Practice Test 2 – Solutions

5) How many solutions are there to the IVP $y'' + \cos(x)y' + \frac{1}{1+x^2}y = 0$ where $y(0) = 2$ and $y'(0) = -1$? What is the domain of each solution?

$\cos(x)$ and $\frac{1}{1+x^2}$ are continuous on $\mathbb{R}$

Since 0, the initial value, is in $\mathbb{R}$
Our existence + uniqueness theorem implies
there is a unique solution to the IVP
and the domain is $\mathbb{R}$, where those functions
and their functions are continuous.

7) Give a general solution to $2y''' + 3y'' + 2y' = 0$.

Char polyn: $2r^3 + 3r^2 + 2r = r(2r^2+3r+2)$

roots: $r = 0$, $\frac{-3 \pm \sqrt{9-4\cdot 2\cdot 2}}{4} = \frac{-3 \pm i\sqrt{5}}{4}$

So genl soln is

$$y = C_1 \underbrace{e^{0t}}_{=1} + C_2 e^{-3/4t} \cos\left(\frac{\sqrt{5}}{4}t\right) + C_3 e^{-3/4t} \sin\left(\frac{\sqrt{5}}{4}t\right)$$

$$\boxed{y = C_1 + e^{-3/4t}\left(C_2 \cos\left(\frac{\sqrt{5}}{4}t\right) + C_3 \sin\left(\frac{\sqrt{5}}{4}t\right)\right)}$$

6) Show that $y_1 = x$ and $y_2 = x \ln x$ are linearly independent solutions to the differential equation $x^2y'' - xy' + y = 0$. Give a general solution. Then find the solution that satisfies the initial conditions $y(1) = 7$ and $y'(1) = 2$.

• Check that they are solutions.

$y_1 = x, \quad y_1' = 1, \quad y_1'' = 0$

So $x^2 \cdot 0 - x \cdot 1 + x = 0$ ✓

* Domain $x \neq 0$

$y_2 = x\ln x \quad y_2' = \ln x + 1 \quad y_2'' = \frac{1}{x}$

$x^2 \cdot \frac{1}{x} - x \cdot (\ln x + 1) + x \ln x = 0$ ✓

• Check that they are lin indep. Wronskian

$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{pmatrix} = x(\ln x + 1) - x\ln x$

$= x \neq 0$ ✓ on domain.

• Genl soln

$$y = C_1 x + C_2 x \ln x.$$

• For initial conditions

$$y' = C_1 + C_2(\ln x + 1)$$

then $7 = y(1) = C_1 \cdot 1 + C_2 \cdot 1 \cdot 0 = C_1$

$2 = y'(1) = C_1 + C_2(0 + 1) = C_1 + C_2$

So $C_1 = 7, \ C_2 = -5$

& soln is $\boxed{y = 7x - 5x\ln x}$

9) Give a differential equation that has $y = 3xe^{-}x+2x^2\cos(x/3)$ as a solution.

That's $y = 3xe^{-x} + 2x^2\cos(\frac{1}{3}x)$

Can see roots -1 w/ multiplicity at least 2

$\rightsquigarrow$ $\frac{1}{3}i$ w/ multiplicity at least 3.

So one $L = (D - -1)^2 \left(D^2 + (\tfrac{1}{3})^2\right)^3$

$L = (D+1)^2 (D^2 + 1/9)^3$

Eqn is $Ly = 0$.

(No need to multiply this out.)

(8) Give general solutions to the following differential equations:

a) • $y'' - 4y' + 4y = \sin(2x)$

b) • $y^{(4)} - 4y'' + 4y = 6e^{2t}$

a)

$$y'' - 4y' + 4y = \sin 2x$$

<u>homogeneous</u>: $L = D^2 - 4D + 4 = (D-2)^2$

$$r^2 - 4r + 4 = (r-2)^2$$

$$h = C_1 e^{2x} + C_2 x e^{2x}$$

<u>Particular</u> $A = D^2 + 4$; $AL = (D^2+4)(D-2)^2$

So $p = k_1 \cos(2x) + k_2 \sin(2x) + \cancel{k_3 e^{2x} + k_4 x e^{2x}}$ in h.

Then determine k_1 & k_2:

$$p' = -2k_1 \sin(2x) + 2k_2 \cos(2x)$$

$$p'' = -4k_1 \cos(2x) + 4k_2 \sin(2x)$$

$$\sin 2x = p'' - 4p' + 4p$$

$$= \underbrace{(\cancel{-4k_1} - 8k_2 + \cancel{4k_1})}_{=0} \cos 2x + \underbrace{(4k_2 + 8k_1 + 4k_2)}_{=1} \sin 2x$$

So $k_2 = 0$ $\quad \Rightarrow k_1 = \frac{1}{8}$

Hence $p = \frac{1}{8} \cos 2x$

$\Rightarrow$ gen'l soln is $y = \frac{1}{8} \cos 2x + C_1 e^{2x} + C_2 x e^{2x}$.

8b) $$y^{(4)} - 4y'' + 4y = 6e^{2t}$$

$$L = D^4 - 4D^2 + 4 = (D^2-2)^2$$

<u>homogeneous</u> char poly $(r^2-2)^2$ roots $\pm\sqrt{2}$, both double roots.

$$h = C_1 e^{\sqrt{2}t} + C_2 t e^{\sqrt{2}t} + C_3 e^{-\sqrt{2}t} + C_4 t e^{-\sqrt{2}t}$$

<u>particular</u> $A = D-2$ ← no roots in common w/ L.

So $p = k_1 e^{2t} \cancel{+h}$.

determine k_1: $p' = 2k_1 e^{2t}$ $p'' = 4k_1 e^{2t}$

$p''' = 8k_1 e^{2t}$ $p^{(4)} = 16 k_1 e^{2t}$

$$6e^{2t} = \cancel{16k_1 e^{2t}} \cancel{-4\cdot 4k_1 e^{2t}} + 4k_1 e^{2t}$$

So $k_1 = \frac{3}{2}$.

General solution is

$$y = \tfrac{3}{2}e^{2t} + C_1 e^{\sqrt{2}t} + C_2 t e^{\sqrt{2}t} + C_3 e^{-\sqrt{2}t} + C_4 t e^{-\sqrt{2}t}$$

(1) For the autonomous differential equation $\frac{dx}{dt} = x^2 - 4x + 3$:

(a) Find all critical points and draw a phase diagram. For each critical point, determine if it is stable, unstable, or semi-stable.

(b) If $x(t)$ is a solution to the IVP $x_0 = x(0)$, determine $\lim_{t\to\infty} x(t)$ in terms of x_0.

(c) Sketch several solution curves on an appropriate domain.

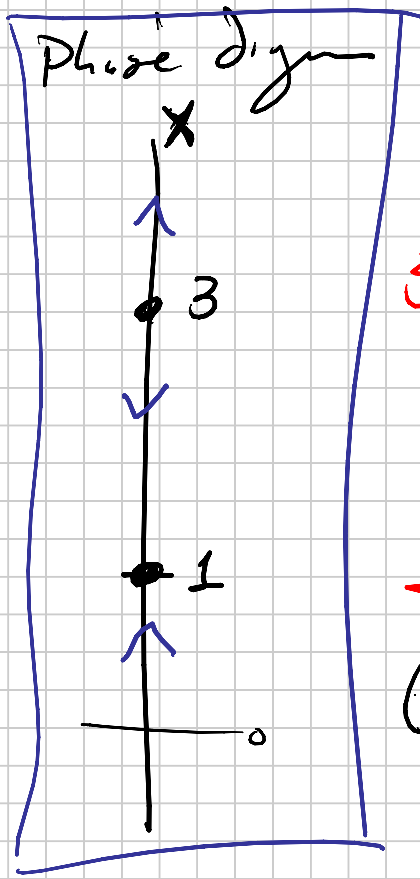
(d) Give a general solution to the differential equation.

a) crit pts where $f(x) = x^2 - 4x + 3 = 0$

$$(x-1)(x-3) = 0$$

So crit pts $x=1$, $x=3$

$(x-1)$	$(x-3)$	$\frac{dx}{dt} = f(x)$	x
+	+	+	incr
+	0	0	const
+	−	−	decr
0	−	0	const
−	−	+	incr



3 is unstable

1 is stable.

(Book draws this horizontal)

b) If

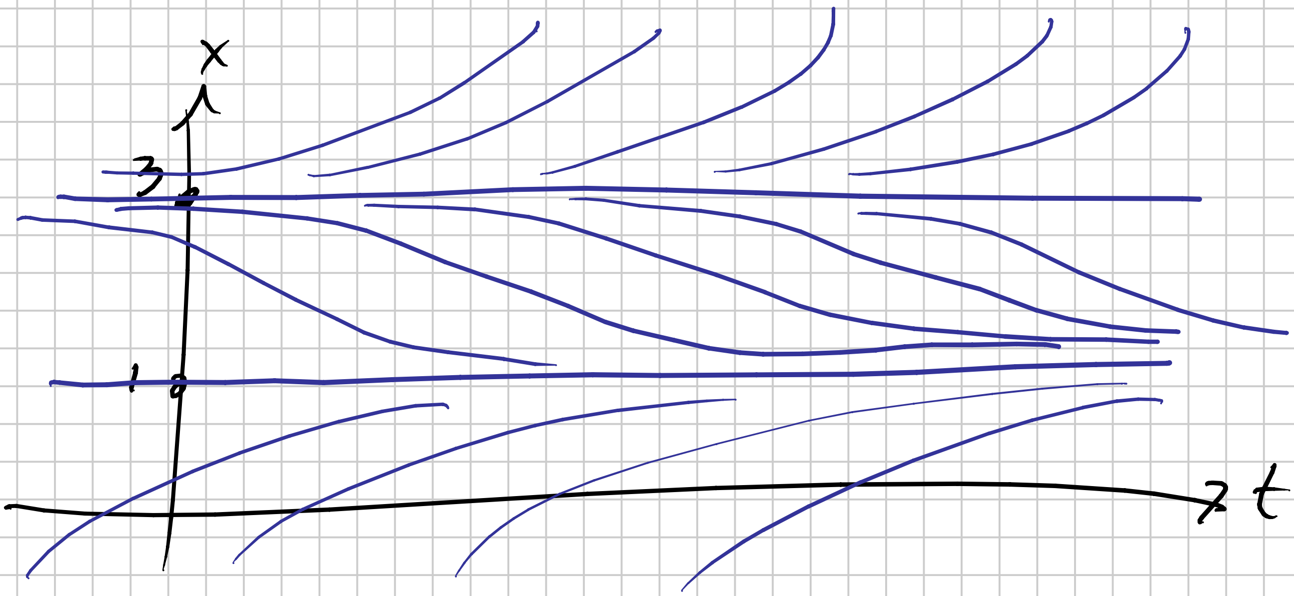
$x_0 > 3$, $\lim_{t\to} x(t) = +\infty$ DNE

$x_0 = 3$, $\lim_{t\to\infty} x(t) = 3$

$x_0 < 3$ $\lim_{t\to\infty} x(t) = 1$

we don't need to make separate cases for $1 < x_0 < 3$, $x_0 = 1$, and $x_0 < 1$ since $\lim_{t\to\infty} x(t) = 1$ for all of them.

e)



d) Solve $\dfrac{dx}{dt} = x^2 - 4x + 3$

Separable.

$$\int \frac{1}{x^2-4x+3}\,dx = \int 1\,dt$$

$$\int \frac{1}{(x-1)(x-3)}\,dx = t + C$$

$$\int \frac{-1/2}{x-1} + \frac{1/2}{x-3}\,dx = t + C.$$

$$\frac{1}{2}\left(-\ln(x-1) + \ln(x-3)\right) = t + C$$

$$\ln \frac{x-3}{x-1} = 2t + 2C \rightsquigarrow \boxed{\frac{x-3}{x-1} = C' e^{2t}}$$

Could solve for x ....

Yes, it's ok to leave this answer like this

Scratch work.

$$\frac{1}{x^2-4x+3} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x-1)}{(x-1)(x-3)}$$

$$= \frac{(A+B)x - (3A+B)}{(x-1)(x-3)}$$

So $A+B=0 \rightsquigarrow B=-A$

$3A+B=-1 \rightsquigarrow 2A=-1$

$A = \frac{-1}{2}$

$B = 1/2$

2

(2) For the autonomous differential equation $\frac{dx}{dt} = x^2(e^{2x-3} - 1)$:

(a) Find all critical point and draw a phase diagram. For each critical point, determine if it is stable, unstable, or semi-stable.

(b) If $x(t)$ is a solution to the IVP $x_0 = x(0)$, determine $\lim_{t\to\infty} x(t)$ in terms of x_0.

(c) Sketch several solution curves on an appropriate domain.

$$\frac{dx}{dt} = f(x) = x^2(e^{2x-3} - 1)$$

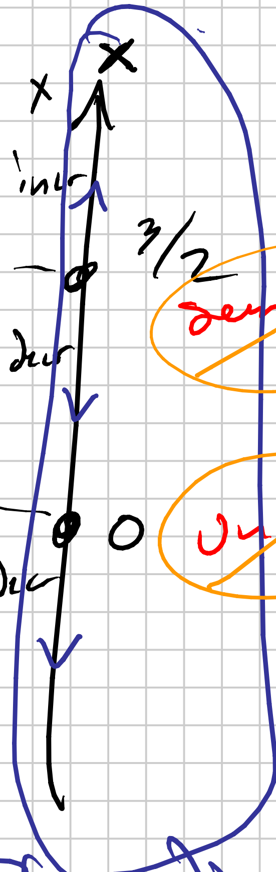
this is 0 if $x = 0$ or $\underbrace{e^{2x-3} - 1 = 0}$

$\hookrightarrow e^{2x-3} = 1$

$2x - 3 = 0$

$x = \frac{3}{2}$

x^2	$e^{2x-3}-1$	f	x
+	+	+	inc
+	0	0	$3/2$
+	−	−	dec
0	−	0	0
+	−	−	dec



~~semistable~~ ← unstable

~~unstable~~ ← semistable

oops. I don't know how I got that swapped

2a

Phase diagram.

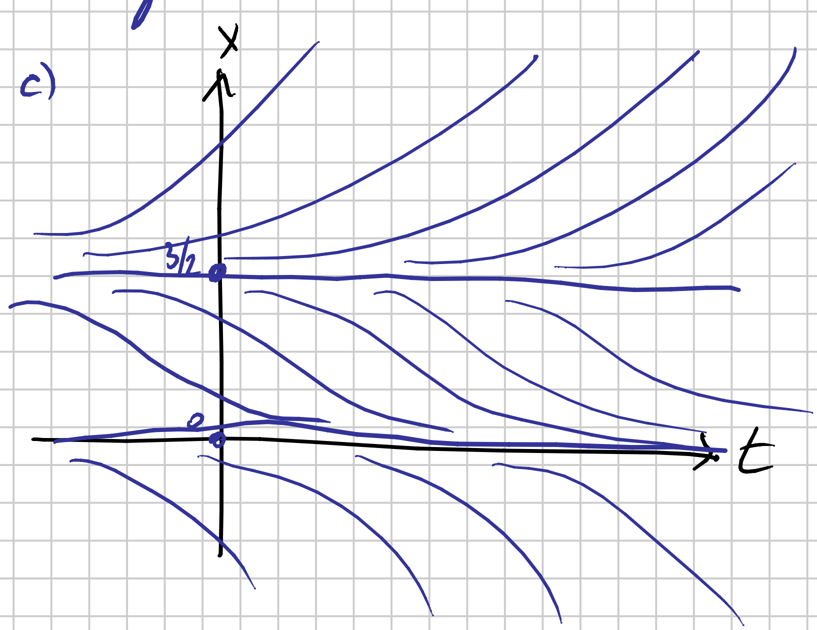
b) $3/2 < x_0$ $\quad \lim_{t\to\infty} (t) = +\infty$

$3/2 = x_0$ $\quad \lim_{t\to\infty} (t) = 3/2$

$0 \le x_0 < 3/2$ $\quad \lim_{t\to\infty} (t) = 0$

$x_0 < 0$ $\quad \lim_{t\to\infty} (t) = -\infty$

c)



(3) Make a differential equation that mathematically models the spread of a rumor in the situation described below. Determine the relevant domains for your variables. Qualitatively describe how the rumor may spread depending on initial conditions.

In a large university with a fixed population of people, the rate of change of the number of those people who have heard a certain rumor is proportional to the number that have not yet heard the rumor.

This is constant, It doesn't depend on P ← k.

P = # of people who heard rumor.

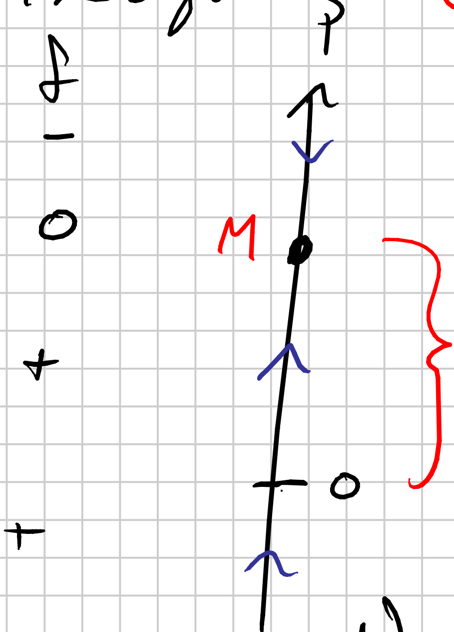
M = total # of people

M − P = # who haven't heard it.

$$\frac{dP}{dt} = \underbrace{\overset{\text{pos. const.}}{k}\,(\overset{\text{pos. const.}}{M}-P)}_{f(P)}$$

assume $k>0$,
$M>0$
and $0 \le P \le M$.

Phase Diagram



Just one crit pt $P=M$.
$f(P)=0$ if $P=M$
$f(P)<0$ if $P>M$
$f(P)>0$ if $P<M$

relevant domain.

Using this model,
Eventually everyone hears the rumor
no matter how many people
initially know it.

(11) Solve the differential equation $Ly = 7e^{-3t}$ where L is the linear differential operator $L = (D^2 - 9)D$.

1.

$$L = (D-3)(D+3)D$$

homogeneous $\quad y_h = C_1 e^{3t} + C_2 e^{-3t} + C_3 e^{0t}$

particular $\quad$ Since $E = 7e^{-3t}$, $\quad A = (D - (-3)) = D+3$

So y_p is a soln to $ALy = 0$

$$(D-3)(D+3)^2 D y = 0$$

$$y_p = \cancel{k_1 e^{3t}} + \cancel{k_2 e^{-3t}} + k_3 t e^{-3t} + \cancel{k_4}$$

in homogeneous soln.

Take $y_p = k t e^{-3t}$, solve for k.

$L = D^3 - 9D$ $\quad$ so need y_p' & $y_p^{(3)}$

$$y_p' = k\left(e^{-3t} + -3te^{-3t}\right) = k\,(1-3t)\,e^{-3t}$$

$$y_p'' = k\left(-3e^{-3t} + (1-3t)(-3)e^{-3t}\right) = k\,(-6+9t)\,e^{-3t}$$

$$y_p^{(3)} = k\left(9e^{-3t} + (-6+9t)(-3)e^{-3t}\right) = k\,(27-27t)\,e^{-3t}$$

$$7e^{-3t} = k(27-27t)e^{-3t} - 9k(1-3t)e^{-3t}$$

$$\leadsto \quad 7 = 18k \qquad k = 7/18.$$

$$\boxed{y = \frac{7}{18} t e^{-3t} + C_1 e^{3t} + C_2 e^{-3t} + C_3}$$

(4) An object moving horizontally experiences resistance due to friction that is:

(a) proportional to the square root of its speed (absolute value of velocity) and

(b) in the direction opposite its motion.

If there are no other forces contributing to its horizontal motion, obtain an equation for its velocity $v(t)$ at time t with initial velocity $v(0) = v_0 > 0$.

Also obtain an equation for its position $x(t)$ with initial position $x(0) = x_0$.

$F_{friction} = -k\sqrt{|v|}$, const $k > 0$.

proport'l to   sq root of speed   in opposite direction

Newton's Law:

$m v' = F_{fric}$ const $m > 0$.

$m v' = -k\sqrt{|v|}$

Assume $v > 0$ since $v_0 > 0$.

then $m v' = -k\sqrt{v}$

So $v^{-\frac{1}{2}} \frac{dv}{dt} = -\frac{k}{m}$ ⇝ $\int v^{-1/2}\,dv = \int -\frac{k}{m}\,dt$

⇝ $2v^{1/2} = -\frac{k}{m}t + C$.   at $t=0$, $2v_0^{1/2} = C$.

$v^{1/2} = -\frac{k}{2m}t + v_0^{1/2}$

Hence $\boxed{v(t) = \left(-\frac{k}{2m}t + v_0^{1/2}\right)^2}$ (whenever $v(t) > 0$.)

Since $x' = v$, $x(t) = \int \left(-\frac{k}{2m}t + v_0^{1/2}\right)^2 dt$

$= \frac{1}{3}\cdot\frac{-2m}{k}\left(-\frac{k}{2m}t + v_0^{1/2}\right)^3 + C'$

at $t=0$, $x_0 = -\frac{2m}{3k}v_0^{3/2} + C'$ so $C' = x_0 + \frac{2}{3}\frac{m}{k}v_0^{3/2}$

Hence $\boxed{x(t) = -\frac{2}{3}\frac{m}{k}\left(-\frac{k}{2m}t + v_0^{1/2}\right)^3 + \frac{2}{3}\frac{m}{k}v_0^{3/2} + x_0}$

Could simplify a little bit, but not wholly worth it...

$x(t) = \frac{2}{3}\frac{m}{k}\left(\left(\frac{k}{2m}t + v_0^{1/2}\right)^3 - v_0^{3/2}\right) + x_0$