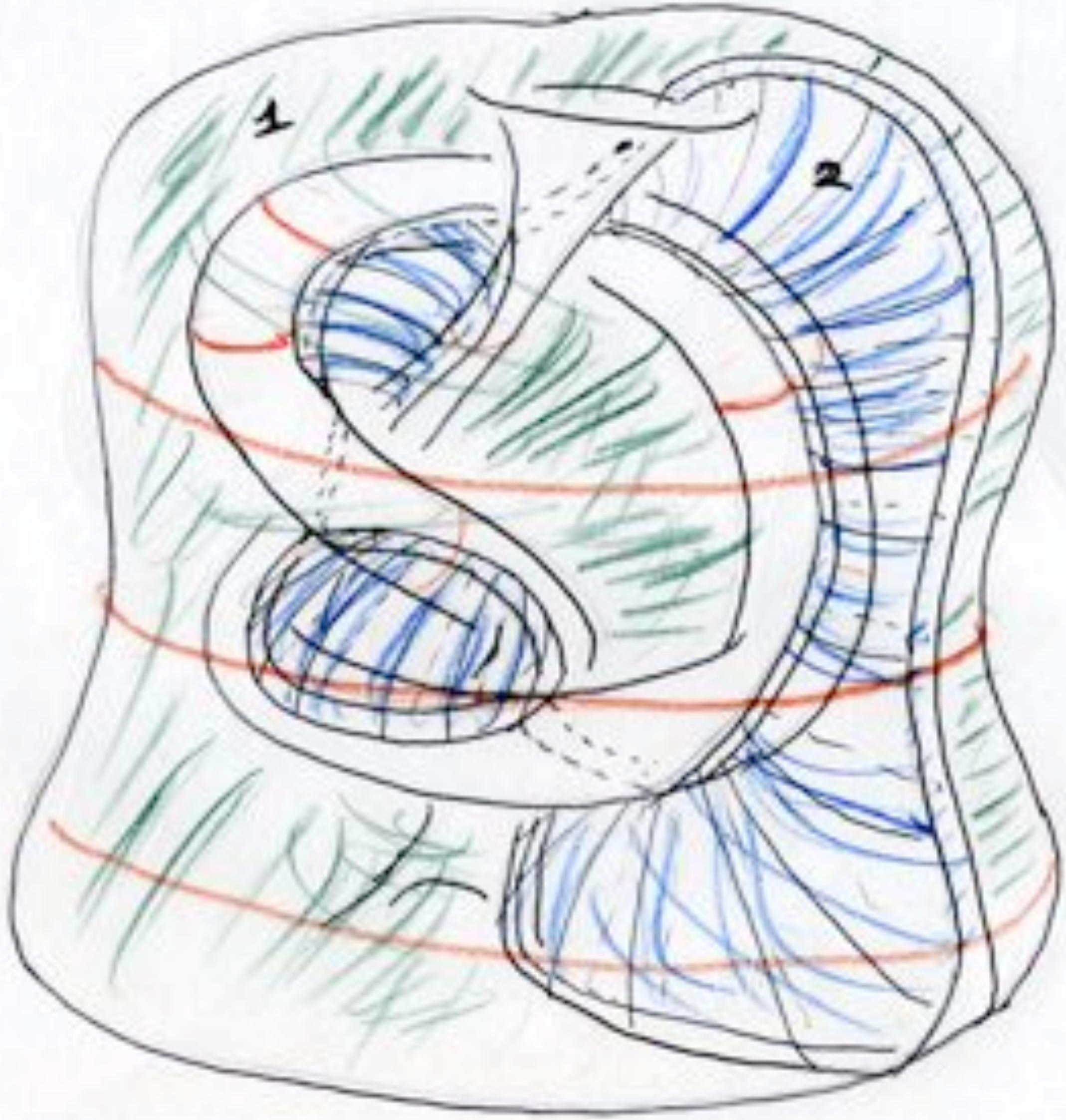
Small Seifert Fibered Spaces (SSFS)

are formed from 3 solid tori glued together along annuli

They have Heegaard genus 2

and are non-Haken or fibered.

Some SSFSs may be obtained
by Dehn surgery on torus knots

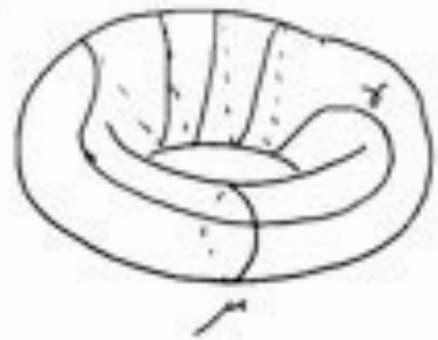


∪

very non hyperbolic.

For all known examples of hyperbolic knots with SSFS surgeries the surgery is <u>integral</u>;

The S^3 meridian μ intersects the surgery slope γ (the SFS meridian) just once



The <u>distance</u> of this surgery is

$$\Delta = \Delta(\mu, \gamma) = 1.$$

<u>Question</u>: Must the distance be 1?

What hyperbolic knots in S^3 admit SSFS surgeries?

Dean and Eudave-Muñoz describe many by generalizing Berge.

- all TN1 and Strongly Invertible.

Mattman, Miyazaki, Motegi describe an interesting family.

- all TN2 and not Strongly Invertible.

$P(-3,3,5) \rightarrow$

<u>Question</u>: Must the tunnel number be ≤ 2?

Theorem (B - Gordon - Luecke)

If a distance $\Delta \geq 3$ surgery on a hyperbolic knot in S^3 produces a SSFS, then $TN \leq 2$

Actually only use Heegaard genus 2 and Non Haken

(Homology $\Rightarrow$ Can't get fibered SSFS by $\Delta \geq 2$ surgery)

Almost Theorem

If a distance $\Delta \geq 3$ surgery
on a hyperbolic knot in S^3
produces a NonHaken, Heegaard genus 2 manifold
then the core of the surgery is 1-bridge
with respect to either
a genus 2 Heegaard Splitting
or
a one-sided genus 3 Heegaard Splitting.

Set up:

exterior of knot

$$X_K$$

Two Surgeries: $S^3 = X_K(\mu)$ $\qquad$ $X_K(\gamma) = M$ non Haken genus 2

$\Delta = \Delta(\mu, \gamma) \geq 3$

non Haken + $\Delta \geq 2$ + S^3 surg $\Rightarrow$ thin is bridge (Thompson)

Put knots in <u>thin position</u>

Thick sphere $\hat{Q}$

Bridge surface $\hat{F}$

$Q = \hat{Q} \cap X_K$ $\qquad$ $F = \hat{F} \cap X_K$

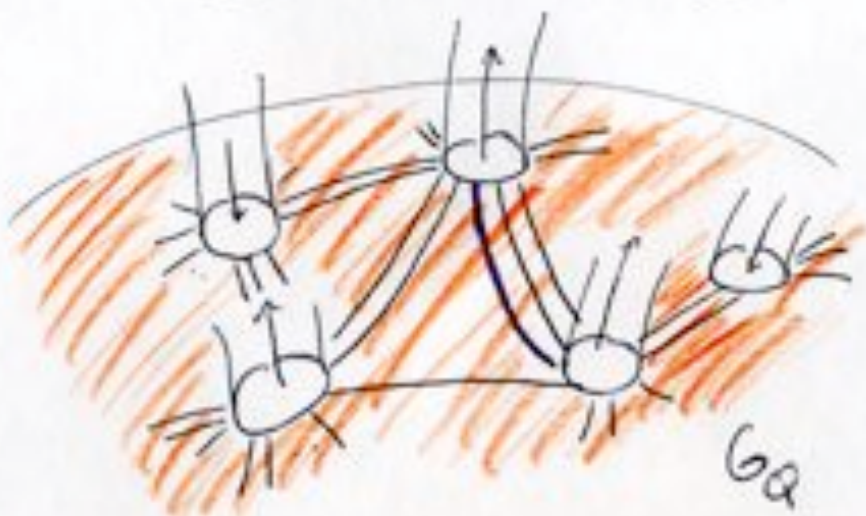
Gabai, Rieck $\Rightarrow$ May take $\hat{Q}, \hat{F}$ so that no arc of $Q \cap F$ is trivial.

Q, F are punctured surfaces.

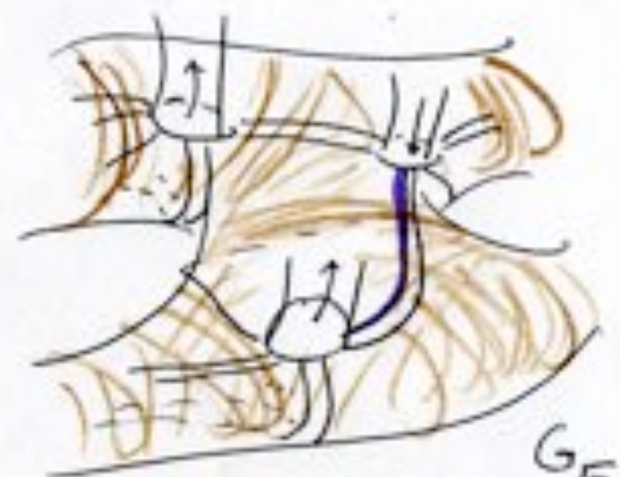
View punctures as (fat) <u>vertices</u> on $\hat{Q}, \hat{F}$

View arcs of $Q \cap F$ as <u>edges</u> on $\hat{Q}, \hat{F}$

Fat vertexed graphs

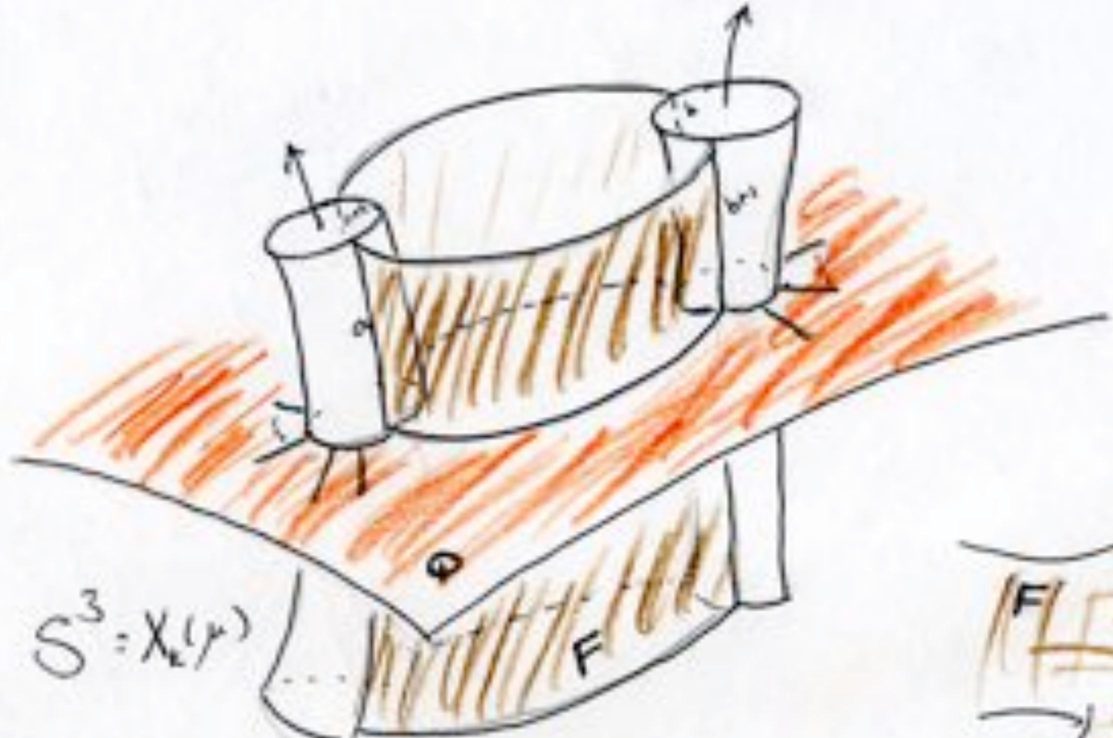


vertices $\leftrightarrow$ copies of μ



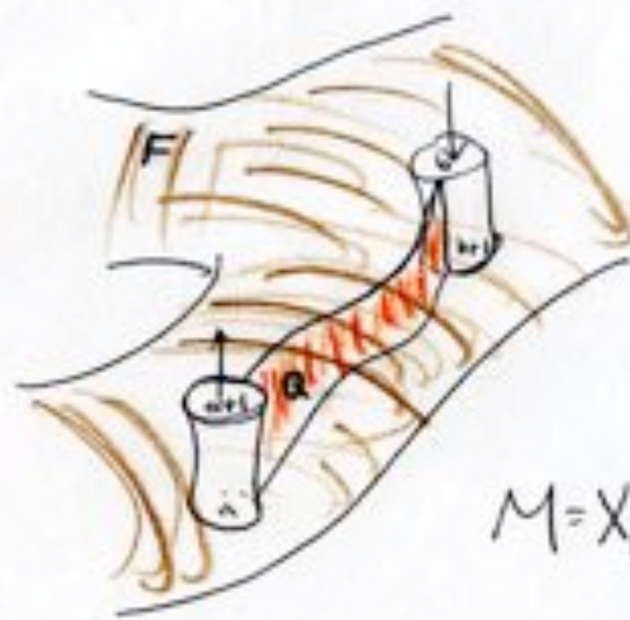
vertices $\leftrightarrow$ copies of γ

Faces of one graph



$S^3 = X_K(\mu)$

Give structure
in the
other manifold



$M = X_K(\gamma)$

Combinatorics of Fat Vertexed Graphs

imply that we will always see

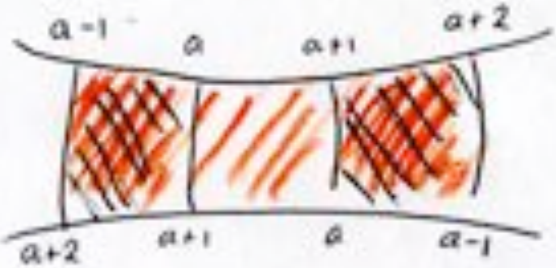
on G_Q

Scharlemann Cycles

which form <u>Mobius bands</u> in M

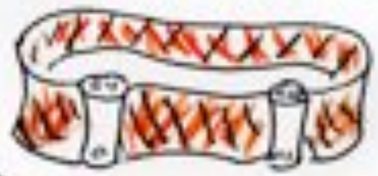
Sometimes we find

Extended Scharlemann Cycles



The Scharlemann cycle makes a Mobius band in one handle body

The other two faces form an annulus in the other handlebody

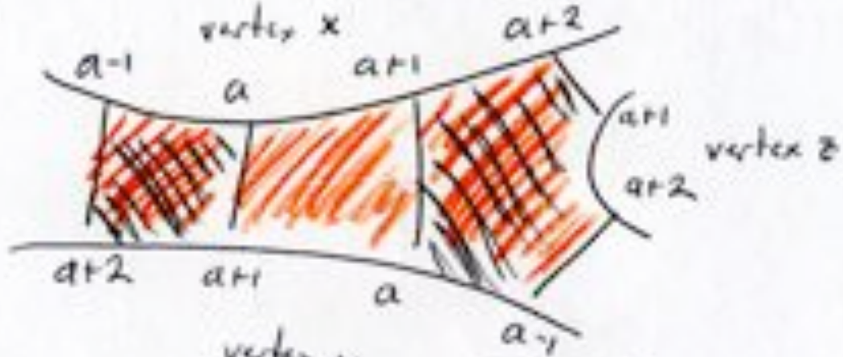


Together they form

a "long Mobius band".

And sometimes we find

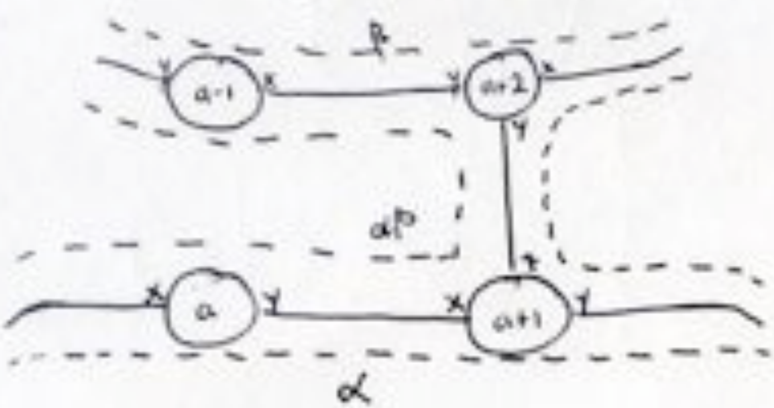
Forked Extended Scharlemann Cycles



Again we have a Möbius band from the Scharlemann Cycle.

How do the other two pieces fit?

Look at graph on G_F:



And we form this.