Grid diagrams and Legendrian links in lens spaces

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Contact Structures

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The standard contact structure on $\mathbb{R}^3$ may be defined as

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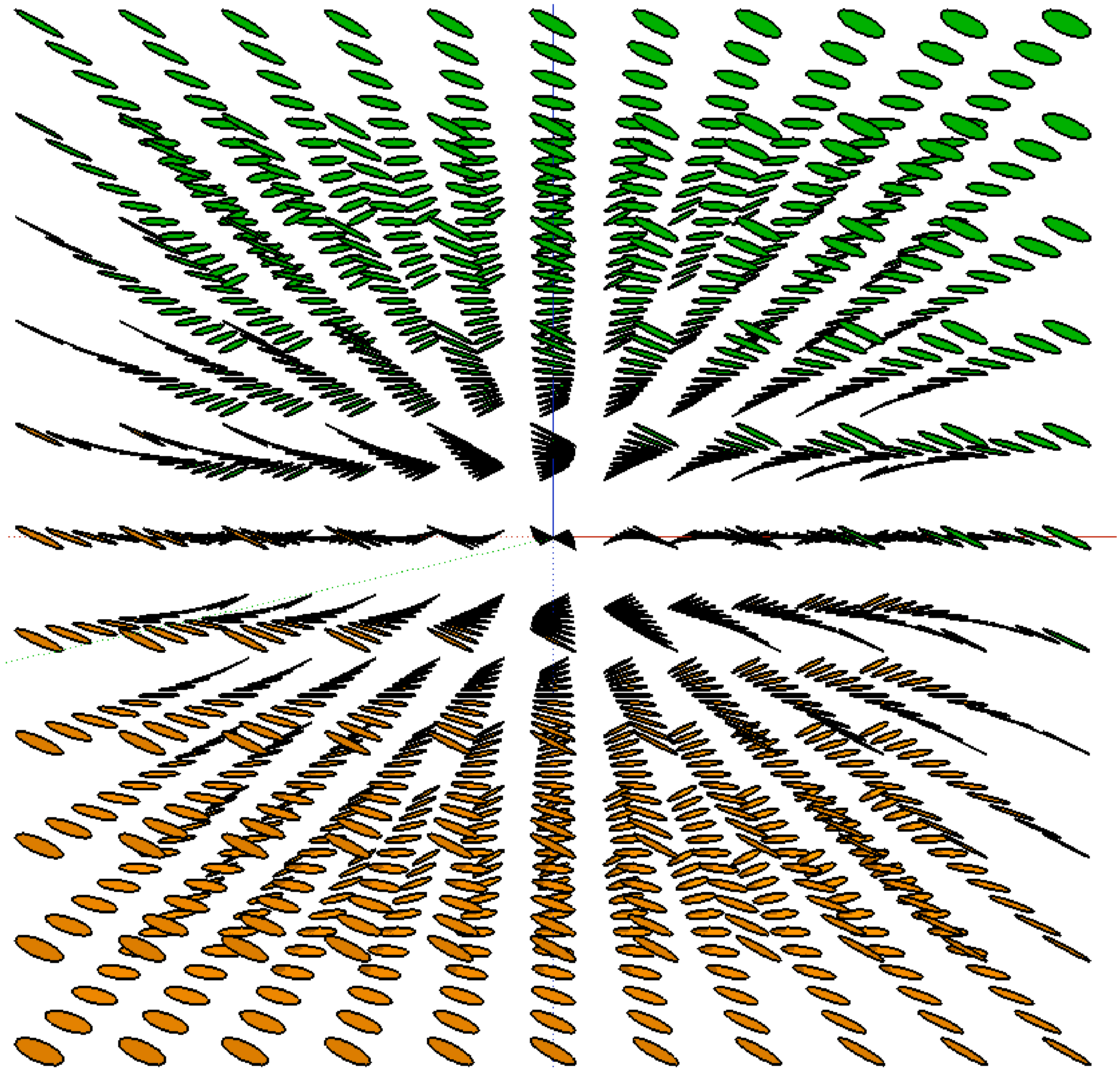

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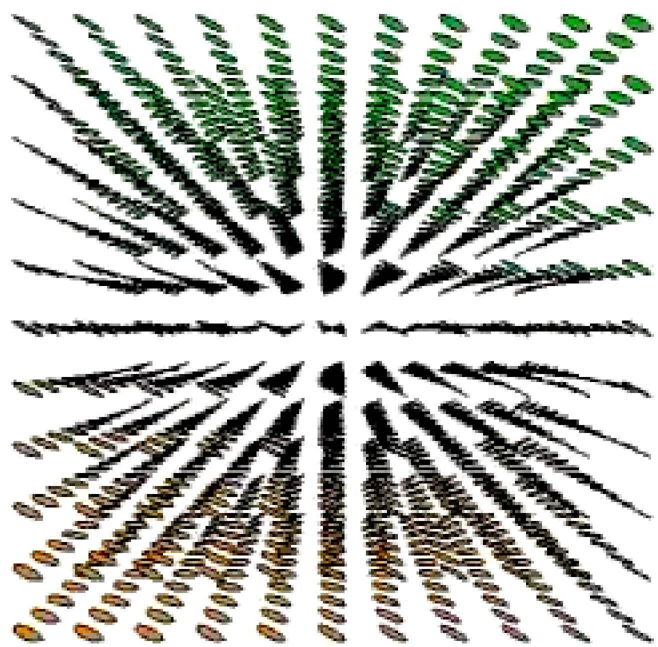
Legendrian links in the contact manifold (Y, ξ) are embedded collections of smooth loops whose tangents lie in ξ.

Locally, they look like the y–axis in the standard model...



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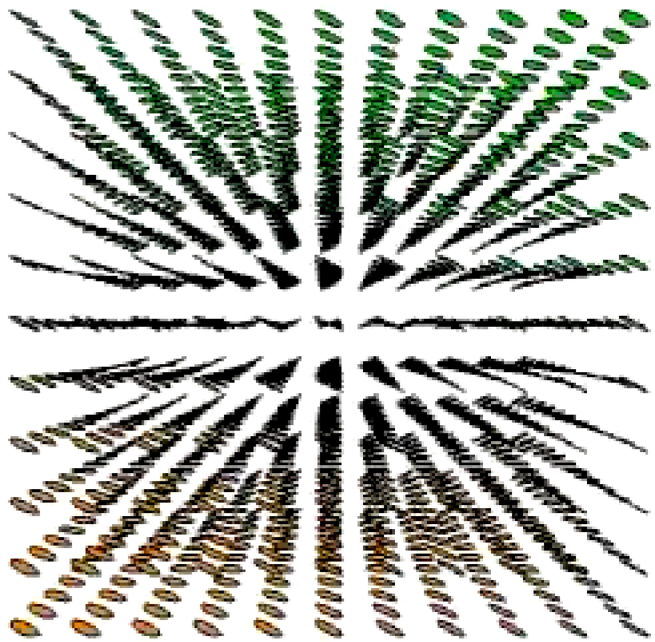

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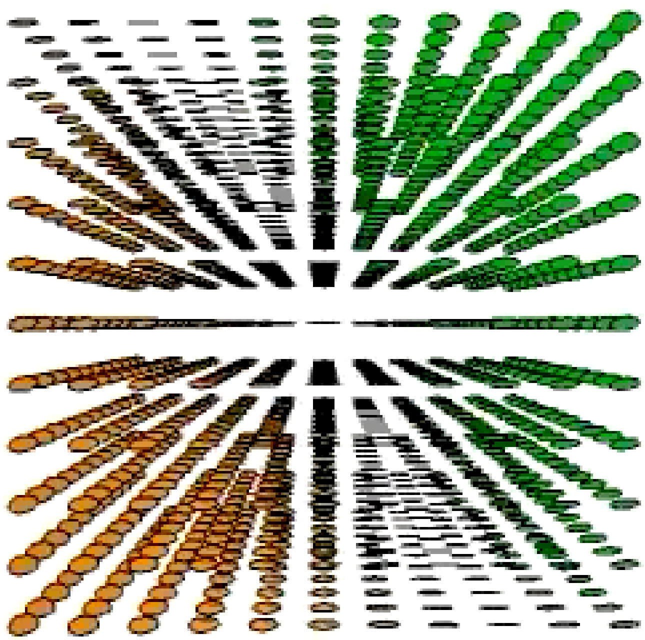

FRONT DIAGRAMS

If $\mathcal{L}$ is a Legendrian curve in $(\mathbb{R}^3, \xi_{\mathbb{R}^3})$,

then $\alpha_{\mathbb{R}^3}$ evaluates to 0 on $T_{\text{pt}}\mathcal{L}$.

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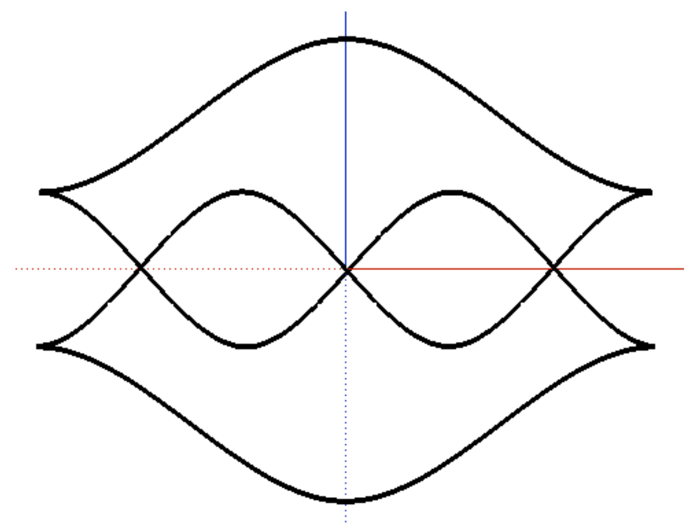
The y–coordinate of a point on $\mathcal{L}$ is the slope of the tangency of $\pi_{xz}(\mathcal{L})$ at the image of that point.



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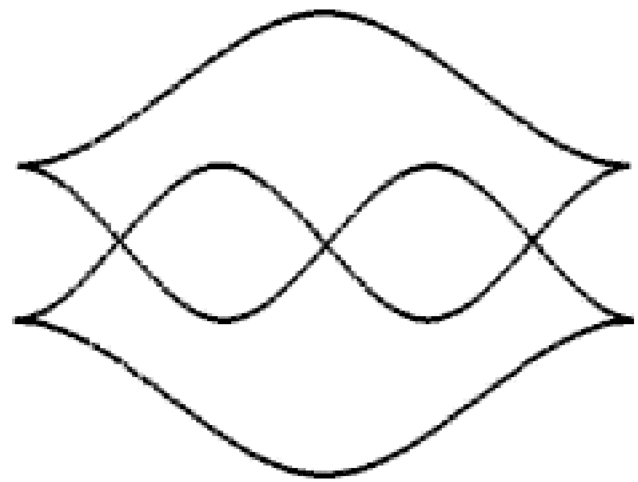
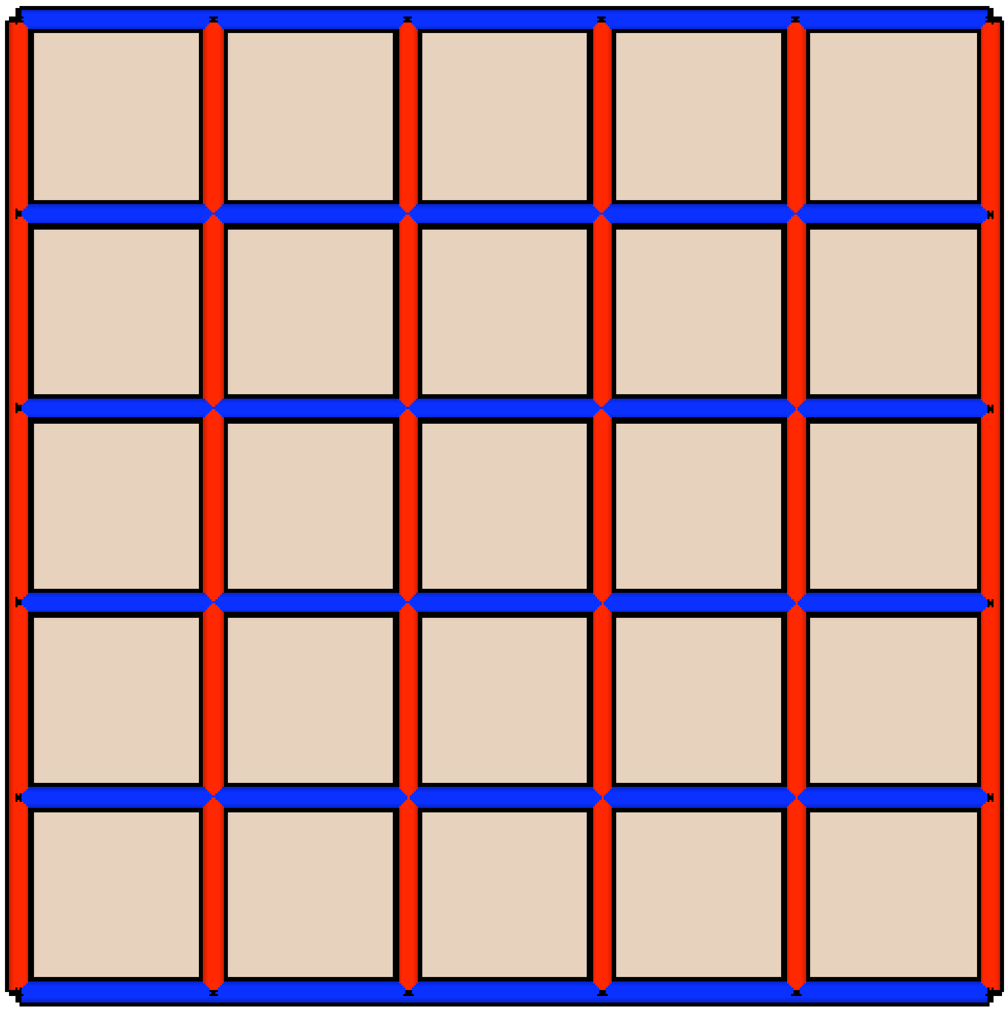


Thus $\mathcal{L}$ may be recovered from its projection $\pi_{xz}(\mathcal{L})$ to the xz–plane.

Such an immersed curve is a front for the Legendrian link $\mathcal{L}$.

Grid Diagrams

Links in $\mathbb{R}^3$ may be combinatorially described with grid diagrams.



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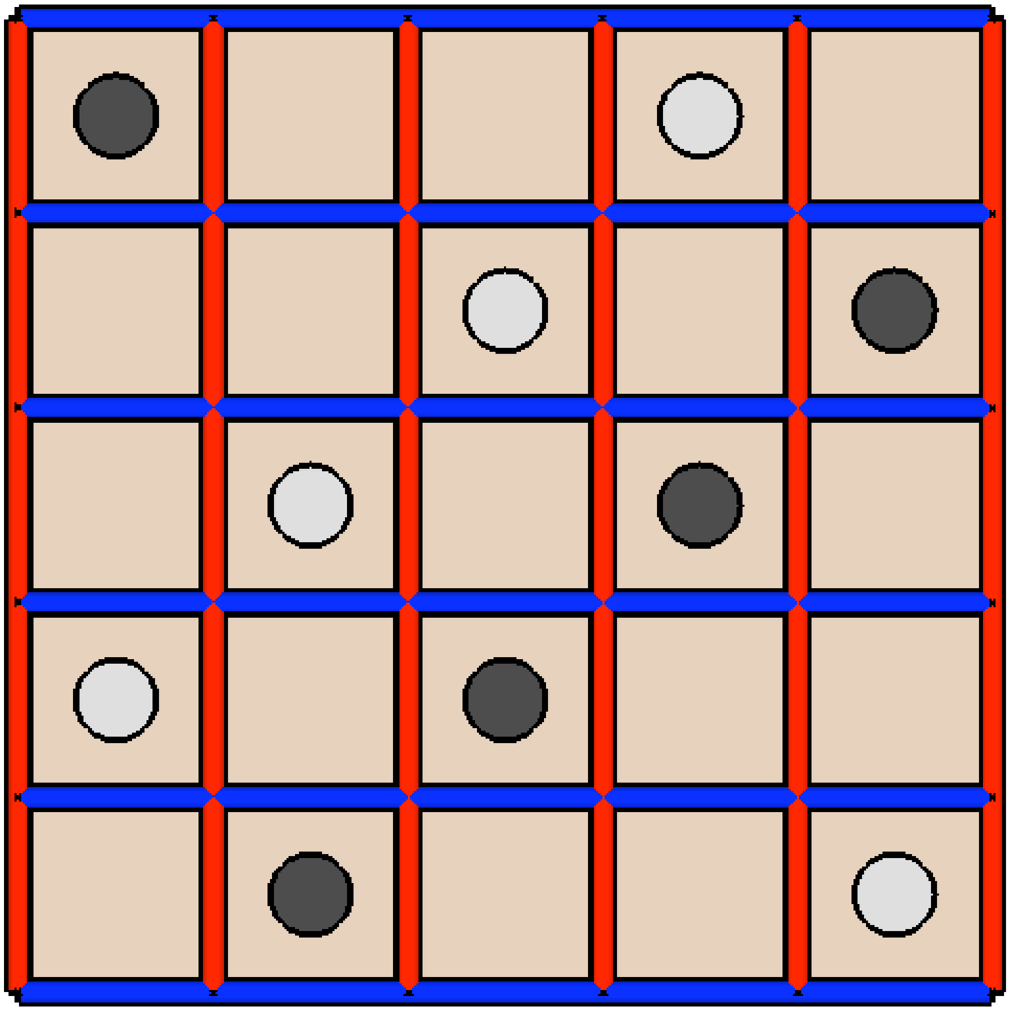

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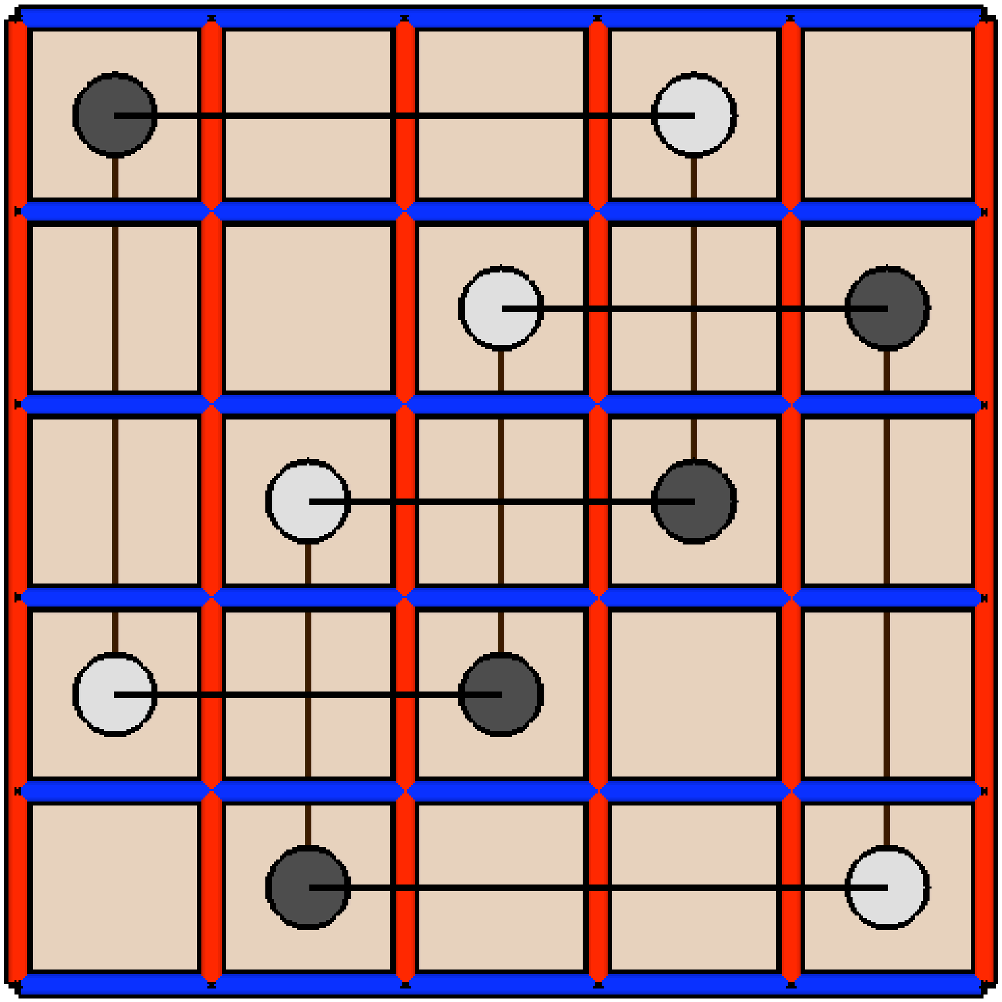

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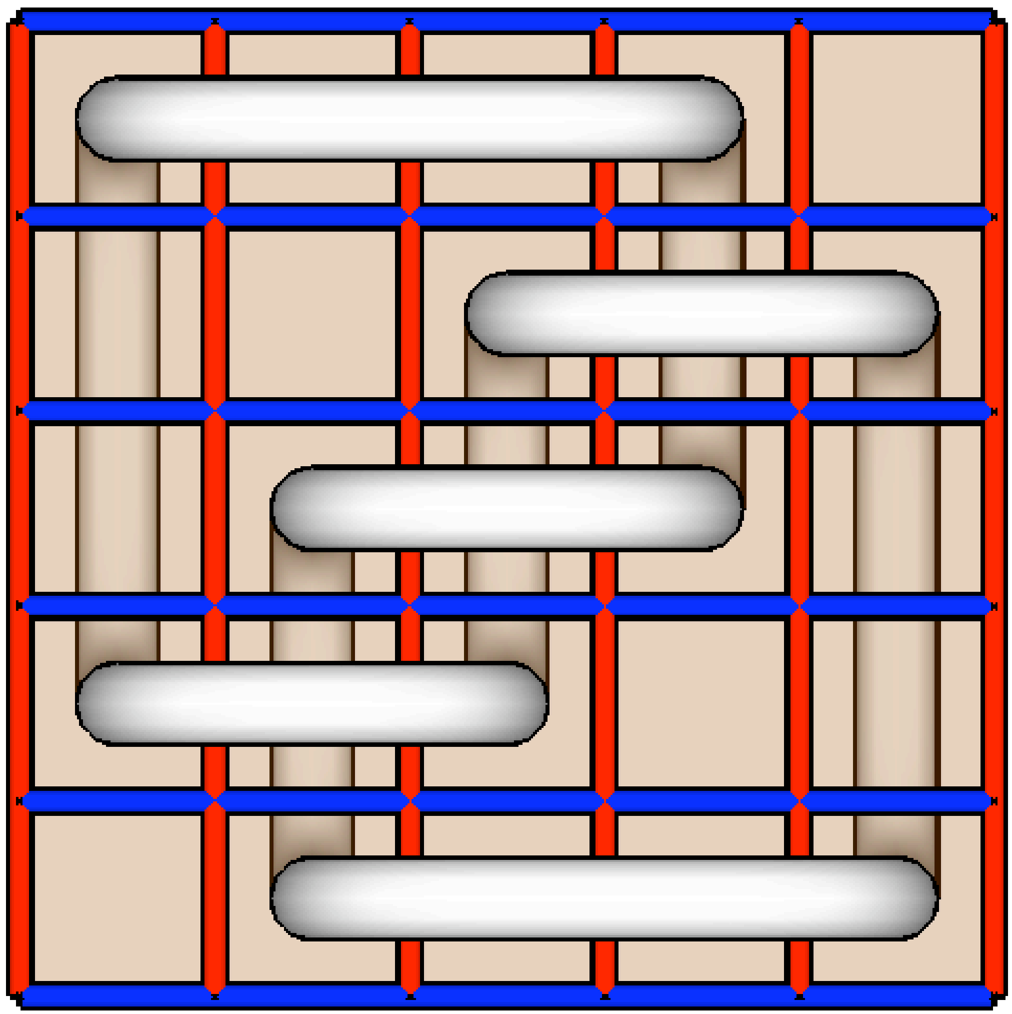

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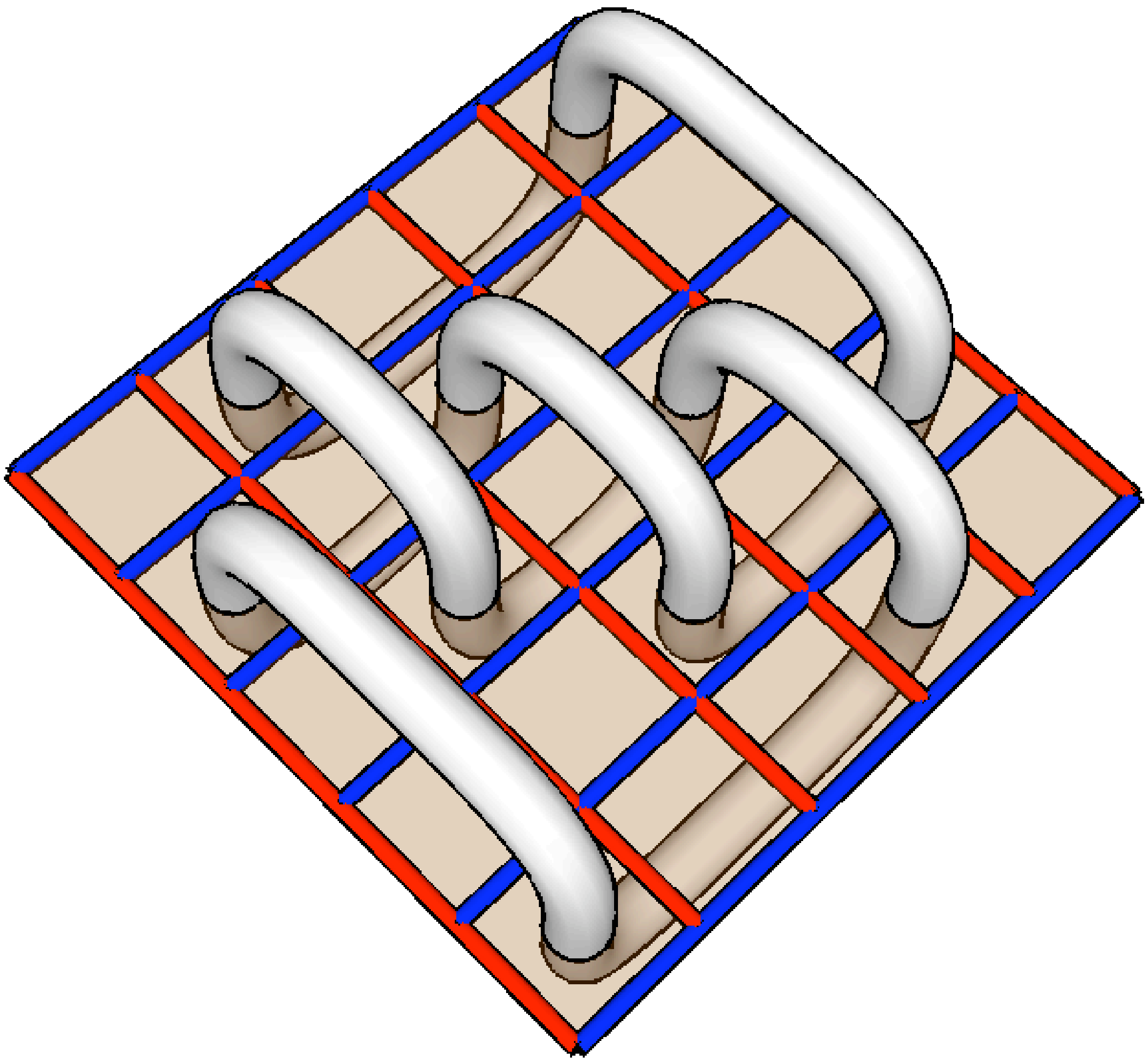
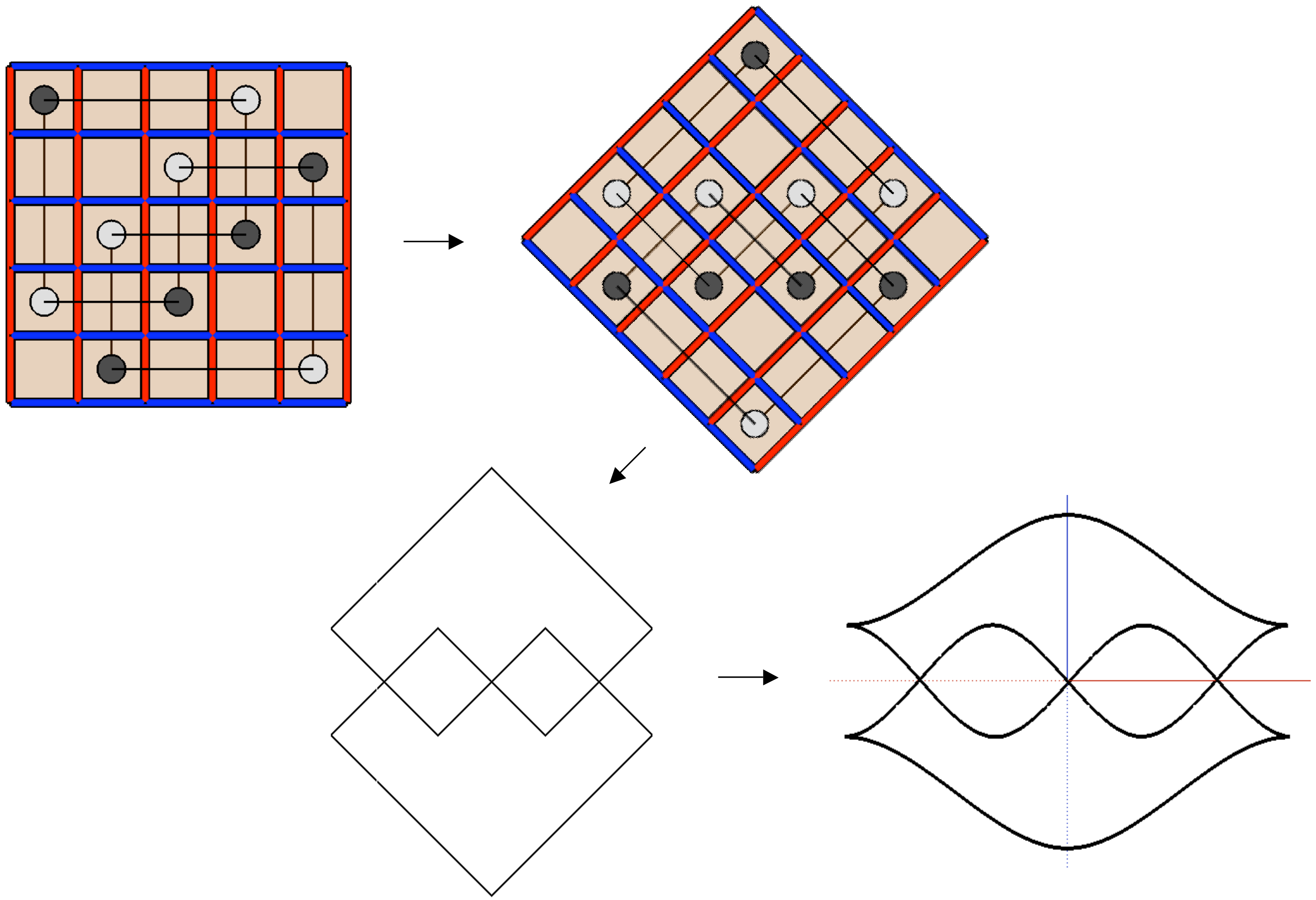


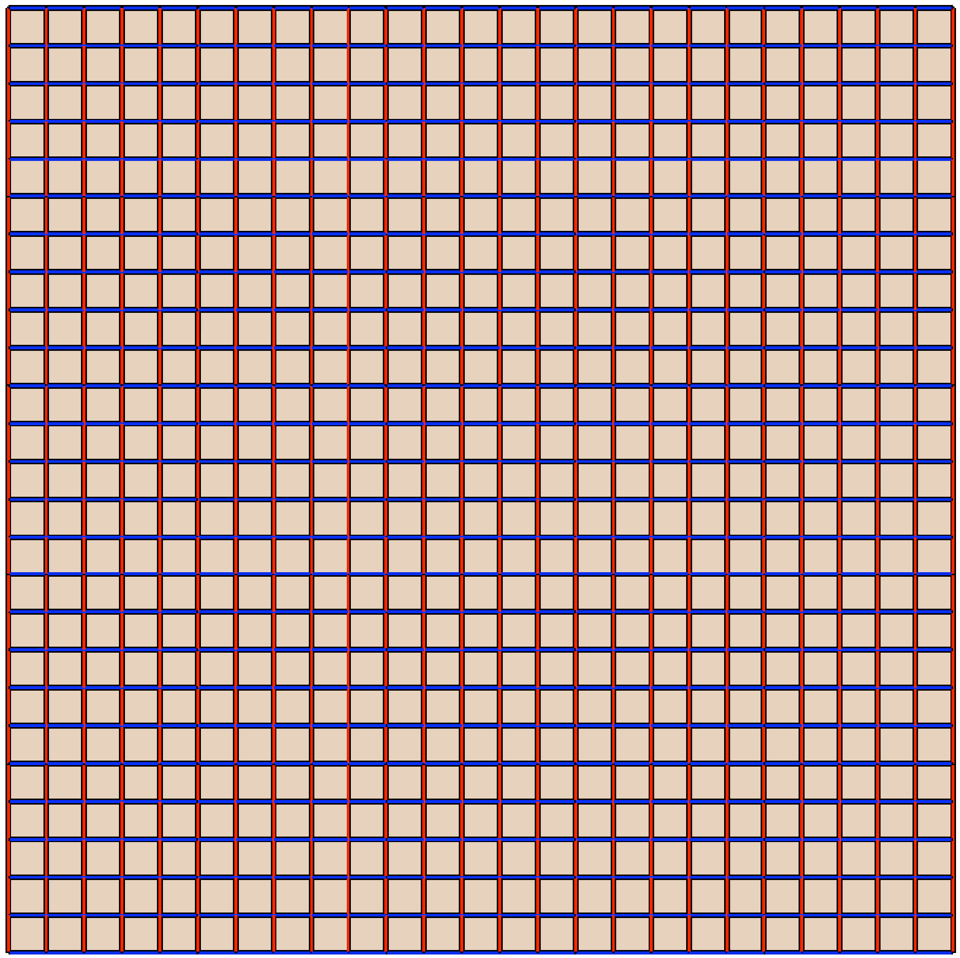
Grid Diagrams to Front Diagrams



Grids on a plane; Grids on a torus

Grids live naturally on the plane $\mathbb{R}^2$

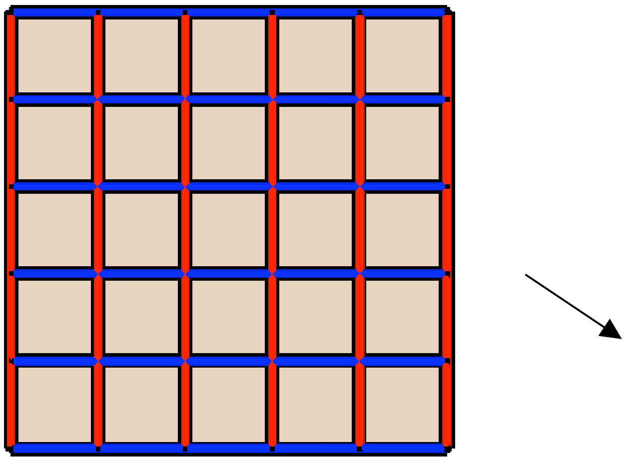
—separates $\mathbb{R}^3$ into half-spaces.

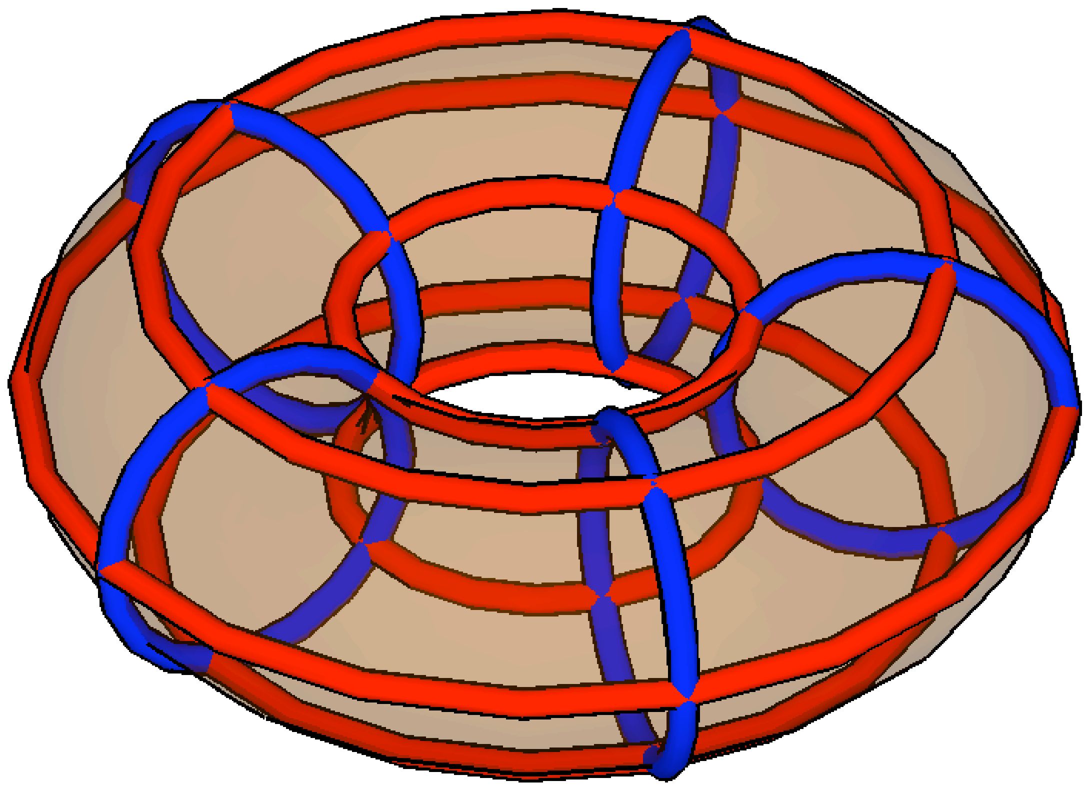


GRIDS ON A PLANE; GRIDS ON A TORUS

Grids live naturally on the torus $\mathbb{R}^2/\mathbb{Z}^2$

— separates $\mathbb{S}^3$ into solid tori.



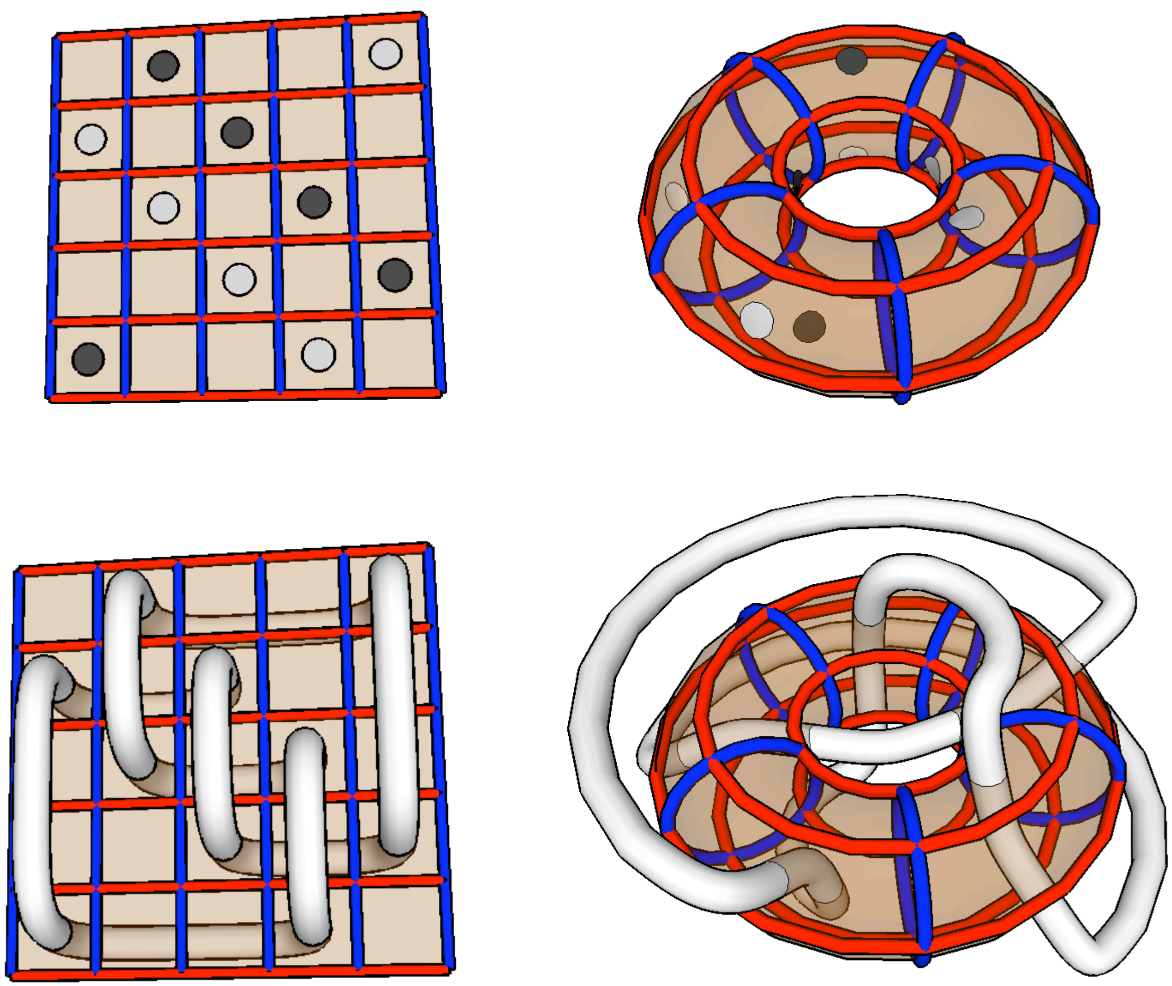


Grids live on a Heegaard torus!

Grid circles are meridians of Heegaard solid tori.

Toroidal Grid Diagrams

Grid diagrams are naturally toroidal...



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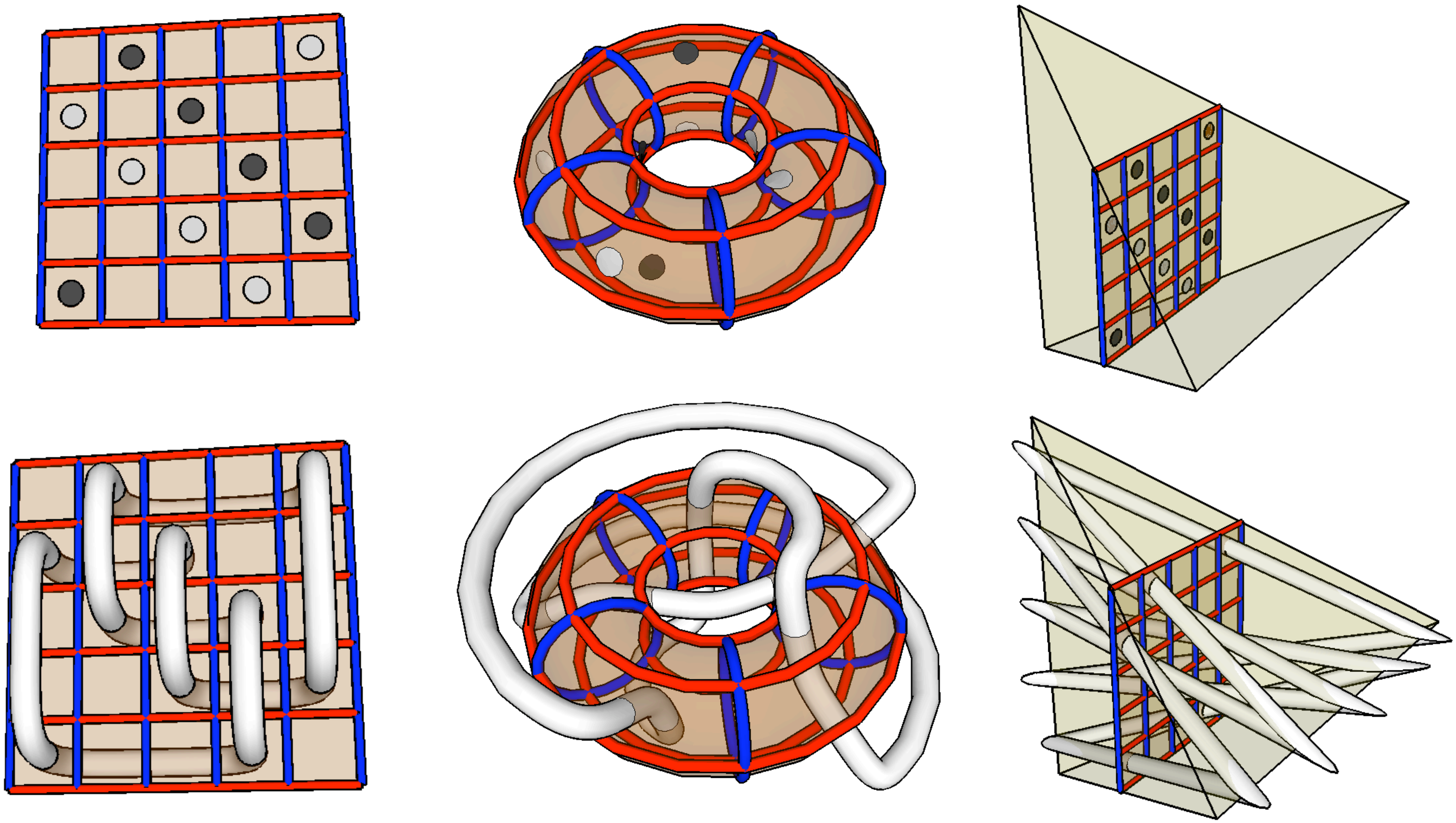
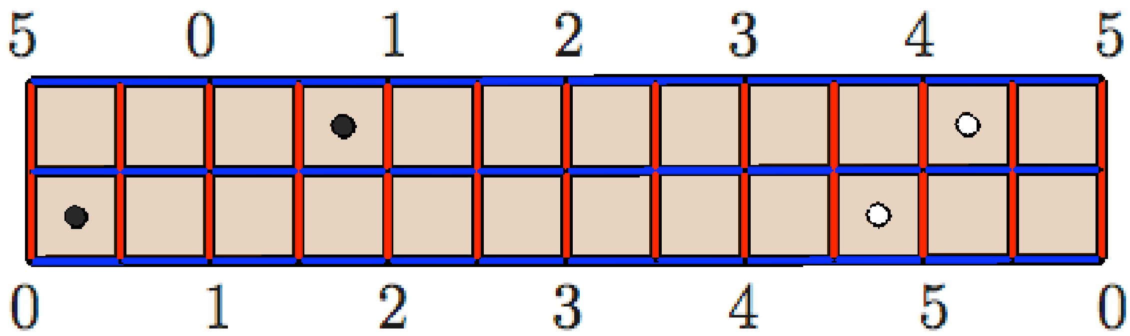
Grid diagrams are naturally toroidal...

... and are nicely viewed in the prism model of $\mathbb{S}^3$.

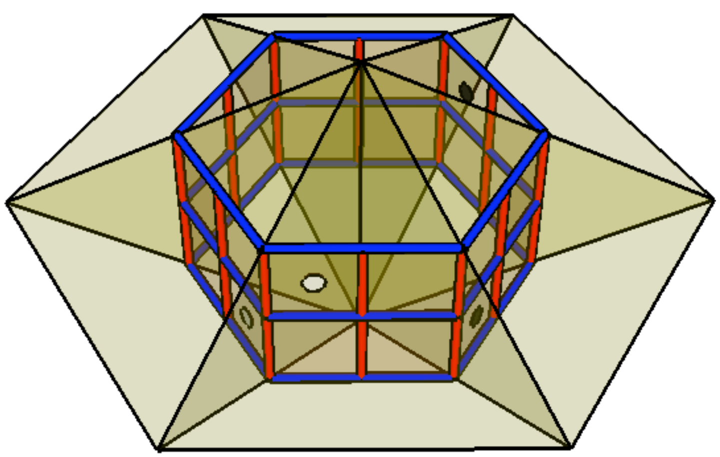


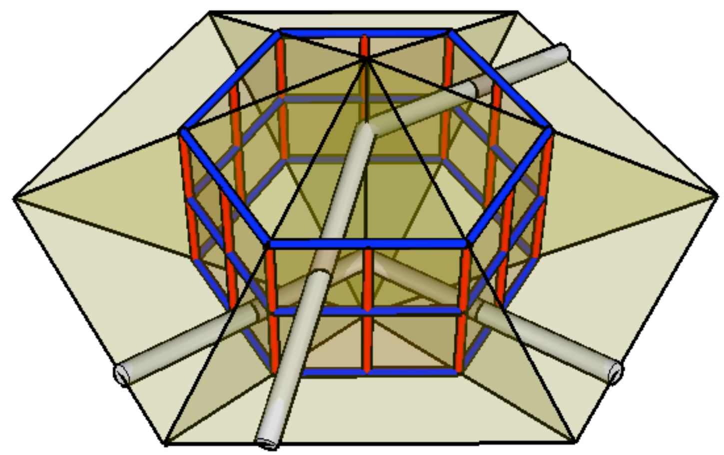
TOROIDAL GRID DIAGRAMS

Links in lens spaces have grid diagrams too.



Here's a GN2 diagram of a knot in $L(6,1)$.





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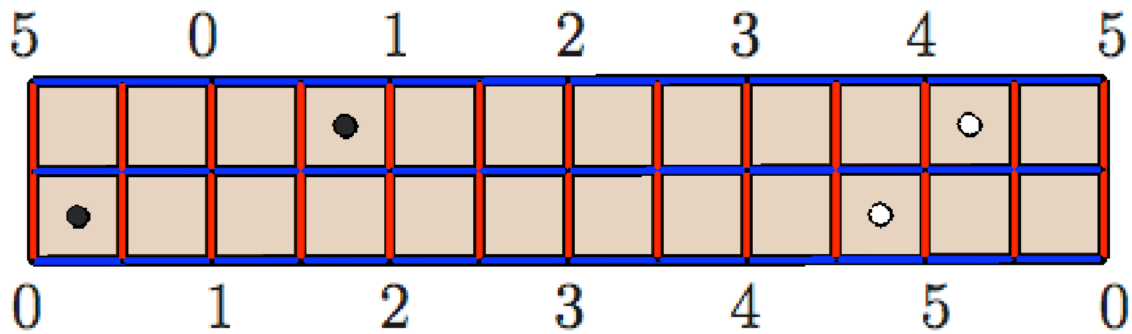

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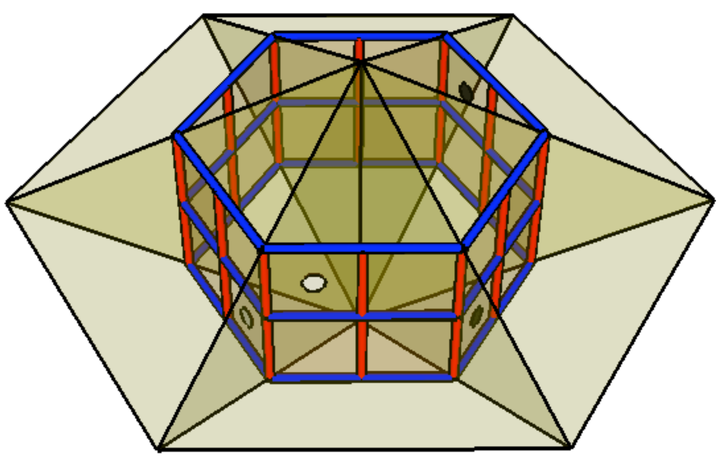
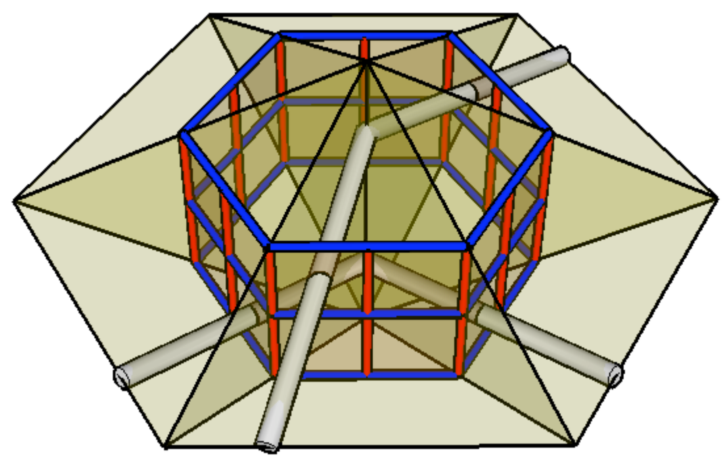




But what of contact structures?

Standard Contact Structure on $\mathbb{S}^3$

View $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$:

$$\mathbb{S}^3 = \{(u_1, u_2) \in \mathbb{C}^2 = \mathbb{C}_1 \times \mathbb{C}_2 : \ |u_1|^2 + |u_2|^2 = 1\}$$

and each $\mathbb{C}_i$ with polar coordinates $u_i = (r_i, \theta_i)$.

The standard contact structure on $\mathbb{S}^3$ is given by

$$\xi_{\mathbb{S}^3} = \ker \alpha_{\mathbb{S}^3} \quad \text{where} \quad \alpha_{\mathbb{S}^3} = r_1^2\, d\theta_1 + r_2^2\, d\theta_2$$

Thinking of $\mathbb{S}^3$ as $\mathbb{R}^3 \cup \{\infty\}$, one may identify

the $r_1 = 1$ circle with the z–axis $\cup\{\infty\}$

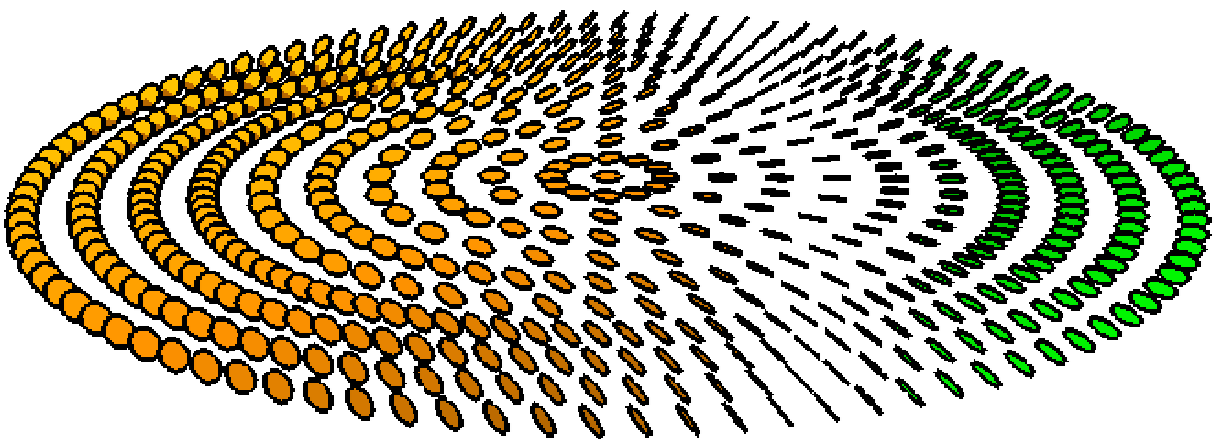
and the $r_2 = 1$ circle with the unit circle in the xy–plane.

Legendrian radii

Radial arcs are Legendrian: $\alpha_{\mathbb{S}^3}(\partial/\partial r_1) = 0$

Cores of solid tori are transverse: $\alpha_{\mathbb{S}^3}(\partial/\partial \theta_1) = 1$ for $r_1 = 1$

$\alpha_{\mathbb{S}^3}(\partial/\partial \theta_2) = 1$ for $r_2 = 1$



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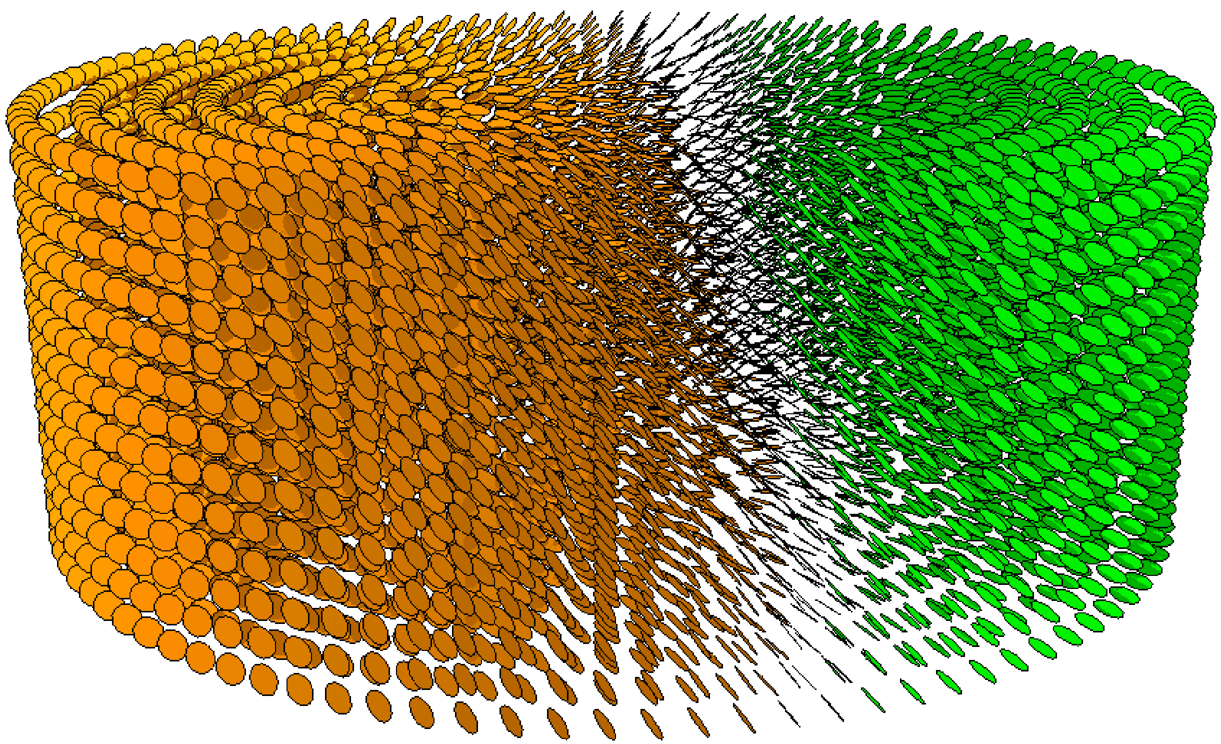

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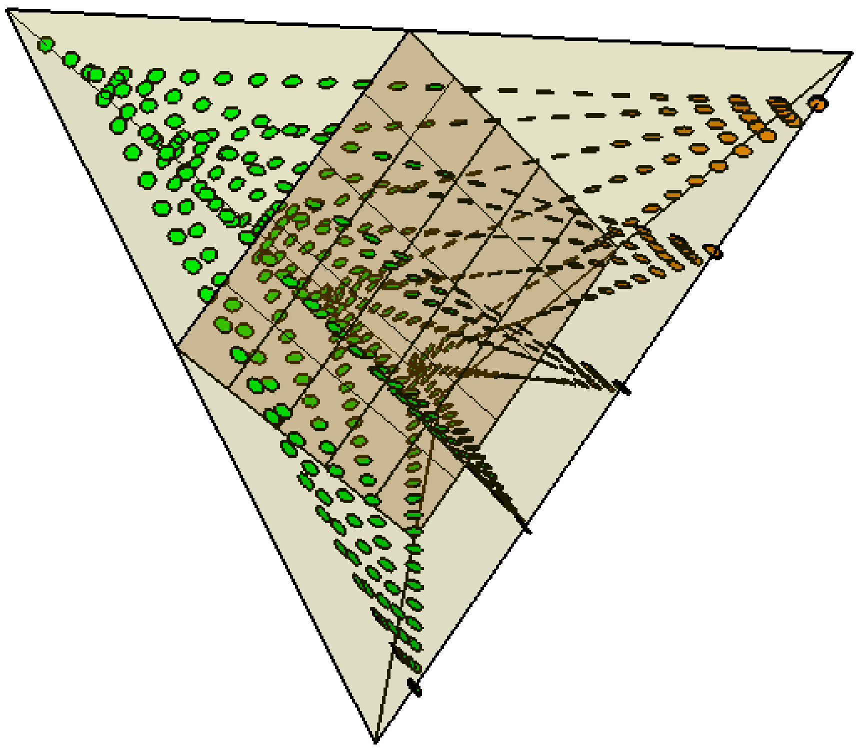
Unfold $(\mathbb{S}^3, \xi_{\mathbb{S}^3})$ into prism model.



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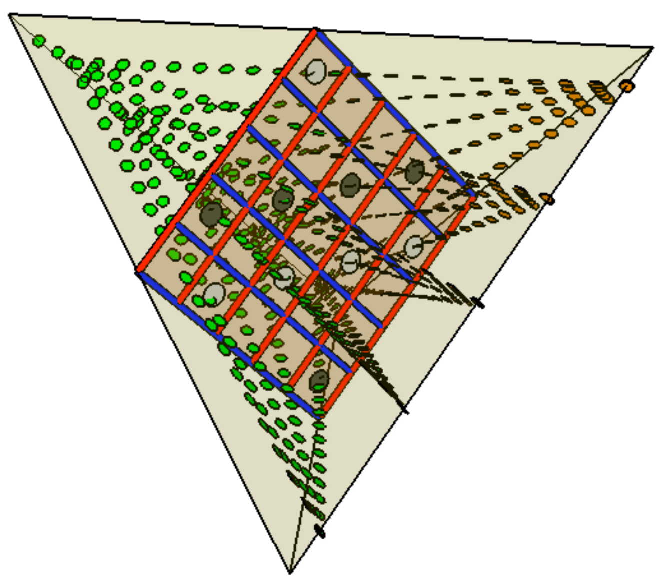
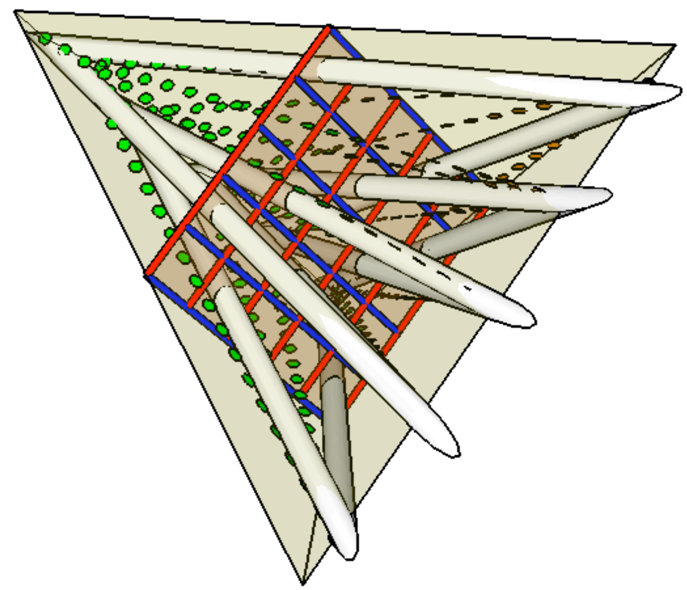




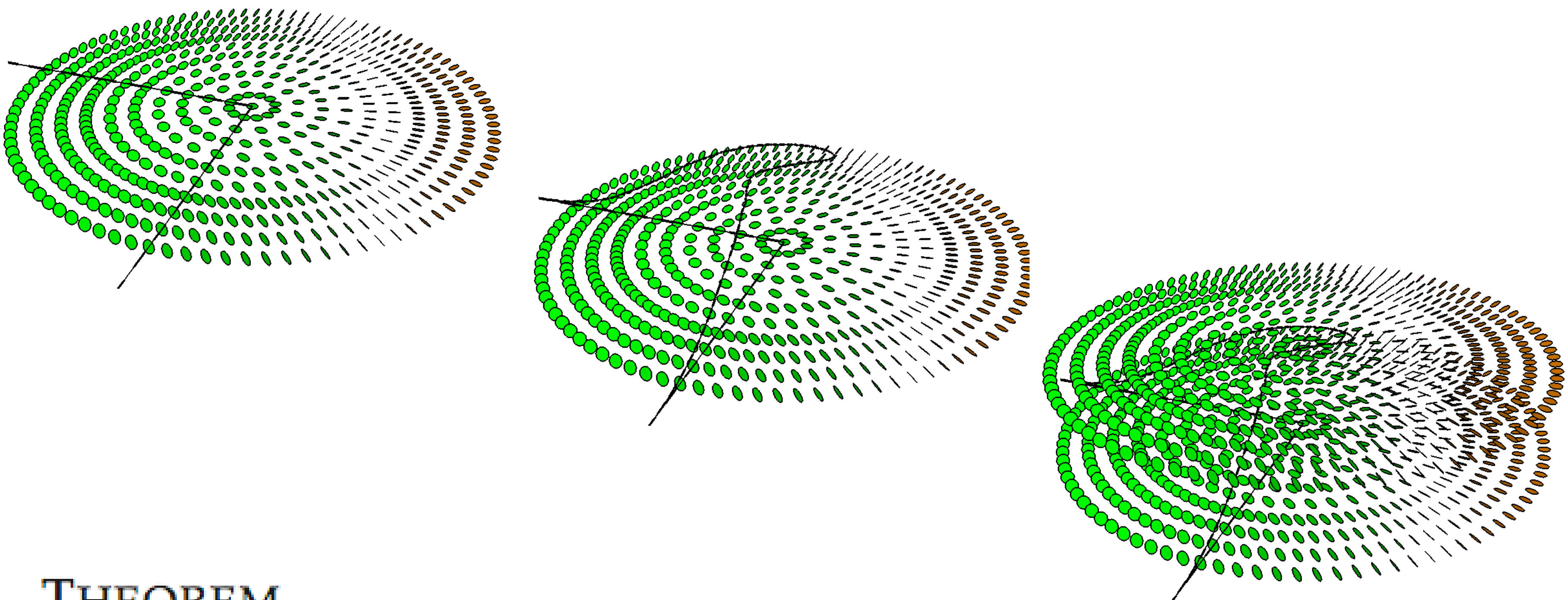
A grid diagram determines a union of Legendrian radii.

But only piecewise smooth!

Smoothing two Legendrian radii

There is a natural way of smoothing the kink between two radii:

Twist (and lift) them around the center to meet along a diameter.



Theorem

A toroidal grid diagram uniquely determines a Legendrian link.

Toroidal Front Diagrams

If $\mathcal{L}$ is a Legendrian curve in $(\mathbb{S}^3, \xi_{\mathbb{S}^3})$,

then $\alpha_{\mathbb{S}^3}$ evaluates to 0 on $T_{\mathrm{pt}}\mathcal{L}$.

Since $r_1^2 + r_2^2 = 1$, $\quad \alpha_{\mathbb{S}^3} = r_1^2\, d\theta_1 + (1 - r_1^2)\, d\theta_2$.

Hence at points of $\mathcal{L}$, setting $m = d\theta_1 / d\theta_2$,

$\frac{r_1^2}{(r_1^2 - 1)} = m$ so that $r_1 = \sqrt{\frac{m}{m-1}}$.

The r_1–coordinate of a point on $\mathcal{L}$ is

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Thus $\mathcal{L}$ may be recovered from its projection $\pi_{\theta_1\theta_2}(\mathcal{L})$

to the $\theta_1\theta_2$–torus.

Such an immersed curve is a toroidal front for the Legendrian link $\mathcal{L}$.

UNIVERSALLY TIGHT CONTACT STRUCTURES ON LENS SPACES

For each coprime $p > 0$ and q,

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Then $\xi_{p,q} = (\pi_{p,q})_*(\xi_{\mathbb{S}^3})$ where $\pi_{p,q}\colon \mathbb{S}^3 \to L(p,q)$ is the covering map.

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THEOREM

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Thurston-Bennequin and Grid Number

Using the connection between fronts and grid diagrams,

Matsuda shows

$$-\overline{\mathrm{tb}}(K) - \overline{\mathrm{tb}}(m(K)) \leq \mathrm{GN}(K)$$

for all links K in $\mathbb{R}^3$.

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PROPOSITION The bound

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Still need to study $\overline{\mathrm{tb}}$...

...could use combinatorial knot Floer homology....

http://sketchesoftopology.wordpress.com