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Actions of $\mathbb{C}^*$ and $\mathbb{C}_+$ on affine algebraic varieties

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1. Introduction

The purpose of this paper is to present recent developments in the study of algebraic group actions on affine algebraic varieties. This study leads inevitably to other subjects in affine algebraic geometry and we touch them as well but in less detail (for a wider overview of affine algebraic geometry we refer to [**Zai99**], [**Kr96**], [**Mi04**]). Among several beautiful results of the last decade the Koras-Russell proof [**KoRu99**] of the Linearization Conjecture in dimension three has a special place. The Conjecture claims that every algebraic $\mathbb{C}^*$-action on $\mathbb{C}^n$ is conjugate to a linear one in the group of polynomial automorphisms of $\mathbb{C}^n$. It was first stated by Kambayashi [**Kam**] but, perhaps, the starting point was the earlier paper of Gutwirth [**Gu62**] who proved it for $n = 2$. Using his previous result with Kraft on linearization of semi-simple group actions Popov showed that the Koras-Russell theorem implies that every algebraic action of a connected reductive group on $\mathbb{C}^3$ is a representation in a suitable polynomial coordinate system [**Po01**]. In general reductive groups admit non-linearizable actions and for every connected reductive group different from a torus such an action exists on $\mathbb{C}^n$ with sufficiently large n, but the question about $\mathbb{C}^*$-actions in higher dimensions remains open. In the case of the similar questions for real Euclidean spaces or holomorphic actions there are counterexamples which will be presented below. While discussing the steps of the solution of the Linearization Conjecture we encounter the Koras-Russell threefolds that are smooth complex affine algebraic varieties diffeomorphic to $\mathbb{R}^6$ and equipped with obviously non-linearizable $\mathbb{C}^*$-actions. The trouble was that all old invariants capable of distinguishing $\mathbb{C}^2$ from smooth contractible surfaces, were the same for these threefolds and for $\mathbb{C}^3$. In particular, these threefolds were viewed as potential counterexamples to linearization until the introduction of the Makar-Limanov invariant [**ML**]. For an affine algebraic variety X this invariant $\mathrm{AK}(X)$ is the subring of the ring of regular functions on X that consists of functions invariant under any $\mathbb{C}_+$-action on X. If X is a Koras-Russell threefold then $\mathrm{AK}(X)$

includes non-constant functions which shows that X is not isomorphic to $\mathbb{C}^3$ since $\mathrm{AK}(\mathbb{C}^n) = \mathbb{C}$. We present the scheme of computation of $\mathrm{AK}(X)$ in the case of the most vivid of the Koras-Russell threefolds which is the Russell cubic - the hypersurface in $\mathbb{C}^4$ given by the equation $x + x^2y + z^2 + t^3 = 0$. In particular, we see that the study of "good guys" (reductive group actions) requires the study of "bad guys" ($\mathbb{C}_+$-actions) that are hidden in the group of automorphisms of X. This leads to the latest developments in the fourteenth Hilbert problem, and some other results on algebraic quotients $X//\mathbb{C}_+$, i.e. we explain why for any nontrivial $\mathbb{C}_+$-action on a smooth affine contractible threefold X the quotient is always a smooth contractible surface. This fact contributes to another theorem in this survey: if in addition the $\mathbb{C}_+$-action on such an X is free then X is isomorphic to $S \times \mathbb{C}$ and the action is a translation on the second factor [**KaSa**]. In particular, every free $\mathbb{C}_+$-action on $\mathbb{C}^3$ is a translation in a suitable polynomial coordinate system [**Ka04**]. At the end we return to $\mathbb{C}^*$-actions on affine algebraic varieties with nontrivial topology and present the coming classification of $\mathbb{C}^*$-actions on smooth affine algebraic surfaces.

2. Preliminaries

Throughout this paper X will be a normal complex affine algebraic variety, $A = \mathbb{C}[X]$ will be the algebra of regular functions on X, and G will be an algebraic group. Recall that a G-action on X is a homomorphism of G into the group of bijections of X which generates, therefore, a map $\Phi : G \times X \to X$. We say that the action is algebraic (resp. holomorphic) if Φ is a morphism (resp. a holomorphic map), i.e. Φ is generated by a homomorphism from G into the group $\operatorname{Aut} X$ of regular (resp. holomorphic) automorphisms of X. Unless we state otherwise every action of an algebraic group G that we discuss below will be algebraic and non-degenerate (i.e. there is a orbit of the same dimension as G). Consider two examples of algebraic $\mathbb{C}^*$-actions and $\mathbb{C}_+$-actions on $\mathbb{C}^n$ crucial for this paper.

Example 2.1. (1) A linear action $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ is given by

$$(\lambda, \bar{x}) \to (\lambda^{k_1} x_1, \dots, \lambda^{k_n} x_n)$$

where $\bar{x} = (x_1, \dots, x_n)$ is a coordinate system on $\mathbb{C}^n$, $\lambda \in \mathbb{C}^*$, and $k_i \in \mathbb{Z}$.

(2) A triangular action $\mathbb{C}_+ \times \mathbb{C}^n \to \mathbb{C}^n$ is given by

$$(t, \bar{x}) \to (x_1, x_2 + tp_2(x_1), \dots, x_n + tp_n(x_1, \dots, x_{n-1}))$$

where $t \in \mathbb{C}_+$ and each p_i is a polynomial in variables $x_1, \dots, x_{i-1}$.

Remark 2.2. Elements of a triangular action are contained in the Jonquière group of automorphisms of $\mathbb{C}^n$ that is

$$\mathcal{J}_n = \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathrm{Aut}\mathbb{C}^n \,|\, \forall i\, \varphi_i \in \mathbb{C}[x_1, \dots, x_i]\}$$

while elements of a linear action are contained in the intersection of $\mathcal{J}_n$ and the subgroup of affine transformation

$$\mathcal{A}_n = \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathrm{Aut}\mathbb{C}^n | \, \forall i \, \varphi_i \in \mathbb{C}[x_1, \dots, x_n], \ \deg \varphi_i = 1\}.$$

Definition 2.3. A point $x \in X$ is a fixed point of a G-action $\Phi : G \times X \to X$ on X if $\Phi(g, x) = x$ for every $g \in G$. The action is called free if it has no fixed points (precaution: this definition of a free action is valid for varieties over $\mathbb{C}$ or other algebraically closed fields of zero characteristics but it must be changed in the absence of algebraic closedness). In the case of $G = \mathbb{C}_+$ we say that the action on X is a translation if X is isomorphic to $Y \times \mathbb{C}$ and the action is generated by a translation on the second factor. Of course, each translation is free.

Example 2.4. The fixed point set for the triangular action in Example 2.1 is given by $p_2 = \ldots = p_n = 0$. That is, it is a cylinder $Y \times \mathbb{C}_{x_n}$ where $\mathbb{C}_z$ (resp. $\mathbb{C}^n_{x_1,\dots,x_n}$) means a line equipped with coordinate z (resp. $\mathbb{C}^n$ equipped with a coordinate system $(x_1, \dots, x_n)$). In particular, the action is free if the polynomials $p_2, \dots, p_n$ have no common roots. In the case of $n = 2$ this means that p_2 is constant and, therefore, any free triangular action on $\mathbb{C}^2$ is a translation.

Definition 2.5. Let $\Phi_i : G \times X \to X$, $i = 1, 2$ be effective algebraic G-actions on X (i.e. no proper subgroup of G acts trivially on X). We say that Φ_1 and Φ_2 are equivalent if there exists $\alpha \in \mathrm{Aut}\, X$ such that $\Phi_2 = \alpha \circ \Phi_1 \circ \alpha^{-1}$.

It is natural, for instance, to ask when a given algebraic $\mathbb{C}^*$-action on $\mathbb{C}^n$ is equivalent to a linear one (that is, it is linear in a suitable coordinate system). One can formulate this question in a more general setting.

Classification Problem. Given two effective algebraic G-actions on X establish whether they are equivalent. More generally, describe equivalence classes of effective algebraic G-actions on X.

The main obstacle for a solution of such a problem may lie in the structure of $\mathrm{Aut}\, X$. When this structure is known the answer may be simple. For $\mathbb{C}^2$, say, it works like this (e.g., see. [**Kr96**]). The group $\mathrm{Aut}\mathbb{C}^2$ is the amalgamated product $\mathcal{A}_2 *_{\mathcal{H}_2} \mathcal{J}_2$ where $\mathcal{H}_n = \mathcal{A}_n \cap \mathcal{J}_n$.[1] Then every algebraic subgroup of $\mathrm{Aut}\mathbb{C}^2$ is of bounded length in this amalgamated product [**Wr**], i.e. the number of factors in the amalgamated decomposition of each element of this subgroup is bounded by the same constant. But any subgroup of bounded length is isomorphic to a subgroup of one of the factors [**Se**]. Using this fact one can reprove the following results of Gutwirth [**Gu62**], [**Gu61**], and Rentschler [**Re**] obtained by other methods.

[1]That is, every $\alpha \in \mathrm{Aut}\,\mathbb{C}^2$ is a composition $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$ where each α_i is contained either in $\mathcal{J}_2 \setminus \mathcal{A}_2$ or in $\mathcal{A}_2 \setminus \mathcal{J}_2$ and $\alpha_i \in \mathcal{J}_2$ iff $\alpha_{i\pm1} \in \mathcal{A}_2$. Furthermore, this composition is unique up to consequent changes of the following type: α_i is replaced by $\alpha_i \circ \gamma$ and α_{i+1} is replaced by $\gamma^{-1} \circ \alpha_{i+1}$ where $\gamma \in \mathcal{H}_2$. In particular, the number n of factors is uniquely determined.

THEOREM 2.6. *Every $\mathbb{C}^*$-action on $\mathbb{C}^2$ is equivalent to a linear one, every $\mathbb{C}_+$-action on $\mathbb{C}^2$ is equivalent to a triangular one. In particular, every free $\mathbb{C}_+$-action on $\mathbb{C}^2$ is a translation.*

Unfortunately, the structure of the automorphism group $\operatorname{Aut}\mathbb{C}^n$ is unknown for $n \geq 3$ and, furthermore, answering the old question of Nagata, Shestakov and Umirbaev [**ShUm04a**], [**ShUm04b**] proved recently that the automorphism of $\mathbb{C}^3$ given by

$$(x, y, z) \to (x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z)$$

is not a composition of Jonquière and affine transformations. Hence one has to use other tools when facing the Classification Problem in higher dimensions and one of them is the notion of algebraic quotient.

3. Algebraic quotient and Hilbert's Fourteenth problem

From now on, given an algebraic G-action on X, we denote by A^G the subalgebra of G-invariant regular functions on X and by $X//G$ the spectrum $\operatorname{Spec}A^G$ which is called the algebraic quotient. Hilbert's fourteenth problem asks when A^G is finitely generated, or, equivalently, when $X//G$ is an affine algebraic variety. The classical result of Nagata says that $X//G$ is affine for a reductive G but for a non-reductive G the answer is negative in general. The complete description of algebraic quotients was obtained by Winkelmann [**Wi03**].

THEOREM 3.1. *For every action of an algebraic group G on X its quotient is quasi-affine. Moreover, for every normal quasi-affine algebraic variety Y there is an algebraic $\mathbb{C}_+$-action on some X such that the algebra of $\mathbb{C}_+$-invariant regular functions on X is isomorphic to the algebra of regular functions on Y.*

The original Nagata's example of an algebraic $\mathbb{C}_+$-action on $\mathbb{C}^n$ with a non-affine quotient was constructed for $n = 32$. Later efforts of Roberts, Daigle, and Freudenburg [**Ro**], [**Fr00**], [**DaiFr99**] reduced this dimension to 5. For $n = 3$ such a quotient is always affine by Zariski's theorem [**Za**] that gives a partial answer to the following version of the Hilbert's fourteenth Problem suggested by Nagata:

Is $F \cap A$ an affine domain for a subfield F of $\operatorname{Frac}(A)$*?*

Zariski's theorem gives a positive answer when the transcendence degree of F (over $\mathbb{C}$) is at most 2 and it has the following corollary (in order to prove it one has to put F equal to the subfield of G-invariant rational functions on X).

THEOREM 3.2. *Suppose that the algebraic quotient $X//G$ of an algebraic G-action on X is of dimension 2. Then $X//G$ is affine. In particular, for a three-dimensional X its algebraic quotients are always affine.*

Recently Kuroda [**Ku**] constructed a counterexample to the question of Nagata in the case when the transcendence degree of $F \cap A$ is 3 or higher. However, $F \cap A$

in his example cannot serve as the ring of G-invariant functions for an algebraic G-action on X. Thus we want to emphasize that the question whether $\mathbb{C}^4//\mathbb{C}_+$ is affine remains open for the fans of the fourteenth Hilbert Problem.

4. Linearization Problem

The problem we are going to discuss is when an algebraic G-action on $\mathbb{C}^n$ is linearizable, i.e. when it is equivalent to a representation. As we mentioned this is so for every algebraic $\mathbb{C}^*$-action on $\mathbb{C}^2$ by the Gutwirth theorem [**Gu62**].

Theorem 4.1. [**KoRu99**] *Every $\mathbb{C}^*$-action on $\mathbb{C}^3$ is equivalent to a linear one.*

Earlier Popov and Kraft proved that an algebraic action of a semi-simple group on $\mathbb{C}^3$ is equivalent to a representation [**KrPo**]. Combining these two results Popov got the following [**Po01**].

Theorem 4.2. *Every action of a connected reductive group on $\mathbb{C}^3$ is linearizable, i.e. it is equivalent to a representation.*

He showed also that for a connected reductive group different from $\mathbb{C}^*$ or $(\mathbb{C}^*)^2$ its algebraic action on $\mathbb{C}^4$ is always linearizable.

The fact that actions of $(\mathbb{C}^*)^4$ and $(\mathbb{C}^*)^3$ on $\mathbb{C}^4$ is linearizable follows from an old result of Bialynicki-Birula [**BB**]. In fact, we have more [**De**], [**BeHa**].

Theorem 4.3. *Let X be a toric variety of dimension n with a canonical action Φ of torus $T=(\mathbb{C}^*)^n$. Then any other effective action of T on X is equivalent to Φ and, furthermore, any effective action of $(\mathbb{C}^*)^{n-1}$ on X is equivalent to the action of an $(n-1)$-dimensional subtorus of T generated by Φ.*

We discuss briefly some elements of proof of the Koras-Russell theorem (for a more detailed exposition of ideas of this proof one can see [**KaKoMLRu**]).

By the end of 1980's the linearization of algebraic $\mathbb{C}^*$-actions on $\mathbb{C}^3$ was established in all cases except for the hyperbolic one. In that case a $\mathbb{C}^*$-action Φ on $\mathbb{C}^3$ has only one fixed point o and the induced linear $\mathbb{C}^*$-action Ψ on $T_o\mathbb{C}^3 \simeq \mathbb{C}^3_{x,y,z}$ is hyperbolic, i.e. it is given by

$$(x,y,z) \to (\lambda^{-a}x, \lambda^b y, \lambda^c z)$$

where integers $a,b,c>0$. In particular, one can see that $T_o\mathbb{C}^3//\Psi \simeq \mathbb{C}^2//\mathbb{Z}_d$ where the last quotient is the result of a linear action on $\mathbb{C}^2$ of a cyclic group $\mathbb{Z}_d$ with $d=a/\mathrm{GCD}(a,b)\mathrm{GCD}(a,c)$. When the action Φ is linearizable then there exists a natural isomorphism $\mathbb{C}^3//\Phi \simeq \mathbb{C}^3//\Psi$.

Koras and Russell discovered a construction of each smooth contractible affine algebraic threefold X equipped with a hyperbolic $\mathbb{C}^*$-action $\tilde{\Phi}$ (i.e., $\tilde{\Phi}$ has only one fixed point $\tilde{o}$ and the induced $\mathbb{C}^*$-action $\tilde{\Psi}$ on $T_{\tilde{o}}X$ is hyperbolic) such that $X//\tilde{\Phi}$ is isomorphic to $T_{\tilde{o}}X//\tilde{\Psi}$. For some of these varieties (which are called now

Koras-Russell threefolds) it was not clear whether they are isomorphic to $\mathbb{C}^3$ but $\tilde{\Phi}$ in this case was obviously non-linearizable. Koras and Russell proved in their previous papers that

Every hyperbolic algebraic $\mathbb{C}^*$-action Φ on $\mathbb{C}^3$ is linearizable provided

(1) *none of Koras-Russell threefolds are isomorphic to $\mathbb{C}^3$;*

(2) *S is isomorphic to $\mathbb{C}^3//\Psi \simeq \mathbb{C}^2/\mathbb{Z}_d$ where Ψ is the induced linear action on the tangent space at the fixed point.*

The introduction of the Makar-Limanov invariant enabled us to remove the first obstacle [**ML**], [**KaML97b**], i.e. Koras-Russell threefolds are, indeed, non-isomorphic to $\mathbb{C}^3$. Then Koras and Russell established some properties of S which include the facts that S is contractible with one singular point s_0 of analytic type $\mathbb{C}^2//\mathbb{Z}_d$, and with the logarithmic Kodaira dimension $\bar{\kappa}(S) = -\infty$. Using the seminal paper of Fujita [**Fu82**] and the results of Miyanishi and Tsunoda [**MiTs**] on the existence of affine rulings and Platonic $\mathbb{C}^*$-fiber spaces for open smooth surfaces with non-connected boundaries and a negative logarithmic Kodaira dimension, Koras and Russell showed first that the second claim is true under additional assumption that $S \setminus s_0$ has logarithmic Kodaira dimension at most 1. Then using among other tools their own deep results and the Kobayashi version of the BMY-inequality they proved that the case when $S \setminus s_0$ is of general type can be excluded and, therefore, removed this additional assumption for (2)[2]. In fact, in their new paper [**KoRu07**] a stronger result is established.

THEOREM 4.4. *Let S be a normal contractible surface of $\bar{\kappa}(S) = -\infty$ with quotient singularities only. Then $\bar{\kappa}(S_{\mathrm{reg}}) = -\infty$. Furthermore, if S has only one singular point then S is isomorphic to the quotient of $\mathbb{C}^2$ with respect to a linear action of a finite group.*

This concludes the brief description of the proof of Linearization Conjecture in dimension 3.

5. The Russell cubic and the scheme of computation of the Makar-Limanov invariant

One of the Koras-Russell threefolds is the Russell cubic R that is the hypersurface R in $\mathbb{C}^4_{x,y,z,t}$ given by $x + x^2y + z^2 + t^3 = 0$ (for construction of a general Koras-Russell threefold we refer again to the survey [**KaKoMLRu**]). Let us explain first why it was difficult to distinguish R from $\mathbb{C}^3$.

[2] A remarkable strengthening of statement (2) is obtained in Gurjar's paper [**Gu07**]. He showed that if X is a smooth contractible affine algebraic variety that admits a dominant morphism from a Euclidean space then for every reductive group G acting on X so that $X//G$ is two-dimensional this quotient is isomorphic to the quotient of $\mathbb{C}^2$ with respect to a linear action of a finite group. His proof is based on Theorem 4.4.

LEMMA 5.1. *The Russell cubic is diffeomorphic to* $\mathbb{R}^6$ *as a real manifold and its logarithmic Kodaira dimension is* $\bar{\kappa}(R) = -\infty$.

PROOF. Note that the natural projection $\rho : R \to \mathbb{C}^3_{x,z,t}$ is an affine modification such that its restriction over $\mathbb{C}^3_{x,z,t} \setminus \{x = 0\} \simeq \mathbb{C}^* \times \mathbb{C}^2$ is an isomorphism. In particular, $\bar{\kappa}(R) = \bar{\kappa}(\mathbb{C}^* \times \mathbb{C}^2) = -\infty$. Furthermore, looking at the exceptional divisor of ρ one can see that it is an affine modification that preserves the fundamental groups and homology ([**KaZa**], Prop. 3.1 and Th. 3.1). Thus R has all trivial senior homotopy groups by the Gurewitch theorem and R is contractible by the Whitehead theorem. Now one can apply the Choudary-Dimca theorem that says that every smooth contractible affine algebraic variety of dimension at least 3 is diffeomorphic to a real Euclidean space [**ChDi**].

□

Ramanujam's theorem [**Ra**] says that if a smooth contractible surface S is homeomorphic to $\mathbb{R}^4$ or the boundary of S at infinity is simply connected then S is isomorphic to $\mathbb{C}^2$. The Gurjar-Miyanishi theorem [**GuMi**] says that a smooth contractible surface of non-positive logarithmic Kodaira dimension is also isomorphic to $\mathbb{C}^2$. Lemma 5.1 shows that the similar invariants cannot help us to prove that R is not isomorphic to $\mathbb{C}^3$. Some of contractible threefolds can be distinguished from $\mathbb{C}^3$ by the absence of dominant morphisms from Euclidean spaces [**KaML97a**]. One of them is a hypersurface suggested by Dimca $\{(x, y, z, t) \in \mathbb{C}^4 | x + x^4 y + y^2 z^3 + t^5 = 0\}$. But the Russell cubic admits a dominant morphism from $\mathbb{C}^3$. Thus we need something new.

DEFINITION 5.2. A derivation ∂ on A is locally nilpotent (LND) if for every $a \in A$ there exists $n = n(a)$ such that $\partial^n(a) = 0$. The set of locally nilpotent derivations on A is denoted by $\mathrm{LND}(A)$. The Makar-Limanov invariant of A is

$$\mathrm{AK}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Ker}\, \partial.$$

Since A is the ring of regular functions on X we may write $\mathrm{AK}(X)$ instead of $\mathrm{AK}(A)$.

REMARK 5.3. There exists one-to-one correspondence between $\mathbb{C}_+$-actions on X and locally nilpotent derivations ∂ on A. Indeed, one can see that $\exp(\partial)$ is a $\mathbb{C}_+$-action while treating a $\mathbb{C}_+$-action as a phase flow with complex time one can check that its generating vector field is locally nilpotent on A. Hence $\mathrm{AK}(X)$ coincides with the ring of regular functions on X that are invariant with respect to any $\mathbb{C}_+$-action. Clearly we have $\mathrm{AK}(\mathbb{C}^n) = \mathbb{C}$ (in order to show that the intersection of kernels of LND consists of constants only it suffices to consider the intersection of kernels of partial derivatives).

We give a sketch of the proof of the following [**ML**].

Theorem 5.4. *The Makar-Limanov invariant of the Russell cubic coincides with* $\mathrm{AK}(R) = \mathbb{C}[x]|_R$*, i.e. R is not isomorphic to $\mathbb{C}^3$.*

Step 1. Introduce an associated affine variety $\hat{X}$ and the affine domain $\hat{A} = \mathbb{C}[\hat{X}]$ for X and A with a map $A \to \hat{A}$, $a \to \hat{a}$ so that for every $\partial \in \mathrm{LND}(A) \setminus 0$ there exists a unique associated $\hat{\partial} \in \mathrm{LND}(\hat{A}) \setminus 0$. Usually, in order to construct $\hat{A}$ one needs to consider the filtration generated by a weighted degree function on A and set $\hat{A}$ equal to the associated graded algebra. Another way is the geometrical construction which we present below [**KaML07**].

Let C be a germ of a smooth curve at $o \in C$, f be a regular function on C with a simple zero at o, $C^* = C \setminus o$, $\rho : \mathcal{X} \to C$ be an affine morphism such that $\mathcal{X}$ is normal, $\hat{X} := \rho^*(o)$ be reduced irreducible, $\mathcal{X}^* := \mathcal{X} \setminus \rho^{-1}(o) \simeq X \times C^*$ over C^*. Then a nonzero LND ∂ on X defines a LND ∂^* on $\mathcal{X}^*$. For some $n \in \mathbb{Z}$ vector field $f^n\partial^*$ extends to a LND δ on $\mathcal{X}$ with $\hat{\partial} = \delta|_{\hat{X}} \neq 0$. We define $\hat{a}$ via a in a similar manner (that is, if $a^* = a \circ \tau$ where $\tau : \mathcal{X}^* \to X$ is the natural projection then for some $n \in \mathbb{Z}$ function $f^n a^*$ extends to a regular function on $\mathcal{X}$ with a nonzero restriction $\hat{a}$ to $\hat{X}$).

Example 5.5. Consider $\rho : \mathcal{R} = \{cx + x^2y + z^2 + t^3 = 0\} \to C \simeq \mathbb{C}$. For $c \neq 0$, $\rho^{-1}(c) \simeq R$ while $\rho^{-1}(0) \simeq \hat{R} = \{x^2y + z^2 + t^3 = 0\}$.

Note that unlike R the threefold $\hat{R}$ has a singular line. This line must be invariant under any $\mathbb{C}_+$-action. This gives hope that it is easier to compute all associated $\mathbb{C}_+$-actions on $\hat{R}$ than $\mathbb{C}_+$-actions on R. Another encouraging fact is that associated LND's are homogeneous (see Definition 10.6) which implies that with an appropriate choice of associated objects the kernels of these LND's are generated by some variables and functions of form $z^2 + ct^3$, $c \in \mathbb{C}^*$. Since it can be shown that when $z^2 + ct^3 \in \mathrm{Ker}\,\partial \setminus \{0\}$ for $\partial \in \mathrm{LND}(A)$ then $z, t \in \mathrm{Ker}\,\partial$, we disregard the last possibility. These observations allow us to find all associated $\mathbb{C}_+$-actions on $\hat{R}$ and LND's on $\mathbb{C}[\hat{R}]$. In fact each of these LND's is proportional to one of the vector fields (with the coefficient of proportionality in the kernel of the LND): $2z\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial z}$ or $3t^2\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial t}$.

Step 2. The rest of computation is based on the existence of the degree function $\deg_\partial(a) = \min\{n | \partial^{n+1}(a) = 0\}$ generated by a LND ∂ on $A \setminus 0$ (i.e., $\mathrm{Ker}\,\partial = \{a \in A | \deg_\partial(a) = 0\}$). The construction of the associated objects implies the inequality $\deg_{\hat{\partial}}(\hat{a}) \leq \deg_\partial(a)$. The computation of associated LND's in Step 1 shows that $\deg_{\hat{\partial}}(\hat{y}) \geq 2$ in $\mathbb{C}[\hat{R}]$ and, therefore, $\deg_\partial(y) \geq 2$ in $\mathbb{C}[R]$. Furthermore, using different (weighted degree functions for the construction of) associated varieties one can extend this inequality to a stronger fact: each $b \in \mathbb{C}[R]$ with $\deg_\partial(b) \leq 1$ is a restriction of $p \in \mathbb{C}[x, z, t]$. Now one has to use the following simple results.

Lemma 5.6. *Let ∂ be a nonzero LND on A. Then*

(1) *for every* $a \in A$, *we have* $a = a_2/a_1$ *where* $a_1 \in \operatorname{Ker} \partial$ *and* a_2 *is from the algebra generated over* $\operatorname{Ker} \partial$ *by* $b \in A$ *with* $\deg_{\partial}(b) = 1$.

(2) *the ring* $\operatorname{Ker} \partial$ *is inept, i.e. if* $a_1 a_2 \in \operatorname{Ker} \partial \setminus 0$ *then* $a_1, a_2 \in \operatorname{Ker} \partial$.

The description of b before Lemma 5.6 implies now that on R we have $y = p(x,z,t)/q(x,z,t)$ with $q(x,z,t) \in \operatorname{Ker} \partial$. On the other hand $y = -(x+z^2+t^3)/x^2$ which yields divisibility of $q(x,z,t)$ by x. Hence, by Lemma 5.6 (2), $x \in \operatorname{Ker} \partial$ for every ∂ on $\mathbb{C}[R]$ and thus $x \in \operatorname{AK}(R)$.

6. Limitation of the Makar-Limanov and Derksen invariants

Definition 6.1. A smooth affine algebraic variety is called an exotic algebraic (resp. holomorphic) structure on $\mathbb{C}^n$ if it is diffeomorphic to $\mathbb{R}^{2n}$ but not isomorphic (resp. biholomorphic) to $\mathbb{C}^n$.

In particular, we proved in the previous section that the Russell cubic is an exotic algebraic structure on $\mathbb{C}^3$ (as well as any other Koras-Russell threefold [**KaML97b**]). However, we do not know any technique that allows to check whether R is an exotic holomorphic structure. Furthermore, the computation described before works very poorly in dimension 4 and higher. We can check, say, that the hypersurface $x + x^2y + z^2 + t^3 + u^5 = 0$ in $\mathbb{C}^5$ is an exotic algebraic structure on $\mathbb{C}^4$ (see [**KaML07**]) but we do not know whether $R \times \mathbb{C}$ is an exotic algebraic structure on $\mathbb{C}^4$. In particular, R is still a potential counterexample to the following.

Cancellation Conjecture (Zariski-Ramanujam). *Let* $X \times \mathbb{C}^k$ *be isomorphic to* $\mathbb{C}^{n+k}$. *Then* X *is isomorphic to* $\mathbb{C}^n$ (for $n \leq 2$ the answer is positive [**Fu79**]).

We still have hope that $\operatorname{AK}(R \times \mathbb{C})$ is nontrivial (i.e. different from $\mathbb{C}$) but this hope is absent in the case of other interesting hypersurfaces [**KaZa**].

Lemma 6.2. *Let* D *be a hypersurface in* $\mathbb{C}^{n+2}_{u,v,\bar{x}}$ *given by* $uv = p(\bar{x})$ *where* $\bar{x} = (x_1, \ldots, x_n)$ *and the zero fiber of the polynomial* $p \in \mathbb{C}[\bar{x}]$ *is smooth reduced. Then* D *has a trivial Makar-Limanov invariant, i.e.* $\operatorname{AK}(D) = \mathbb{C}$. *If* $p^{-1}(0)$ *is also contractible then* D *is diffeomorphic to* $\mathbf{R}^{2n+2}$.

Example 6.3. One of such contractible hypersurfaces is given by $uv + x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^6_{x,y,z,t,u,v}$. A topological computation shows that nonzero fibers of $uv + x + x^2y + z^2 + t^3$ are not contractible.

That is, the hypersurface from this example is a potential counterexample to the following.

Embedding Conjecture (Abhyankar-Sathaye). *Every algebraic embedding of* $\mathbb{C}^{n-1}$ *into* $\mathbb{C}^n$ *is rectifiable. That is, the image of* $\mathbb{C}^{n-1}$ *is a coordinate hyperplane in a suitable polynomial coordinate system* (for $n = 2$ the answer is positive - the AMS theorem).

REMARK 6.4. (1) There is another potential counterexample to the Embedding Conjecture that would be much more spectacular; it is the Vénéreau polynomial[3] $v_1 := y + x(xz + y(yt + z^2)) = y + x^2z + xy^2t + xyz^2$ on $\mathbb{C}^4$. All fibers of this polynomial are isomorphic to $\mathbb{C}^3$ but nobody has found a way to rectify its zero fiber. In smaller dimensions the similar problem was solved positively by Sathaye [**Sat**] (in combination with [**BaCoWr**]). Even more, every polynomial on $\mathbb{C}^3$ with infinite number of fibers isomorphic to $\mathbb{C}^2$ is a variable in a suitable polynomial coordinate system (see [**DaiKa**], [**Ka02**], [**Mi87**]). It is worth mentioning that if one considers the Vénéreau polynomial on $\mathbb{C}^n$ (whose coordinate system contains x, y, z, t as variables) then for a sufficiently large n the zero fiber of v_1 becomes rectifiable. This is a consequence of the Asanuma [**As87**] and Bass-Connell-Wright theorems [**BaCoWr**].

(2) There is a more general form of this conjecture which says that each algebraic embedding $\mathbb{C}^k \hookrightarrow \mathbb{C}^{n+k}$ is rectifiable. This conjecture is true for $n \geq k + 2$ [**Ka91**] (see also [**Sr**] and Nori, unpublished). In particular, each algebraic embedding $\mathbb{C} \hookrightarrow \mathbb{C}^n$ is rectifiable for $n \neq 3$ (for $n = 2$ this is the statement of the AMS theorem and for $n \geq 4$ it was proven earlier by Craighero [**Cr**] and Jelonek [**Je**]). It is also known that each algebraic embedding $\mathbb{C} \hookrightarrow \mathbb{C}^3$ is rectifiable by means of analytic automorphisms of $\mathbb{C}^3$ [**Ka92**]. Precaution: there are proper holomorphic embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^n$, $n \geq 2$ non-rectifiable by holomorphic automorphisms.

In fact, D from Lemma 6.2 is m-transitive for any $m > 0$ (see [**KaZa**]) and it has the Andersén-Lempert property, i.e. the Lie algebra generated by completely integrable algebraic vector fields coincides with Lie algebra of all algebraic vector fields on it [**KaKu**]. This implies the validity of the Oka-Grauert-Gromov principle for D and, in particular, each point of D has a (Fatou-Bieberbach) neighborhood biholomorphic to a Euclidean space but it is still unknown whether D itself is biholomorphic to a Euclidean space.

DEFINITION 6.5. Derksen suggested a modification of the Makar-Limanov invariant which is easier to compute in some cases [**DeKr**] (it almost eliminates Step 2 in the computation for the Russell cubic). His new invariant $\mathrm{Dr}(A)$ is the subring of A generated by kernels of all nonzero LND's on A. We say that this invariant is trivial if $\mathrm{Dr}(A) = A$. This is so, of course, in the case of $A = \mathbb{C}^{[n]}$.

The Derksen and Makar-Limanov invariants are not equivalent, i.e. one may be trivial while the other is not [**CrMa**]. For the Russell cubic R the Derksen invariant coincides with the subring of $\mathbb{C}[R]$ that consists of the restrictions of all polynomials from $\mathbb{C}[x, z, t]$ to R but in the case of other hypersurfaces in this section it is as ineffective as the Makar-Limanov invariant.

[3] Freudengurg informed us that slightly more complicated polynomials with similar properties were discovered earlier by Bhadwadekar and Dutta [**BhDu**] and a general method of constructing such polynomials will appear in [**DaiFr07**].

7. Asanuma's construction and other counterexamples to the Linearization Problem

The first example of a non-linearizable action of $O(2)$ on $\mathbb{C}^4$ was constructed by Schwarz [**KrSh**]. As we mentioned in the introduction for every connected reductive group different from a torus there exists a non-linearizable action on some $\mathbb{C}^n$ where n depends on the group [**Kn**]. For finite groups non-linearizable actions on $\mathbb{C}^4$ were constructed by Jauslin-Moser, Masuda, Petrie, and Freudenburg [**MaPe**], [**MaMJPe**], [**FrMJ**]. These examples of non-linearizable G-actions were extracted from the existence of non-trivial G-vector bundles on representation spaces of G. New ideas were brought by Asanuma [**As99**].

Theorem 7.1. *There exists a non-linearizable algebraic* $\mathbf{R}^*$*-action on* $\mathbf{R}^5$.

Applying the technique of Asanuma, Derksen and Kutzschebauch constructed a counterexample to linearization in the holomorphic category [**DeKu**].

Theorem 7.2. *There exist non-linearizable holomorphic* $\mathbb{C}^*$*-actions on* $\mathbb{C}^n$ *for any* $n \geq 4$.

Remark 7.3. Asanuma's construction works if there are non-rectifiable algebraic embeddings of a line into a Euclidean space. Say, one can find such an embedding $\mathbb{R} \hookrightarrow \mathbb{R}^3$ since every unbounded knot in $\mathbb{R}^3$ is isotopic to an algebraic embedding $\mathbb{R} \hookrightarrow \mathbb{R}^3$ [**Sh**]. There are also non-rectifiable proper holomorphic embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^n$, $n \geq 2$ (e.g., see [**FoGlRo**], [**Ka92**]). As we mentioned before it is unknown whether there exists an algebraic embedding $\mathbb{C} \hookrightarrow \mathbb{C}^3$ non-rectifiable by algebraic automorphisms. However we consider this construction in the case of complex algebraic varieties since it may have very interesting consequences (its reformulation in the real, higher dimensional, or holomorphic cases is obvious which yields Theorems 7.1 and 7.2).

Definition 7.4. Let $f \in A \setminus 0$, $D = f^*(0) \subset X$, and C be a closed subvariety of D whose defining ideal in A is I. We suppose for simplicity that D and C are reduced irreducible, that X, D, and C are smooth, and that C is at least of codimension 2 in X. Then one can consider the blowing-up $\mathrm{Bl}_C(X)$. We call the variety $Y = \mathrm{Bl}_C(X) \setminus D'$ the affine modification of X with center C along D (where D' is the strict transform of D in $\mathrm{Bl}_C(X)$).

The following easy facts can be found in [**KaZa**].

Proposition 7.5. (1) *The affine modification* Y *is a smooth affine variety with the algebra of regular functions* $\mathbb{C}[Y] = A[I/f]$ *where* $A[I/f]$ *is the algebra generated over* A *by rational functions* $b_1/f, \ldots, b_n/f$ *such that* $f, b_1, \ldots, b_n$ *are generators of* I.

(2) *If* X, D, C *are contractible, so is* Y.

REMARK 7.6. If C is a strict complete intersection in X given by $f = b_1 = \ldots = b_k = 0$ then the affine modification Y is isomorphic to the subvariety of $X \times \mathbb{C}^k_{v_1,\ldots,v_k}$ given by $fv_1 - b_1 = \ldots = fv_k - b_k = 0$. In this case we call Y a simple modification of X.

EXAMPLE 7.7. Let $X = \mathbb{C}^{n+1}_{x_1,\ldots,x_n,t}$, D be the hyperplane $t = 0$ in X, and C be given by $x_1 = \ldots x_k = t = 0$ where $k \leq n-1$. Then by Remark 7.6 $Y \subset \mathbb{C}^{n+k+1}_{x_1,\ldots,x_n,t,v_1\ldots,v_k}$ is given by $tv_1 - x_1 = \ldots = tv_k - x_k = 0$. In particular, $Y \simeq \mathbb{C}^{n+1}$.

DEFINITION 7.8. Let $X = \mathbb{C}^4_{x,y,z,t}$, $f = t$, $D = \{t = 0\} \simeq \mathbb{C}^3_{x,y,z}$, and C is the image of $\mathbb{C}$ in D under a polynomial embedding. Then the affine modification Y of X with center at C along D is called an Asanuma fourfold. Proposition 7.5 and the Choudary-Dimca theorem imply that it is diffeomorphic to $\mathbb{R}^8$.

Asanuma's construction is based on the following elegant fact.

LEMMA 7.9. *Let X, D, C, Y be as in Definition 7.4. Set $\tilde{X} = X \times \mathbb{C}$, $\tilde{D} = D \times \mathbb{C}$, and $\tilde{C} = C \times 0 \subset \tilde{D}$. Then the affine modification $\tilde{Y}$ of $\tilde{X}$ with center at $\tilde{C}$ along $\tilde{D}$ is isomorphic to $Y \times \mathbb{C}$.*

PROOF. Let f and I be as in Definition 7.4 and $f, b_1, \ldots, b_k$ be generators of I. That is, $\mathbb{C}[Y]$ is generated over A by $b_1/f, \ldots, b_k/f$. Hence $\mathbb{C}[Y \times \mathbb{C}_v]$ is generated over A by $b_1/f, \ldots, b_k/f$ and an independent variable v. Then $f, b_1, \ldots, b_k, u$ are generators of the defining ideal of $\tilde{C}$ in $\tilde{X} = X \times \mathbb{C}_u$. Hence $\mathbb{C}[\tilde{Y}]$ is generated over A by $b_1/f, \ldots, b_k/f$, and u/f. Sending $v \to u/f$ we get an isomorphism $\mathbb{C}[\tilde{Y}] \simeq \mathbb{C}[Y \times \mathbb{C}]$.

□

PROPOSITION 7.10. *Let $X = \mathbb{C}^4$, $D \simeq \mathbb{C}^3$ be a coordinate hyperplane in X and $C \subset D$ be isomorphic to $\mathbb{C}$. Suppose that Y is the affine modification of X with center at C along D. Then*

(1) *$Y \simeq \mathbb{C}^4$ provided C is rectifiable in D;*

(2) *$Y \times \mathbb{C} \simeq \mathbb{C}^5$;*

(3) *Y is biholomorphic to $\mathbb{C}^4$;*

(4) *in the case of a non-rectifiable C there exist non-linearizable $\mathbb{C}^*$-actions on Y and on $\mathbb{C}^5$.*

PROOF. Example 7.7 implies (1). By Remark 6.4 every algebraic embedding of $\mathbb{C}$ into $\mathbb{C}^4$ is rectifiable. Hence (2) follows from Lemma 7.9 and Example 7.7. Statement (3) follows from the same argument since every algebraic embedding of $\mathbb{C}$ into $\mathbb{C}^3$ is rectifiable by a holomorphic automorphism. In (4) consider first the $\mathbb{C}^*$-action on $\mathbb{C}^4_{x,y,z,t}$ given by $(x, y, z, t) \to (x, y, z, \lambda t)$ and a neighborhood U of any point $\zeta \in C$ which is of form $U = U_0 \times \mathbb{C}_t$ where $U_0 \subset \mathbb{C}^3_{x,y,z}$ is open in the standard topology. We restrict the $\mathbb{C}^*$-action to U and since C is smooth we can suppose that it is given in U by $\tilde{y} = \tilde{z} = t = 0$ where $(\tilde{x}, \tilde{y}, \tilde{z})$ is a local holomorphic coordinate system on U_0 (in particular, $\zeta = (c_0, 0, 0, 0)$). Let $\sigma : Y \to X$ be the natural projection. Then $\sigma^{-1}(U)$ is a hypersurface in $U \times \mathbb{C}^2_{v_1,v_2}$ given by $tv_1 - \tilde{y} = tv_2 - \tilde{z} = 0$, i.e. the action can be lifted to a $\mathbb{C}^*$-action Φ on $\sigma^{-1}(U)$ as $(\tilde{x}, \tilde{y}, \tilde{z}, t, v_1, v_2) \to (\tilde{x}, \tilde{y}, \tilde{z}, \lambda t, \lambda^{-1} v_1, \lambda^{-1} v_2)$ (clearly, Φ can be extended to Y). The only $\mathbb{C}^*$-orbit in X, whose proper transforms are non-closed $\mathbb{C}^*$-orbits in Y, are of form $\tilde{x} - c_0 = \tilde{y} = \tilde{z} = 0$ and the closures of these orbits contain fixed points of Φ (given by $t = v_1 = v_2 = 0$). The algebra of $\mathbb{C}^*$-invariant functions on Y consists of polynomials in x, y, and z, i.e. there is a natural isomorphism between $Y//\Phi$ and D. Hence, if $\pi : Y \to D$ is the quotient morphism then $\pi^{-1}(\zeta)$ contains a fixed point for $\zeta \in D$ iff $\zeta \in C$. Assume that $Y \simeq \mathbb{C}^4$ and Φ is linearizable. Looking at the induced action at fixed points one can see that Φ is equivalent to $(x, y, z, t) \to (x, \lambda^{-1} y, \lambda^{-1} z, \lambda t)$. The quotient of this linear action is $\mathbb{C}^3$ but the image of the fixed point set in this quotient is a rectifiable line which is the desired contradiction.

For the last statement of (4) consider the $\mathbb{C}^*$-action on Y generated by the action $(x, y, z, t) \to (x, y, z, \lambda^2 t)$ on $\mathbb{C}^4$. Extend it to the $\mathbb{C}^*$-action Ψ on $\mathbb{C}^5 = Y \times \mathbb{C}_u$ so that $u \to \lambda^{-1} u$. Then $\mathbb{C}^5//\Psi \simeq D \times \mathbb{C} \simeq \mathbb{C}^4$ and for the quotient morphism $\rho : \mathbb{C}^5 \to D \times \mathbb{C}$ the image of the fixed point set is $C \times 0$ is contained in $D \times 0 = \rho(Y \times 0)$. For any orbit O of Ψ the action $\Psi|_O$ is effective iff O is not in $Y \times 0$. Assume that Ψ is linearizable. Looking at the induced action at fixed points one can see that Ψ is equivalent to $(x, y, z, t, u) \to (x, \lambda^{-2} y, \lambda^{-2} z, \lambda^2 t, \lambda^{-1} u)$. The quotient of this linear action is $\mathbb{C}^4$ but the image of the orbits with non-effective action Ψ on them (resp. the fixed point set) is $\mathbb{C}^3$ (resp. C') and C' is a rectifiable line is this $\mathbb{C}^3$ which is the desired contradiction.

□

REMARK 7.11. If we could prove that one of Asanuma's fourfold is an exotic algebraic structure on $\mathbb{C}^4$ this would produce a counterexample to the Zariski-Ramanujam Conjecture in dimension 4, a non-rectifiable embedding $\mathbb{C} \hookrightarrow \mathbb{C}^3$, and an exotic algebraic structure biholomorphic to $\mathbb{C}^4$ (examples of exotic algebraic structures that are not exotic analytic structures are not known).

8. Free $\mathbb{C}_+$-actions

We mentioned that any $\mathbb{C}_+$-action on $\mathbb{C}^2$ is triangular and, therefore, any free $\mathbb{C}_+$-action is a translation (see Theorem 2.6).

There are non-triangular $\mathbb{C}_+$-actions on $\mathbb{C}^3$ [**Ba**], (and, furthermore, such actions can be constructed on any $\mathbb{C}^n$ with $n \geq 3$ [**Po87**]). Indeed, consider the action $\Phi : \mathbb{C}_+ \times \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$(t, x_1, x_2, x_3) = (x_1, x_2 + t x_1 u, x_3 - 2 t x_2 u - t^2 x_1 u^2)$$

where $u = x_1x_3 + x_2^2$. Then its fixed point set $x_1x_3 + x_2^2 = 0$ is not a cylinder as required for triangular actions by Remark 2.4.

Starting with $n = 4$ not all free $\mathbb{C}_+$-actions on $\mathbb{C}^n$ are translations since it may happen that $\mathbb{C}^4//\mathbb{C}_+$ is not homeomorphic to the standard topological quotient $\mathbb{C}^4/\mathbb{C}_+$ ([**Wi90**], known also to M. Smith). In Winkelmann's example the standard quotient $\mathbb{C}^4/\mathbb{C}_+$ of a free action is not Hausdorff[4] while in the case of translations on $X \simeq Y \times \mathbb{C}$ the standard topological quotient $X/\mathbb{C}_+$ is homeomorphic to $X//\mathbb{C}_+ \simeq Y$ and, therefore, is affine.

The remaining question about translations on $\mathbb{C}^3$ was tackled by the following [**Ka04**].

Theorem 8.1. *Let Φ be a $\mathbb{C}_+$-action on a factorial three-dimensional X with $H_2(X) = H_3(X) = 0$. Suppose that the action is free and $S = X//\Phi$ is smooth.*

Then Φ is a translation, i.e. X is isomorphic to $S \times \mathbb{C}$ and the action is generated by a translation on the second factor.

Indeed, since $\mathbb{C}^3//\mathbb{C}_+ \simeq \mathbb{C}^2$ for any nontrivial $\mathbb{C}_+$-action [**Mi80**] we have the long-expected result.

Corollary 8.2. *A free $\mathbb{C}_+$-action on $\mathbb{C}^3$ is a translation in a suitable coordinate system.*

Equivalently, every nowhere vanishing (as a vector field) locally nilpotent derivation on $\mathbb{C}^{[3]}$ is a partial derivative in a suitable coordinate system.

Actually in the case of a smooth contractible X the assumption on smoothness of S in Theorem 8.1 is superfluous by virtue of the following generalization of Miyanishi's theorem [**KaSa**].

Theorem 8.3. *Let Φ be a nontrivial $\mathbb{C}_+$-action on a three-dimensional smooth contractible X. Then the quotient $S = X//\Phi$ is a smooth contractible surface.*

This result has an application to the Van de Ven's conjecture which says that

Every smooth contractible affine algebraic manifold X is rational.

Gurjar and Shastri established the validity of this conjecture in the case of $\dim X = 2$. Combining this result with Theorem 8.3 we get the following.

Corollary 8.4. *If a smooth affine contractible threefold X admits a nontrivial $\mathbb{C}_+$-action, then X is rational.*

Remark 8.5. The rationality of a smooth affine contractible X equipped with a nontrivial $\mathbb{C}^*$-action was establishes by Gurjar, Shastri, and Pradeep [**GuSh**], [**GuPr**], [**GuPrSh**].

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[4] More precisely, the action is given by $(t, x_1, x_2, x_3, x_4) \to (x_1, x_2+tx_1, x_3+tx_2+t^2x_1/2, x_4+t(x_2^2 - 2x_1x_3 - 1))$. One can see that the points $(0, 1, 0, 0)$ and $(0, -1, 0, 0)$ do not belong to the same orbit while for $\varepsilon \neq 0$ the points $(\varepsilon, 1, 0, 0)$ and $(\varepsilon, -1, 0, 0)$ are in the same orbit (just take $t = 2/\varepsilon \in \mathbb{C}_+$).

9. Elements of the proof of Theorem 8.3

The crucial fact about surjectivity of the quotient morphism $\pi : X \to S$ was established in [**Ka04**]. Hence if s_0 is a singularity of S then there is a finite morphism from a germ of a smooth surface in X onto a germ of S at s_0. This implies that s_0 is at worst a quotient singularity [**Br**]. Furthermore, surjectivity of π implies (via algebraic topology) that S is contractible and the link of S is a homology sphere. Thus by another Brieskorn's result s_0 is at worst an E_8–singularity (i.e. a singularity of type $x^2 + y^3 + z^5 = 0$) and its link is a Poincaré homology 3-sphere $\mathcal{L}_0$. Algebraic topology implies also that the link $\mathcal{L}_\infty$ of S at infinity is also a homology 3-sphere.

Assume first for simplicity that s_0 is the only singularity of S. The the part of S between the links yields a cobordism between $\mathcal{L}_0$ and $\mathcal{L}_\infty$. Moreover, surjectivity of π in combination with algebraic topology implies that it is a simply connected homology cobordism. But this type of cobordisms between $\mathcal{L}_0$ and another homology 3-sphere is forbidden by a theorem of Taubes [**Ta**]. Thus S cannot have one singular point.

In the case of several singularities $s_0, s_1, \ldots, s_k$ one has to consider links $\mathcal{L}_i$ around s_i and drill "holes" joining $\mathcal{L}_\infty$ with $\mathcal{L}_1, \ldots, \mathcal{L}_k$. Then we get again a simply connected homology cobordism between $\mathcal{L}_0$ and another component of the boundary which is the connected sum of $\mathcal{L}_\infty, \mathcal{L}_1, \ldots, \mathcal{L}_k$. But a connected sum of homology 3-spheres is again a homology 3-sphere. Thus we get a contradiction with the theorem of Taubes once more.

It remains to note that being a quotient of a normal space S is normal itself, i.e. the set of its singular points is at worst finite. Thus we get the smoothness of S.

10. Partial results on classification of $\mathbb{C}_+$-actions on affine contractible threefolds

In this section X is a smooth contractible affine threefold equipped with a nontrivial $\mathbb{C}_+$-action and $\pi : X \to S = X//\mathbb{C}_+$ is its algebraic quotient morphism. It is well-known that there is a curve $\Gamma \subset S$ such that $\pi^{-1}(S \setminus \Gamma)$ is isomorphic to $(S \setminus \Gamma) \times \mathbb{C}$ over $S \setminus \Gamma$. We have the following result from [**Ka04**].

Theorem 10.1. *Each component of the (smallest possible) curve* Γ *as before is a polynomial curve, i.e. its normalization is* $\mathbb{C}$.

Remark 10.2. In all known examples Γ, and therefore, each of its components, is contractible. The author suspects that this is always so. If this fact were correct then, using Zaidenberg's theorem [**Zai88**] about the absence of curves with Euler characteristics 1 on smooth contractible surfaces of general type, one would get the following: if logarithmic Kodaira dimension $\bar{\kappa}(S) = 2$ then $X \simeq S \times \mathbb{C}$ and the action is proportional to the the natural translation on X over S.

In the case when $\bar{\kappa}(S) = 1$ one can give a complete classification of such $\mathbb{C}_+$-actions based on the description of polynomial curves in smooth contractible surfaces of logarithmic Kodaira dimension 1. It was established by Zaidenberg, Gurjar and Miyanishi that such a surface S contains a unique curve L isomorphic to a line [**Zai88**], [**GuMi**]. We can claim more [**KaML96**].

THEOREM 10.3. *Let S and L be as before (i.e. S is smooth contractible with $\bar{\kappa}(S) = 1$ and L is the line in it). Then L is the only polynomial curve in S.*

PROOF. We use the result of tom Dieck and Petrie [**tDPe**] (see also [**KaML97a**]) which says that there exists an affine modification $\rho : S \to \mathbb{C}^2_{x,y}$ such that (1) ρ is an isomorphism over $\mathbb{C}^2 \setminus P$ where P is the curve given by $x^k - y^l = 0$ with $k \geq 2$ and $l \geq 2$ being relatively prime, (2) $\rho(S)$ coincides with the union of $\mathbb{C}^2 \setminus P$ and point $(1,1)$, and (3) $\rho^{-1}(1,1) = L$. Assume there is a polynomial curve in S different from L. Then its image C in $\mathbb{C}^2$ is a polynomial curve that meets P at most at one point $(1,1)$. If C does not meet P then it is contained in a nonzero fiber of $x^k - y^l$ which has a nonzero genus but this is impossible for a polynomial curve. Thus $C \cap P = (1,1)$. Let C be given by the zero locus of a polynomial $p(x,y)$ and $\mathbb{C}_z \to \mathbb{C}^2$, $z \to (z^l, z^k)$ be the normalization of P. Then $p(z^l, z^k) = 0$ must have the only root at $z = 1$ (since the image of $z = 1$ under this normalization is $(1,1)$), i.e. up to a constant factor $p(z^l, z^k) = (z-1)^m$. Now one can check that the right-hand side of the last equality has a zero derivative at $z = 0$ while the similar derivative of the left-hand side is nonzero. This contradiction concludes the proof. $\square$

REMARK 10.4. Suppose that X, π, S, and Γ are as in the beginning of this section. It follows from [**Ka04**] that the quotient morphism π factors through a surjective affine modification $\rho : X \to S \times \mathbb{C}$ which generates an isomorphism over $(S \setminus \Gamma) \times \mathbb{C}$. In the case when Γ is isomorphic to a line (or a union of disjoint lines), Theorem 3.1 from [**Ka02**] implies that ρ is a composition of a finite number of simple affine modifications (in the sense of Remark 7.6) whose centers are contractible irreducible curves (they are also smooth when $X \simeq \mathbb{C}^3$). In combination with Theorem 10.3 this gives a construction of all smooth affine contractible threefolds equipped with a nontrivial $\mathbb{C}_+$-action such that the logarithmic Kodaira dimension of the quotient is 1. However, Freudenburg constructed a $\mathbb{C}_+$-action on $\mathbb{C}^3$ such that Γ consists of two lines in $S \simeq \mathbb{C}^2$ that meets transversally at one point [**Fr98**]. It can be shown that in this case ρ cannot be presented as a composition of simple modifications.

EXAMPLE 10.5. (1) Consider, for instance, the $\mathbb{C}_+$-action on the Russell cubic R whose associated LND is $\partial = 2z\frac{\partial}{\partial y} - x^2\frac{\partial}{\partial z}$. The kernel of ∂ is $\mathbb{C}[x,t]$. Hence the quotient space $S = \mathbb{C}^2_{x,t}$ and the affine modification $\rho : R \to S \times \mathbb{C}$ described in Remark 10.4 is nothing but the natural projection $R \to \mathbb{C}^3_{x,z,t}$. The role of $\Gamma \subset S$ is played by the line $x = 0$. Consider the simple affine modification $X_1 \to \mathbb{C}^3_{x,z,t}$ along the (z, t)-coordinate plane with center at the polynomial curve $z^2 + t^3 = 0$. By Remark 7.6 X_1 can be viewed as the hypersurface $xv = z^2 + t^3$ in $\mathbb{C}^4_{x,z,t,v}$. Take the next affine modification $X_2 \to X_1$ along the divisor $\{x = 0\} \subset X_1$ with center at the polynomial curve $v + 1 = 0$. Again X_2 can be viewed as the hypersurface in $X_1 \times \mathbb{C}_y$ given by $xy = -(v + 1)$. Multiplying this equality by x we get $x + x^2y + z^2 + t^3 = 0$, i.e. $X_2 \simeq R$ and we have the desired decomposition of ρ into simple affine modifications.

(2) We describe Freudenburg's example as a LND on $\mathbb{C}[x, y, z]$. Consider polynomials F, G, R defined as follows:

$$F = xz - y^2;\ R = x^3 - Fy;\ G = zF^2 - 2x^2yF + x^5.$$

In particular, $xG = F^3 + R^2$. Consider the derivation ∂ such that $\partial(F) = \partial(G) = 0$ (i.e. $\operatorname{Ker}\partial = \mathbb{C}[F, G]$) and $\partial(R) = -FG$. Since $x = (F^3 + R^2)/G$, $y = (X^3 - R)/F$, and

$$z = (G - x^5 + 2x^2yF)/F^2 = (G - 2x^2R + x^5)/F^2$$

one can see that ∂ is a well-defined LND on $\mathbb{C}^3_{x,y,z}$. The curve Γ is the cross $FG = 0$ in the quotient $S = \mathbb{C}^2_{F,G}$.

Note that in Freudenburg's example the coordinates F and G in the quotient plane $\mathbb{C}^2_{F,G}$ are homogeneous polynomials on $\mathbb{C}^3_{x,y,z}$. This is a special case of a homogeneous LND.

Definition 10.6. Consider a weighted degree function ω on $\mathbb{C}[x, y, z]$ such that the weights $\omega(x), \omega(y), \omega(z)$ are relatively prime natural numbers, i.e. ω generates a weighted projective space $\mathbb{P}^2_\omega$. By Miyanishi's theorem the kernel of any nontrivial LND ∂ on $\mathbb{C}[x, y, z]$ is of form $\mathbb{C}[f, g]$ with $f, g \in \mathbb{C}[x, y, z]$. We call a LND ∂ homogenous if f and g are homogeneous with respect to some weighted degree function ω as before.

Theorem 10.7. ([**Dai**]) *Let ω be as in Definition 10.6 and f and $g \in \mathbb{C}[x, y, z]$ be two ω-homogeneous polynomials with relatively prime ω-degrees. Suppose that C_f and C_g are the curves in $\mathbb{P}^2_\omega$ generated by zeros of f and g respectively. Then the existence of a homogeneous LND on $\mathbb{C}[x, y, z]$ with kernel $\mathbb{C}[f, g]$ is equivalent to the fact that $\mathbb{P}^2_\omega \setminus (C_f \cup C_g) \simeq \mathbb{C} \times \mathbb{C}^*$.*

Remark 10.8. Thus the problem of classification of homogeneous LND on $\mathbb{C}^{[3]}$ (and the associated homogenous $\mathbb{C}_+$-actions on $\mathbb{C}^3$) is equivalent to the problem of finding all curves C_f and C_g as in Theorem 10.7. Such curves were completely classified by Daigle and Russell [**DaiRu01a**] and [**DaiRu01b**] but their description is a bit long for this survey.

11. $\mathbb{C}^*$-actions on affine surfaces.

In this section we consider the Classification Problem for effective $\mathbb{C}^*$-actions on smooth affine algebraic surfaces. As we mentioned any algebraic $\mathbb{C}^*$-action on $\mathbb{C}^2$ is equivalent to a linear one: $(x, y) \to (\lambda^k x, \lambda^l y)$ where (x, y) are coordinates on $\mathbb{C}^2$, $\lambda \in \mathbb{C}^*$, and $k, l \in \mathbb{Z}$. We require that k and l are relatively prime since otherwise the action is not effective. It is easy to check that any other linear action $(x, y) \to (\lambda^n x, \lambda^m y)$ is equivalent to this one if the unordered pairs $\{k, l\}$ and $\{n, m\}$ coincide. This yields the classification of all equivalence classes of $\mathbb{C}^*$-actions on $\mathbb{C}^2$. Applying Theorem 4.3 we see that for the other smooth toric surfaces $\mathbb{C} \times \mathbb{C}^*$ and $\mathbb{C}^* \times \mathbb{C}^*$ every $\mathbb{C}^*$-action is also equivalent to a linear one, i.e. we have a classification in this case as well.

In the case of a normal non-toric affine surface S we need some canonical effective $\mathbb{C}^*$-actions on S (similar to linear actions on $\mathbb{C}^2$) such that any other action is equivalent to one of those. They are provided by the Dolgachev-Pikhman-Demazure (DPD) presentation, introduced by Flenner and Zaidenberg [**FlZa03**], [**FlZa05a**], which allows also to describe all normal affine surfaces that admit nontrivial $\mathbb{C}^*$-actions.

Remark 11.1. Their approach was generalized in [**AlHa**] with $\mathbb{C}^*$-actions replaced by actions of higher dimensional tori. Another beautiful extension of this DPD-presentation can be found in [**Ko**].

Recall that the existence of a non-trivial algebraic $\mathbb{C}^*$-action Φ on S is equivalent to the existence of a nontrivial $\mathbb{Z}$-grading $B = \oplus_{i \in \mathbb{Z}} B_i = B_{\geq 0} \oplus_{B_0} B_{\leq 0}$ of the algebra $B = \mathbb{C}[S]$ of regular functions on S where each $b \in B_i$ is an λ^i-eigenvector of the isomorphism of B generated by the action of $\lambda \in \mathbb{C}^*$.

If $S//\Phi$ is a curve, we set $C = S//\Phi$. Otherwise there exists only one fixed (attractive) point o of Φ and we set $C = (S \setminus o)/\Phi$. The DPD-presentation describes the $\mathbb{Z}$-grading in terms of $\mathbb{Q}$-divisors on C and it distinguishes three types of action.

Elliptic type: Φ has an attractive fixed point o. Then C is a smooth projective curve and there exists an ample $\mathbb{Q}$-divisor D on C so that $B = \oplus_{i \geq 0} H^0(C, \mathcal{O}(\lfloor iD \rfloor) u^i$ where u is an unknown and $\lfloor E \rfloor$ is the integral part of a $\mathbb{Q}$-divisor E. (If $B = \oplus_{i \leq 0} H^0(C, \mathcal{O}(\lfloor iD \rfloor) u^i$ then switching from Φ to Φ^{-1} one can make indices nonnegative.)

Parabolic type: Φ contains a curve of fixed points. Then this curve is smooth affine and isomorphic to C, and there exists a $\mathbb{Q}$-divisor D on C so that $B = \oplus_{i \geq 0} H^0(C, \mathcal{O}(\lfloor iD \rfloor) u^i$. (One may need to replace Φ by Φ^{-1} as before to get such a presentation for B.)

Hyperbolic type: a finite number of fixed points none of which are attractive. Then C is affine smooth and there exist $\mathbb{Q}$-divisors D_+ and D_- on C so that $D_+ + D_- \leq 0$; $B_{\geq 0} = \oplus_{i \geq 0} H^0(C, \mathcal{O}(\lfloor iD_+ \rfloor) u^i$, $B_{\leq 0} = \oplus_{i \leq 0} H^0(C, \mathcal{O}(\lfloor -iD_- \rfloor) u^i$, and $B = B_{\geq 0} \oplus_{B_0} B_{\leq 0}$. To replace of Φ by Φ^{-1} in this case is the same as to interchange D_+ and D_-.

Vice versa: taking the spectrum of the algebra B, that appears in the description of these three types of actions, one can see that any smooth affine curve C with $\mathbb{Q}$-divisors D_+ and D_- on it such that $D_+ + D_- \leq 0$ (resp. a smooth affine or projective curve C with an appropriate $\mathbb{Q}$-divisor D on it) corresponds to a normal affine algebraic surface S equipped with an effective hyperbolic (resp. parabolic or elliptic) $\mathbb{C}^*$-action.

Remark 11.2. It is worth mentioning some geometrical features of S in the hyperbolic case which is the most interesting for us. Suppose that $\{p_i\}$ (resp. $\{q_j\}$) is the set of points of C for which $D_+(p_i) + D_-(p_i) < 0$ (resp. $D_+(q_j) = -D_-(q_j) \notin \mathbb{Z}$). By technical reasons we put $D_\pm(p_i) = \frac{\mp e_i^\pm}{m_i^\pm}$ where $(e_i^\pm, m_i^\pm)$ is a pair of relatively prime integers and $\pm m_i^\pm > 0$. We put also $D_+(q_j) = \frac{-e_j}{m_j}$ with $m_j > 0$. The quotient morphism $\pi : S \to C = S//\mathbb{C}^*$ admits a proper extension $\tilde{S} \to C$ such that $\tilde{S} \setminus S$ consists of two sections C_+ and C_- of this morphism. Denote by $p_i^\pm$ (resp. $q_j^\pm$) the image of p_i (resp. q_j) under the natural isomorphism $C \simeq C_\pm$. Every singularity of $\tilde{S}$ is automatically a quotient singularity and the only singularities of $\tilde{S}$ on $C_\pm$ are points from $\{p_i^\pm, q_j^\pm\}$ with each $p_i^\pm$ being of type $(\pm m_i^\pm, -e_i^\pm)$ [5] and each $q_j^\pm$ being of type $(m_j, \mp e_j)$. A complete description of the quotient morphism fibers and the other singularities of $\tilde{S}$ can be also obtained from this data. Say, every fiber $\pi^*(p_i)$ is not irreducible and every fiber $\pi^*(q_j)$ is not reduced.

Consider an effective $\mathbb{C}^*$-action Φ' on our non-toric surface S and suppose that a curve C' plays the same role for Φ' as C for Φ. It can be shown that Φ and Φ' are of the same type and $C \simeq C'$.

Theorem 11.3. ([**FlZa03**]) *Let D (resp. D') be the $\mathbb{Q}$-divisor on $C \simeq C'$ that appears in the DPD-presentation of Φ (resp. Φ') in the elliptic or parabolic cases. In the hyperbolic case we denote the similar divisors on C by D_+ and D_- (resp. D'_+ and D'_-). Then Φ is equivalent to Φ' if and only if an isomorphism $C \simeq C'$ can be chosen so that*

(1) *D is linearly equivalent to D' in the elliptic and parabolic cases;*

(2) *$D'_+ = D_+ + P$ and $D'_- = D_- - P$ for a principal divisor P in the hyperbolic case.*

Thus in order to find all equivalence classes of effective $\mathbb{C}^*$-actions one needs to classify all possible DPD-presentations for a fixed surface S up to a linear

[5]Recall that a cyclic quotient singularity of type (m, e) is biholomorphic to the singularity of $\mathbb{C}^2/\mathbb{Z}_m$ where $\mathbb{Z}_m$ is the cyclic group generated by a primitive m-root ζ of unity that acts on $\mathbb{C}^2_{x,y}$ by formula $(x, y) \to (\zeta x, \zeta^e y)$ with e and m being relatively prime.

equivalence (in the parabolic and elliptic case) or up to a principal divisor (in the hyperbolic case). Flenner and Zaidenberg established a uniqueness of DPD-presentations for non-toric surfaces with a non-trivial Makar-Limanov invariant [**FlZa05a**], [**FlZa05b**].

THEOREM 11.4. *Let Φ and Φ' be effective $\mathbb{C}^*$-actions on a non-toric affine surface S such that $\mathrm{AK}(S) \neq \mathbb{C}$. Then, replacing Φ by Φ^{-1} if necessary, one can choose an isomorphism between the curves C and C' so that D and D' (resp. $D_\pm$ and $D'_\pm$) in the DPD-presentations satisfy condition (1) (resp. (2)) of Theorem 11.3 (in particular, there are at most two equivalence classes of effective $\mathbb{C}^*$-actions on S). Moreover, Φ (or Φ^{-1}) is conjugated to Φ' by an element of a $\mathbb{C}_+$-action*[6].

For a non-toric surface with a trivial Makar-Limanov invariant effective $\mathbb{C}^*$-actions are automatically hyperbolic [**FKZ07b**] and the curve C is always isomorphic to $\mathbb{C}$. In this case the situation is much more complicated and Russell found examples of such smooth surfaces with more than two equivalence classes of effective $\mathbb{C}^*$-actions.

DEFINITION 11.5. Let $\mathbb{F}_n \to \mathbb{P}^1$ be a Hirzebruch surface over $\mathbb{P}^1$ and L be its section with $L^2 = k+1$. If L is ample (i.e. $k \geq n$) then $\mathbb{F}_n \setminus L$ is a Danilov-Gizatullin surface[7].

THEOREM 11.6. ([**FKZ07b**]) *There are exactly k equivalence classes of effective $\mathbb{C}^*$-actions on a surface from Definition 11.5. More precisely, every action is hyperbolic and its DPD-presentation is of form $C \simeq \mathbb{C}$, $D_+ = -\frac{1}{r}[p_0]$, $D_- = -\frac{1}{k+1-r}[p_1]$ for distinct $p_0, p_1 \in C$ and $r = 1, \dots, k$.*

Danilov-Gizatullin surfaces are contained in the wider class of Gizatullin surfaces.

DEFINITION 11.7. A normal affine algebraic surface S is a Gizatullin one if it has a simple normal crossing completion $\bar{S}$ such its boundary divisor $\bar{S} \setminus S$ consists of rational curves and the dual weighted graph Γ of this divisor is linear[8] (this graph is called a zigzag). If S admits also a nontrivial $\mathbb{C}^*$-action we say that it is a Gizatullin $\mathbb{C}^*$-surface.

REMARK 11.8. It is worth mentioning that with the exception of $\mathbb{C} \times \mathbb{C}^*$ the set of Gizatullin surfaces coincides with the set of surfaces with a trivial Makar-Limanov

[6] Thus, if $\mathrm{AK}(S) = \mathbb{C}[S]$ then either Φ or Φ^{-1} is equal to Φ'.

[7] We use this name because of the theorem of Danilov and Gizatullin which states that two of such surfaces are isomorphic if and only if they have the same k in their construction [**DaGi**]. This fact was crucial for Russell.

[8] Recall that each irreducible component of $\bar{S} \setminus S$ is viewed as a vertex of Γ with its weight being the selfintersection number of the component, and two vertices are connected by an edge if the corresponding components meet.

invariant. Each of these surfaces is also quasi-homogeneous, i.e. it contains an orbit of the algebraic automorphism group whose complement is at most finite (the only quasi-homogeneous surface that is not a Gizatullin one is $\mathbb{C}^* \times \mathbb{C}^*$). In particular, such surfaces possess huge automorphism groups. That is why the Classification Problem was the most difficult in this case.

A simple normal crossing completion $\bar{S}$ of a non-toric S can be always chosen so that its zigzag is standard, i.e. Γ consists of vertices C_i, $i = 0, 1, \ldots, n$ (where $n \geq 1$) of weights w_i such that $w_0 = w_1 = 0$ and $w_i \leq -2$ for the rest of indices. The standard zigzag is unique up to the reversion of its nonzero part and smooth Gizatullin $\mathbb{C}^*$-surfaces can be divided into the following three collections ([**FKZ07b**]):
(1) Danilov-Gizatullin surfaces (for these surfaces $w_i = -2$ for all $i \geq 2$ in their standard zigzags);
(2) special $\mathbb{C}^*$-Gizatullin surfaces whose standard zigzags have $w_i = -2$ for all $i \geq 2$ except for one index $2 \leq k \leq n$;
(3) the rest of smooth Gizatullin $\mathbb{C}^*$-surfaces.

Collection (3) was classified in [**FKZa**], and we have the following uniqueness result.

Theorem 11.9. *For every surface S from the third collection up to interchange of D_+ and D_- any two DPD-presentations are equivalent, i.e. their divisors satisfy condition (2) of Theorem 11.3. In particular, there are at most two equivalence classes of $\mathbb{C}^*$-actions.*

The most difficult remaining case of the second collection will be settled in the coming paper [**FKZb**] which concludes the classification of effective $\mathbb{C}^*$-actions on smooth affine surfaces. Compared with the other cases the answer is really amazing but first we need a few remarks.

It can be shown [**FKZ07b**] that for any Gizatullin $\mathbb{C}^*$-surface the support of the fractional part $\{D_+\} = D_+ - \lfloor D_+ \rfloor$ of D_+ (resp. $\{D_-\}$ of D_-) is concentrated at most at one point p_- (resp. p_+). For special Gizatullin surfaces these fractional parts are both nonzero and $p_+ \neq p_-$ unless $k = 2$ or n in the description of collection (2). In the case of $k = 2$ or n either $\{D_-\}$ or $\{D_+\}$ vanishes and the support of the fractional part of $D_+ + D_-$ is concentrated at most at one point p (when $n = 2$ both fractional parts vanish).

Theorem 11.10. *Let S and S' be special Gizatullin surfaces with the same standard zigzag and with DPD-presentations D_+, D_- and D'_+, D'_- on curves $C \simeq \mathbb{C}$ and $C' \simeq \mathbb{C}$ respectively. Then S and S' are isomorphic if and only if an isomorphism $C \simeq C'$ can be chosen so that $\lfloor D_+ + D_- \rfloor = \lfloor D'_+ + D'_- \rfloor =: E$. In particular, (by Theorem 11.3) up to automorphisms of $\mathbb{C}$ preserving E each equivalence class of $\mathbb{C}^*$-actions on S is determined uniquely by the continuous parameters p_+ and p_-*

in the case of $k \neq 2$ or n, and by the continuous parameter p in the case of $k = 2$ or $k = n > 2$.

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