ALGEBRAIC VOLUME DENSITY PROPERTY OF AFFINE ALGEBRAIC MANIFOLDS

SHULIM KALIMAN AND FRANK KUTZSCHEBAUCH

Abstract. We introduce the notion of algebraic volume density property for affine algebraic manifolds and prove some important basic facts about it, in particular that it implies the volume density property. The main results of the paper are producing two big classes of examples of Stein manifolds with volume density property. One class consists of certain affine modifications of $\mathbb{C}^n$ equipped with a canonical volume form, the other is the class of all Linear Algebraic Groups equipped with the left invariant volume form.

1. Introduction

In this paper we study a less developed part of the Andersén-Lempert theory ([1], [3], [8], [22], [20], [21],[7]) namely the case of volume preserving maps. Recall that Andersén-Lempert theory describes complex manifolds such that among other things the local phase flows on their holomorphically convex compact subsets can be approximated by global holomorphic automorphisms which leads to construction of holomorphic automorphisms with prescribed local properties. Needless to say that this implies some remarkable consequences for such manifolds (e.g., see [22], [23], [12]). It turns out that a complex manifold has such approximations if it possesses the following density property introduced by Varolin.

1.1. **Definition.** A complex manifold X has the density property if in the compact-open topology the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X. An affine algebraic manifold X has the algebraic density property if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by completely integrable algebraic vector fields on it coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}(X)$ of all algebraic vector fields on it (clearly, the algebraic density property implies the density property).

The algebraic density property was established for a wide variety of affine algebraic manifolds, including all connected linear algebraic groups except for $\mathbb{C}_+$ and complex

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tori by the authors [12]. Furthermore, in the coming paper of Donzelli, Dvorsky and the first author [6] it will be extended to connected homogeneous spaces of form G/R, where R is a reductive subgroup of a linear algebraic group G, with exception of $\mathbb{C}_+$ and complex tori.

However Andersén, Lempert, Forstneric, Rosay and Varolin considered also another property which has similar consequences for automorphisms preserving a volume form.

1.2. **Definition.** Let a complex manifold X be equipped with a holomorphic volume form ω (i.e. ω is a nowhere vanishing section of the canonical bundle). We say that X has the volume density property with respect to ω if in the compact-open topology the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}^{\omega}$ generated by completely integrable holomorphic vector fields ν such that $L_\nu(\omega) = 0$ (where L_ν is the Lie derivative), is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}^{\omega}(X)$ of all holomorphic vector fields that annihilate ω (note that condition $L_\nu(\omega) = 0$ is equivalent to the fact that ν is of zero ω-divergence).

Compared with the density property, the class of complex manifolds with established volume density property has been quite narrow. It was essentially described by the original result of Andersén and Lempert [1], [3] who proved it for Euclidean spaces plus a few other examples found by Varolin [22]. In particular he proved that $SL_2(\mathbb{C})$ has volume density property with respect to the Haar form but he was unable to decide whether the following hypersurface given by a similar equation like $SL_2(\mathbb{C})$

$$\Sigma^3 = \{(a, b, c, d) \in \mathbb{C}^4 \ : \ a^2c - bd = 1\}$$

had volume density property or not ([24] section 7).

In order to deal with this lack of examples we introduce like in the previous pattern the following.

1.3. **Definition.** If X is affine algebraic we say that X has the algebraic volume density property with respect to an algebraic volume form ω if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}^{\omega}$ generated by completely integrable algebraic vector fields ν such that $L_\nu(\omega) = 0$, coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}^{\omega}(X)$ of all algebraic vector fields that annihilate ω.

It is much more difficult to establish the algebraic volume density property than the algebraic density property. This is caused, perhaps, by the following difference which does not allow to apply the most effective criterion for the algebraic density property (see [13]): $\mathrm{VF}_{\mathrm{alg}}^{\omega}(X)$ is not a module over the ring $\mathbb{C}[X]$ of regular functions on X while $\mathrm{VF}_{\mathrm{alg}}(X)$ is. Furthermore, some features that are straightforward for the algebraic density property are not at all clear in the volume-preserving case. For instance, it is not quite obvious that the algebraic volume density property implies the volume density property and that the product of two manifolds with algebraic volume density property has again the algebraic volume density property. We shall show in this paper the validity of these two facts among other results that enable us to enlarge the class of examples of Stein manifolds with the volume density property substantially. In particular we establish the following.

Theorem 1. *Let X' be a hypersurface in $\mathbb{C}^{n+2}_{u,v,\bar{x}}$ given by an equation of form $P(u,v,\bar{x}) = uv - p(\bar{x}) = 0$ where p is a polynomial on $\mathbb{C}^n_{\bar{x}}$ with a smooth reduced zero fiber C such that in the case of $n \geq 2$ the reduced cohomology $\hat{H}^{n-2}(C,\mathbb{C}) = 0$ (for $n = 1$ no additional assumption is required). Let Ω be the standard volume form on $\mathbb{C}^{n+2}$ and ω' be a volume form on X' such that $\omega' \wedge \mathrm{d}P = \Omega|_{X'}$. Then X' has the algebraic ω'-volume density property.*

This gives, of course, an affirmative answer to Varolin's question mentioned before. The next theorem is our main result.

Theorem 2. *Let G be a linear algebraic group. Then G has the algebraic volume density property with respect to the left (or right) invariant volume form.*

Let us describe briefly the content of the paper and the main steps in the proof of these facts.

In Section 2 we remind some standard facts about divergence.

In Section 3 we deal with Theorem 1 in a slightly more general situation. Namely we consider a hypersurface X' in $X \times \mathbb{C}^2_{u,v}$ given by an equation $P := uv - p(x) = 0$ where X is a smooth affine algebraic variety and $p(x)$ is a regular function on X. We suppose that X is equipped with a volume form ω and establish the existence of a volume form ω' on X' such that $\Omega|_{X'} = \mathrm{d}P \wedge \omega'$ where $\Omega = \mathrm{d}u \wedge \mathrm{d}v \wedge \omega$. Then we prove (Proposition 3.3) that X' has the ω'-volume algebraic density property provided two technical conditions (A1) and (A2) hold.

Condition (A2) is easily verifiable for X' and condition (A1) is equivalent to the following (Lemma 3.5): the space of algebraic vector fields on X with ω-divergence zero, that are tangent to the zero fiber C of p, is generated by vector fields of form $\nu_1(fp)\nu_2 - \nu_2(fp)\nu_1$ where ν_1 and ν_2 are commuting completely integrable algebraic vector fields of ω-divergence zero on X and f is a regular function on X.

Then we notice the duality between the spaces of zero ω-divergence vector fields on X and closed $(n-1)$-forms on X which is achieved via the inner product that assigns to each vector field ν the $(n-1)$-form $\iota_\nu(\omega)$ (Lemma 3.6). This duality allows to reformulate condition (A1) as the following:

(i) the space of algebraic $(n-2)$-forms on X is generated by the forms of type $\iota_{\nu_1}\iota_{\nu_2}(\omega)$ where ν_1 and ν_2 are as before; and

(ii) the outer differentiation sends the space of $(n-2)$-forms on X that vanish on C to the set of $(n-1)$-form whose restriction to C yield the zero $(n-1)$-form on C.

In the case of X isomorphic to a Euclidean space (i) holds automatically with ν_1 and ν_2 running over the set of partial derivatives.

If the reduced cohomology $\hat{H}^{n-2}(C,\mathbb{C}) = 0$ and also $H^n(X,\mathbb{C}) = 0$ the validity of (ii) is a consequence of the Grothendieck theorem (see Proposition 3.9) that states that the complex cohomology can be computed via the De Rham complex of algebraic forms on a smooth affine algebraic variety which concludes the proof of Theorem 1.

We end Section 3 with an important corollary of Theorem 1 which will be used in the proof of Theorem 2 : the groups $SL_2(\mathbb{C})$ (already proved by Varolin as mentioned above) and $PSL_2(\mathbb{C})$ have the algebraic volume density property with respect to the invariant volume (Propositions 3.11 and 3.12). The proof is based on the fact that $SL_2(\mathbb{C})$ is isomorphic to the hypersurface in $\mathbb{C}^4_{u,v,x_1,x_2}$ given by $uv - x_1x_2 - 1 = 0$.

Section 4 contains two general facts about the algebraic volume density property with short but non-trivial proofs. The first of them (Proposition 4.1) says that the algebraic volume density property implies the volume density property (in the holomorphic sense). It is also based on the Grothendieck theorem mentioned before. The second one (Proposition 4.3) states that the product $X \times Y$ of two affine algebraic manifolds X and Y with the algebraic volume density property (with respect to volumes ω_X and ω_Y) has also the algebraic volume density property (with respect to $\omega_X \times \omega_Y$). As a consequence of this result we establish the algebraic volume density property for all tori which was also established earlier by Varolin [22] (recall that the density property is not established for higher dimensional tori yet and the algebraic density property does not hold for these objects [2]).

We start Section 5 with discussion of a phenomenon which makes the proof of Proposition 4.3 about the algebraic volume density of $X \times Y$ non-trivial and prevents us from spreading it directly to locally trivial fibrations. More precisely, consider the subspace F_Y of $\mathbb{C}[Y]$ generated by the images $\operatorname{Im}\delta$ with δ running over $\operatorname{Lie}^{\omega}_{\text{alg}}(Y)$. In general $F_Y \neq \mathbb{C}[Y]$ and the absence of equality here is the source of difficulties. Nevertheless one can follow the pattern of the proof of Proposition 4.3 in the case of fibrations when the span of F_Y and constants yields $\mathbb{C}[Y]$. This is so-called property (C) for Y which turns out to be true for SL_2 and PSL_2. We introduce also the notion of a volume fibration $p : W \to X$ which a generalization of the product situation and has nicely related volume forms of the fiber of p, the base X, and the total space W. The main result in Section 5 is Theorem 4 saying that the total space of a volume fibration satisfying some additional assumptions (such as the algebraic density property and property (C) for the fiber and the algebraic volume density property for the base) has the algebraic volume density property as well.

Section 6 contains basic knowledge about invariant volume forms on linear algebraic groups. Of further importance will be Corollary 6.8 about the Mostow decomposition of a linear algebraic group as the product of Levi reductive subgroup and its unipotent radical. We end it with an important example of a volume fibration that (as we shall see later) satisfies the assumption of Theorem 4 - the quotient map of a reductive group by its Levi semi-simple subgroup (see Lemma 6.11).

Section 7 prepares the proof of Theorem 2 in the case of a semi-simple group. The central notion discussed in that section is a p-compatible vector field $\sigma' \in \operatorname{Lie}^{\omega}_{\text{alg}}(W)$ for a locally trivial fibration $p : W \to X$. Its most important property is that $\operatorname{Span}\operatorname{Ker}\sigma' \cdot \operatorname{Ker}\delta'$ coincides with the algebra $\mathbb{C}[W]$ of regular functions for any $\delta' \in \operatorname{VF}^{\omega}_{\text{alg}}(W)$ tangent to the fibers of p. It is established that for any at least three-dimensional semi-simple group G and its SL_2- or PSL_2-subgroup S corresponding to a root of the Dynkin diagram the fibration $q : G \to G/S$ admits a sufficiently large family q-compatible vector fields. The existence of such a family (in combination with the fact that SL_2 and PSL_2 have the algebraic volume density property and property (C)) leads to the claim that q satisfies all assumptions of Theorem 4 but the algebraic volume density for the base. This enables us to use properties of such fibrations established earlier in Proposition 5.16 (but not Theorem 4 since it is unknown whether G/S has the algebraic volume density property).

Section 8 contains the proof of Theorem 2. The general case follows easily from a semi-simple one (via Lemma 6.11, Theorem 4, and Corollary 6.8). The idea of the proof in the latter case is the following. We consider SL_2- or PSL_2-subgroups $S_0, \ldots, S_m$ corresponding to the simple roots of a Dynkin diagram of a semi-simple group G and fibrations $p_i : G \to G/S_i$ with $i = 0, \ldots, m$. Using results of Section 7 we establish that there is a sufficiently big collection Θ of completely integrable fields θ of zero divergence that are of p_i-compatible for every i. Furthermore up to an element of $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(G)$ every algebraic vector field of zero divergence can be presented as a finite sum $\sum h_i \theta_i$ where $\theta_i \in \Theta$ and $h_i \in \mathbb{C}[G]$. Then we consider a standard averaging operator av_j on $\mathbb{C}[G]$ that assigns to each $h \in \mathbb{C}[G]$ a regular function $\mathrm{av}_j(h)$ invariant with respect to the natural S_j-action on G and establish the following relation: $\sum h_i\theta_i \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(G)$ if and only if $\sum \mathrm{av}_j(h_i)\theta_i \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(G)$. We show also that a consequent application of operators $\mathrm{av}_0, \ldots, \mathrm{av}_m$ leads to a function invariant with respect to each S_j, $j = 0, \ldots, m$. Since the only functions invariant under the natural actions of all such subgroups are constants we see that $\sum h_i\theta_i \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(G)$ because $\sum c_i\theta_i \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(G)$ for constant coefficients c_i which concludes the proof of Theorem 2.

The appendix contains definition of strictly semi-compatible fields and refinements of two Lemmas about it from our previous work [12].

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2. Preliminaries

Recall that a holomorphic vector field $\nu \in \mathrm{VF}_{hol}(\mathbb{C}^n)$ is completely (or globally) integrable if for any initial value $z \in \mathbb{C}^n$ there is a global holomorphic solution of the ordinary differential equation

$$\dot{\gamma}(t) = \nu(\gamma(t)), \quad \gamma(0) = z. \tag{1}$$

In this case the phase flow (i.e. the map $\mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(t, z) \mapsto \gamma_z(t)$) is a holomorphic action of the additive group $\mathbb{C}_+$ on $\mathbb{C}^n$, where index z in γ_z denotes the dependence on the initial value. It is worth mentioning that this action is not necessarily algebraic in the case of an algebraic vector field $\nu \in \mathrm{VF}_{\mathrm{alg}}(\mathbb{C}^n)$.

A holomorphic (resp. algebraic) volume form on a complex (resp. affine algebraic) manifold X of dimension n is a nowhere vanishing holomorphic (resp. algebraic) n-form. Let us discuss some simple properties of the divergence $\mathrm{div}_\omega(\nu)$ of a vector field ν on X with respect to this volume form ω. The divergence is defined by the equation

$$\mathrm{div}_\omega(\nu)\omega = L_\nu(\omega) \tag{2}$$

where L_ν is the Lie derivative. Here is another useful formula

$$\mathrm{div}_\omega(f\nu) = f\,\mathrm{div}_\omega(\nu) + \nu(f) \tag{3}$$

for any holomorphic function f on X. Furthermore, for any vector fields ν_1, ν_2 on X we have the following relation between divergence and Lie bracket

$$\mathrm{div}_\omega([\nu_1,\nu_2]) = L_{\nu_1}(\mathrm{div}_\omega(\nu_2)) - L_{\nu_2}(\mathrm{div}_\omega(\nu_1)). \tag{4}$$

In particular, when $\mathrm{div}_\omega(\nu_1) = 0$ we have

$$\mathrm{div}_\omega([\nu_1,\nu_2]) = L_{\nu_1}(\mathrm{div}_\omega(\nu_2)). \tag{5}$$

2.1**. Lemma.** *Let Y be a Stein complex manifold with a volume form Ω on it, and X be a submanifold of Y which is a strict complete intersection (that is, the defining ideal of X is generated by holomorphic functions $P_1, \ldots, P_k$ on Y, where k is the codimension of X in Y). Suppose that ν is a vector field on X and μ is its extension to Y such that $\mu(P_i) = 0$ for every $i = 1, \ldots, k$. Then*

(i) *there exists a volume form ω on X such that $\Omega|_X = \mathrm{d}P_1 \wedge \ldots \wedge \mathrm{d}P_k \wedge \omega$; and*

(ii) $\mathrm{div}_\omega(\nu) = \mathrm{div}_\Omega(\mu)|_X$.

Proof. Let $x_1, \ldots, x_n$ be a local holomorphic coordinate system in a neighborhood of a point in X. Then $P_1, \ldots, P_k, x_1, \ldots, x_n$ is a local holomorphic coordinate system in a neighborhood of this point in Y. Hence in that neighborhood $\Omega = h\mathrm{d}P_1 \wedge \ldots \wedge \mathrm{d}P_k \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ where h is a holomorphic function. Set $\omega = h|_X \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. This is the desired volume form in (i).

Recall that $L_\nu = \mathrm{d} \circ \imath_\nu + \imath_\nu \circ \mathrm{d}$ where $\imath_\nu$ is the inner product with respect to ν ([15], Chapter 1, Proposition 3.10). Since $\mu(P_i) = 0$ we have $L_\mu(dP_i) = 0$. Hence by formula (2) we have $\mathrm{div}_\Omega(\mu)\Omega|_X =$

$$L_\mu\Omega|_X = L_\mu(\mathrm{d}P_1 \wedge \ldots \wedge \mathrm{d}P_k \wedge \omega)|_X = \mathrm{d}P_1 \wedge \ldots \wedge \mathrm{d}P_k|_X \wedge L_\nu\omega + L_\mu(\mathrm{d}P_1 \wedge \ldots \wedge \mathrm{d}P_k)|_X \wedge \omega$$
$$= \mathrm{div}_\omega(\nu)(\mathrm{d}P_1 \wedge \ldots \wedge \mathrm{d}P_k)|_X \wedge \omega = \mathrm{div}_\omega(\nu)\Omega|_X$$

which is (ii).

□

2.2**. Remark.**

(1) Lemma 2.1 remains valid in the algebraic category

(2) Furthermore, it enables us to compute the divergence of a vector field on X via the divergence of a vector field extension on an ambient space. It is worth mentioning that there is another simple way to compute divergence on X which leads to the same formulas in Lemma 2.5 below. Namely, X that we are going to consider will be an affine modification $\sigma : X \to Z$ of another affine algebraic manifold Z with a volume form ω_0 (for definitions of affine and pseudo-affine modifications see [14]). In particular, for some divisors $D \subset Z$ and $E \subset X$ the restriction of σ produces an isomorphism $X \setminus E \to Z \setminus D$. One can suppose that D coincides with the zero locus of a regular (or holomorphic) function α on Z. In the situation, we are going to study, the function $\tilde{\alpha} = \alpha \circ \sigma$ has simple zeros on E. Consider the form $\sigma^*\omega_0$ on X. It may vanish on E only. Dividing this form by some power $\tilde{\alpha}^k$ we get a volume form on X. In order

to compute divergence of a vector field on X it suffices to find this divergence on the Zariski open subset $X \setminus E \simeq Y \setminus D$, i.e. we need to compute the divergence of a vector field ν on $Y \setminus D$ with respect to a volume form $\beta\omega_0$ where $\beta = \alpha^{-k}$. The following formula relates it with the divergence with respect to ω_0:

$$\operatorname{div}_{\beta\omega_0}(\nu) = \operatorname{div}_{\omega_0}(\nu) + L_\nu(\beta)/\beta. \tag{6}$$

In the cases, we need to consider, β will be often in the kernel of ν, i.e. $\operatorname{div}_{\beta\omega_0}(\nu) = \operatorname{div}_{\omega_0}(\nu)$ in these cases.

The condition in Lemma 2.1 that an algebraic field ν on X has an extension μ on Y with $\mu(P_i) = 0$ is also very mild. We consider it in the case of hypersurfaces only.

2.3. **Lemma.** *Let X be a smooth hypersurface in a complex Stein (resp. affine algebraic) manifold Y given by zero of a reduced holomorphic (resp. algebraic) function P on Y. Then every holomorphic (resp. algebraic) vector field ν on X has a similar extension μ to Y such that $\mu(P) = 0$.*

Proof. Consider, for instance the algebraic case, i.e. P belongs to the ring $\mathbb{C}[Y]$ of regular functions on Y. Since μ must be tangent to X we see that $\mu(P)$ vanishes on X, i.e. $\mu(P) = PQ$ where $Q \in \mathbb{C}[Y]$. Any other algebraic extension of ν is of form $\tau = \mu - P\theta$ where $\theta \in \mathrm{VF}_{\mathrm{alg}}(Y)$. Thus if $\theta(P) = Q$ then we are done.

In order to show that such θ can be found consider the set $M = \{\theta(P) | \theta \in \mathrm{VF}_{\mathrm{alg}}(Y)\}$. One can see that M is an ideal of $\mathbb{C}[Y]$. Therefore, it generates a coherent sheaf $\mathcal{F}$ over Y. The restriction $Q|_{Y\setminus X}$ is a section of $\mathcal{F}|_{Y\setminus X}$ because $Q = \mu(P)/P$. Since X is smooth for every point $x \in X$ there are a Zariski open neighborhood U in Y and an algebraic vector field ∂ such that $\partial(P)$ does not vanish on U. Hence $Q|_U$ is a section of $\mathcal{F}|_U$. Since $\mathcal{F}$ is coherent this implies that Q is a global section of $\mathcal{F}$ and, therefore, $Q \in M$ which is the desired conclusion. $\square$

2.4. **Terminology and Notation.** In the rest of this section X is a closed affine algebraic submanifold of $\mathbb{C}^n$, ω is an algebraic volume form on X, p is a regular function on X such that the divisor $p^*(0)$ is smooth reduced, X' is the hypersurface in $Y = \mathbb{C}^2_{u,v} \times X$ given by the equation $P := uv - p = 0$.[1] Note that X' is smooth and, therefore, Lemma 2.3 is applicable. We shall often use the fact that every regular function f on X' can be presented uniquely as the restriction of a regular function on Y of the form

$$f = \sum_{i=1}^{m} (a_i u^i + b_i v^i) + a_0 \tag{7}$$

where $a_i = \pi^*(a_i^0), b_i = \pi^*(b_i^0)$ are lift-ups of regular functions a_i^0, b_i^0 on X via the natural projection $\pi : Y \to X$ (as we mentioned by abusing terminology we shall say that a_i and b_i themselves are regular functions on X).

[1]By abusing notation we treat p in this formula as a function on Y, and, if necessary, we treat it as a function on X'. Furthermore, by abusing notation, for any regular function on X we denote its lift-up to Y or X' by the same symbol.

Let $\Omega = \mathrm{d}u \wedge \mathrm{d}v \wedge \omega$, i.e. it is a volume form on Y. By Lemma 2.1 there is a volume form ω' on X' such that $\Omega|_{X'} = \mathrm{d}P \wedge \omega'$. Furthermore, for any vector field μ such that $\mu(P) = 0$ and $\nu' = \mu|_{X'}$ we have $\mathrm{div}_{\omega'}(\nu') = \mathrm{div}_\Omega(\mu)|_X$. Note also that any vector field ν on X generates a vector field κ on Y that annihilates u and v. We shall always denote $\kappa|_{X'}$ by $\tilde{\nu}$. It is useful to note for further computations that $u^i\pi^*(\mathrm{div}_\omega(\nu)) = \mathrm{div}_\Omega(u^i\kappa)$ for every $i \geq 0$. Note also that every algebraic vector field λ on X' can be written uniquely in the form

$$\lambda = \tilde{\mu}_0 + \sum_{i=1}^{m}(u^i\tilde{\mu}_i^1 + v^i\tilde{\mu}_i^2) + f_0\partial/\partial u + g_0\partial/\partial v \tag{8}$$

where μ_0, μ_i^j are algebraic vector fields on X, and f_0, g_0 are regular functions on X'.

For any algebraic manifold Z with a volume form ω we denote by $\mathrm{Lie}_{\mathrm{alg}}(Z)$ (resp. $\mathrm{Lie}_{\mathrm{alg}}^\omega(Z)$) the Lie algebra generated by algebraic completely integrable vector fields on Z (resp. that annihilates ω) and by $\mathrm{VF}_{\mathrm{alg}}(Z)$ we denote the Lie algebra of all algebraic vector fields on Z. We have a linear map

$$\widetilde{\mathrm{Pr}} : \mathrm{VF}_{\mathrm{alg}}(X') \to \mathrm{VF}_{\mathrm{alg}}(X)$$

defined by $\widetilde{\mathrm{Pr}}(\lambda) = \mu_0$ where λ and μ_0 are from formula (8). As it was mentioned in [13] the following facts are straightforward calculations that follow easily from Lemma 2.1.

2.5. **Lemma.** *Let ν_1, ν_2 be vector fields on X, and f be a regular function on X. For $i \geq 0$ consider the algebraic vector fields*

$$\nu_1' = u^{i+1}\tilde{\nu}_1 + u^i\nu_1(p)\partial/\partial v, \quad \nu_2' = v^{i+1}\tilde{\nu}_2 + v^i\nu_2(p)\partial/\partial u$$

and $\mu_f = f(u\partial/\partial u - v\partial/\partial v)$ on Y. Then

(i) *ν_i' and μ_f are tangent to X' (actually they are tangent to fibers of $P = uv - p(x)$), i.e., they can be viewed as vector fields on X';*

(ii) *μ_f is always completely integrable on X', and ν_i' is completely integrable on X' if ν_i is completely integrable on X;*

(iii) $\mathrm{div}_{\omega'}(\mu_f) = 0$, $\mathrm{div}_{\omega'}(\nu_1') = u^{i+1}\,\mathrm{div}_\omega(\nu_1)$, $\mathrm{div}_{\omega'}(\nu_2') = v^{i+1}\,\mathrm{div}_\omega(\nu_2)$, *and*

$$\mathrm{div}_{\omega'}([\mu_f, \nu_1']) = (i+1)u^{i+1}f\,\mathrm{div}_\omega(\nu_1),\ \mathrm{div}_{\omega'}([\nu_2', \mu_f]) = (i+1)v^{i+1}f\,\mathrm{div}_\omega(\nu_2);$$

(iv) *we have the following Lie brackets*

$$[\mu_f, \nu_1'] = (i+1)u^{i+1}f\tilde{\nu}_1 + \alpha_1\partial/\partial u + \beta_1\partial/\partial v,$$
$$[\nu_2', \mu_f] = (i+1)v^{i+1}f\tilde{\nu}_2 + \alpha_2\partial/\partial u + \beta_2\partial/\partial v,$$

where α_i and β_i are some regular functions on X';

(v) *more precisely, if $i = 0$ in formulas for ν_1' and ν_2' then*

$$[\mu_f, \nu_1'] = fu\tilde{\nu}_1 - u^2\nu_1(f)\partial/\partial u + \nu_1(fp)\partial/\partial v,$$
$$[\nu_2', \mu_f] = fv\tilde{\nu}_2 - v^2\nu_2(f)\partial/\partial v + \nu_2(fp)\partial/\partial u;$$

and

$$\widetilde{\Pr}([[\mu_f,\nu_1'],\nu_2']) = \nu_1(fp)\nu_2 - \nu_2(fp)\nu_1 + fp[\nu_1,\nu_2]. \tag{9}$$

3. The proof of Theorem 1.

3.1**. Additional Notation.** For every affine algebraic manifold Z let $\mathbb{C}[Z]$ be the algebra of its regular functions, $\mathrm{IVF}_{\mathrm{alg}}(Z)$ be the set of completely integrable algebraic vector fields on Z. If there is a volume form ω on Z then we denote by $\mathrm{Div}_Z : \mathrm{VF}_{\mathrm{alg}}(Z) \to \mathbb{C}[Z]$ the map that assigns to each vector field its divergence with respect to ω, and set $\mathrm{IVF}_{\mathrm{alg}}^{\omega}(Z) = \mathrm{Ker}\ \mathrm{Div}_Z \cap \mathrm{IVF}_{\mathrm{alg}}(Z)$, $\mathrm{VF}_{\mathrm{alg}}^{\omega}(Z) = \mathrm{Ker}\ \mathrm{Div}_Z \cap \mathrm{VF}_{\mathrm{alg}}(Z)$. For a closed submanifold C of Z denote by $\mathrm{VF}_{\mathrm{alg}}^{\omega}(Z,C)$ the Lie algebra of algebraic vector fields of divergence zero on Z that are tangent to C. Formula (7) yields a monomorphism of vector spaces $\iota : \mathbb{C}[X'] \hookrightarrow \mathbb{C}[Y]$ and the natural embedding $X \hookrightarrow X \times (0,0) \subset Y$ generates a projection $\Pr : \mathbb{C}[Y] \to \mathbb{C}[X]$. Note that $\Pr(\iota(f)) = a_0$ in the notation of formula (7).

3.2**. Lemma.** *Let λ be a vector field on $X' \subset X \times \mathbb{C}^2_{u,v}$ given by formula (8). Suppose that ω_0 is a volume form on X and a volume form ω on X' coincides with the pull-back of the volume form $\omega_1 := (\omega_0 \wedge \mathrm{d}u)/u$ on $Z := X \times \mathbb{C}^*_u$ under the natural projection (i.e. ω constructed as in Remark 2.2). Then* $\Pr(\mathrm{div}_{\omega}(\lambda)) = \mathrm{div}_{\omega_0}(\mu_0)$. *In particular, if* $\mathrm{div}_{\omega}\,\lambda = 0$ *then* $\mathrm{div}_{\omega_0}(\mu_0) = 0$.

Proof. The natural projection $\sigma : X' \to Z$ is an affine modification whose restriction over $X \times \mathbb{C}^*_u$ is an isomorphism. Hence λ is the pull-back of the following vector field

$$\kappa = \tilde{\mu}_0 + \sum_{i=1}^{m} u^i(\tilde{\mu}_i^1 + \tilde{\mu}_i^2/p^i) + f_0\partial/\partial u$$

on Z. Thus it suffices to show that $\mathrm{div}_{\omega_0}(\mu_0) = T_0(\mathrm{div}_{\omega_1}(\kappa))$ where $T_0 : \mathbb{C}(X)[u,u^{-1}] \to \mathbb{C}(X)$ assigns to each Laurent polynomial in u its constant term. By (6) $\mathrm{div}_{\omega_1}(\kappa) = \mathrm{div}_{\omega_0\wedge du}(\kappa) - \kappa(u)/u = \mathrm{div}_{\omega_0\wedge du}(\kappa) - f_0/u$. Hence

$$T_0(\mathrm{div}_{\omega_1}(\kappa)) = \mathrm{div}_{\omega_0}(\mu_0) + T_0(\partial f_0/\partial u) - T_0(f_0/u).$$

The desired conclusion follows now from the obvious fact that $T_0(\partial f_0/\partial u) = T_0(f_0/u)$. □

3.3**. Proposition.** *Let C be the smooth zero locus of p in X. Suppose also that the following conditions hold:*

(A1) *the linear space* $\mathrm{VF}_{\mathrm{alg}}^{\omega}(X,C)$ *is generated by vector fields that are of the form* $\widetilde{\Pr}([[\mu_f,\nu_1'],\nu_2'])$ *where μ_f and ν_i' are as in formula (9) from Lemma 2.5 with* $\nu_i \in \mathrm{IVF}_{\mathrm{alg}}^{\omega}(X)$*;*

(A2) $\mathrm{VF}_{\mathrm{alg}}(X)$ *is generated by* $\mathrm{IVF}_{\mathrm{alg}}^{\omega}(X)$ *as a module over* $\mathbb{C}[X]$.

Then $\mathrm{Lie}_{\mathrm{alg}}^{\omega}(X')$ *coincides with* $\mathrm{VF}_{\mathrm{alg}}^{\omega}(X')$*, i.e., X' has the algebraic volume density property.*

Proof. Let λ, f_0, and g_0 be as in formula (8) and $\Lambda = \iota(\lambda)$ be the extension of λ to Y also given by formula (8). By formula (7) f_0 and g_0 can be written uniquely in the form

$$f_0 = \sum_{i=1}^{m} (a_i u^i + b_i v^i) + a_0 \text{ and } g_0 = \sum_{i=1}^{m} (\hat{a}_i u^i + \hat{b}_i v^i) + \hat{a}_0$$

where $a_i, \hat{a}_i, b_i, \hat{b}_i \in \mathbb{C}[X]$.

Since Λ is a vector field tangent to $X' = P^{-1}(0)$ we have $\Lambda(P)|_{X'} = 0$. Thus $0 = \mathrm{Pr}(\iota(\Lambda(P)|_{X'})) = p(a_1 + \hat{b}_1) - \mu_0(p)$ (recall that $P = uv - p(x)$). Hence $\mu_0(p)$ vanishes on C, i.e. by Lemma 3.2 $\mu_0 \in \mathrm{VF}_{\mathrm{alg}}^{\omega}(X, C)$. Let $\mu_f, \nu_i' \in \mathrm{IVF}_{\mathrm{alg}}^{\omega}(X')$ be as in Lemma 2.5. Condition (A1) implies that adding elements of the form $[[\mu_f, \nu_1'], \nu_2']$ to λ we can suppose that $\mu_0 = 0$. Using condition (A2) and Lemma 2.5 (iv) we can make $\mu_i^j = 0$ by adding fields of the form $[\mu_f, \nu_i']$ with $\nu_i \in \mathrm{IVF}_{\mathrm{alg}}^{\omega}(X)$. Note that this addition leaves not only μ_0 equal to 0 but also $\mathrm{div}_{\omega'}(\lambda)$ equal to 0, since $\mathrm{div}_{\omega'}([\mu_f, \nu_i']) = 0$ as soon as $\mathrm{div}_{\omega}(\nu_i) = 0$. Hence $\lambda = f\partial/\partial u + g\partial/\partial v$ and $\Lambda(P)|_{X'} = fv + gu = 0$.

Using formula (7) one can see that f must be divisible by u, and g by v. That is, there exists a regular function h on X' for which $f = uh$ and $g = -vh$. Hence $\lambda = h(u\partial/\partial u - v\partial/\partial v)$. Note that $\Lambda(P) = 0$ now. Thus by Lemma 2.1

$$0 = \mathrm{div}_{\omega'}(\lambda) = \mathrm{div}_{\Omega}(\Lambda)|_{X'} = (u\frac{\partial h}{\partial u} - v\frac{\partial h}{\partial v})|_{X'}.$$

Taking h as in formula (7) we see that h is independent of u and v. Thus λ is integrable and of zero divergence by Lemma 2.5 (ii)-(iii).

□

3.4**. Remark.** The proof of Theorem 1 in the case of $n = 1$ (that is, when X' is a Danielevski surface) is complete. Indeed, we have $X = \mathbb{C}_z$. Any divergence-free vector field on $\mathbb{C}_z$ is of form $c\partial/\partial z$ where $c \in \mathbb{C}$. Thus if it vanishes on C it is identically zero, i.e. μ_0 from the above proof is zero which implies Condition (A1). Condition (A2) is also straightforward. Hence from now on we assume $n = \dim X \geq 2$.

Taking vector fields ν_1 and ν_2 from $\mathrm{IVF}_{\mathrm{alg}}^{\omega}(X)$ in formula (9) with $[\nu_1, \nu_2] = 0$ we have the following.

3.5**. Lemma.** *Condition (A1) in Proposition* 3.3 *holds if* $\mathrm{VF}_{\mathrm{alg}}^{\omega}(X, C)$ *is generated as a linear space by vector fields of the form* $\nu_1(fp)\nu_2 - \nu_2(fp)\nu_1$ *where the vector fields* $\nu_1, \nu_2 \in \mathrm{IVF}_{\mathrm{alg}}^{\omega}(X)$ *commute.*

It is more convenient for us to reformulate this new condition in terms of differential forms for which we need some extra facts. Let ι_ν be the inner product with a vector field ν on X. Recall the following relations between the outer differentiation d, the Lie derivative L_ν and ι_ν

$$L_\nu = \mathrm{d} \circ \iota_\nu + \iota_\nu \circ \mathrm{d} \text{ and } [L_{\nu_1}, \iota_{\nu_2}] = \iota_{[\nu_1, \nu_2]}. \tag{10}$$

Then by formula (2) we have

$$\mathrm{div}_{\omega}(\nu)\omega = \mathrm{d} \circ \iota_\nu(\omega) + \iota_\nu \circ \mathrm{d}(\omega) = \mathrm{d}(\iota_\nu(\omega)).$$

Thus we have the first statement of the following.

3.6. **Lemma.**

(1) *A vector field* ν *is of zero divergence if and only if the form* $\iota_\nu(\omega)$ *is closed.*

(2) *Furthermore, for a zero divergence field* ν *and every regular function* f *on* X *we have* $\mathrm{d}(\iota_{f\nu}(\omega)) = \nu(f)\omega$.

(3) *Let* $\nu_1, \nu_2 \in \mathrm{IVF}^\omega_{\mathrm{alg}}(X)$ *commute and* $\kappa = \nu_1(fp)\nu_2 - \nu_2(fp)\nu_1$. *Then* $\mathrm{d}(\iota_{\nu_1} \circ \iota_{\nu_2}(fp\omega)) = \iota_\kappa(\omega)$ *where* $p, f \in \mathbb{C}[X]$.

Proof. Indeed, by (10)

$$\mathrm{d}(\iota_{f\nu}(\omega)) = L_{f\nu}(\omega) - \iota_{f\nu} \circ \mathrm{d}(\omega) = L_{f\nu}(\omega) = \mathrm{div}_\omega(f\nu)\omega = (f\,\mathrm{div}_\omega(\nu) + \nu(f))\omega = \nu(f)\omega$$

which is (2).

Again by (10) we have

$$\mathrm{d} \circ \iota_{\nu_1} \circ \iota_{\nu_2}(fp\omega) = L_{\nu_1} \circ \iota_{\nu_2}(fp\omega) - \iota_{\nu_1} \circ \mathrm{d} \circ \iota_{\nu_2}(fp\omega).$$

Then

$$L_{\nu_1} \circ \iota_{\nu_2}(fp\omega) = L_{\nu_1}(fp)\iota_{\nu_2}(\omega) + fpL_{\nu_1} \circ \iota_{\nu_2}(\omega)$$

and

$$L_{\nu_1} \circ \iota_{\nu_2}(\omega) = \iota_{\nu_2} L_{\nu_1}(\omega) + \iota_{[\nu_1,\nu_2]}(\omega) = 0$$

since $[\nu_1, \nu_2] = 0$ and $L_{\nu_i}(\omega) = 0$. Similarly

$$\iota_{\nu_1} \circ \mathrm{d} \circ \iota_{\nu_2}(fp\omega) = L_{\nu_2}(fp)\iota_{\nu_1}(\omega) + fp\iota_{\nu_1} \circ L_{\nu_2}(\omega) - \iota_{\nu_1} \circ \iota_{\nu_2} \circ \mathrm{d}(fp\omega)) = L_{\nu_2}(fp)\iota_{\nu_1}(\omega).$$

Therefore,

$$\mathrm{d} \circ \iota_{\nu_1} \circ \iota_{\nu_2}(fp\omega) = L_{\nu_1}(fp)\iota_{\nu_2}(\omega) - L_{\nu_2}(fp)\iota_{\nu_1}(\omega) = \nu_1(fp)\iota_{\nu_2}(\omega) - \nu_2(fp)\iota_{\nu_1}(\omega)$$

which yields the desired conclusion.

□

Suppose that $\Omega^q(X)$ is the sheaf of algebraic q-forms on X, $\Omega^q_i(X)$ is its subsheaf that consists of forms that vanish on C with multiplicity at least i for $i \geq 1$, and vanish on all elements $\Lambda^{n-1}TC \subset \Lambda^{n-1}TX$ for $i = 0$ where $\Lambda^q TX$ is the q-h wedge-power of TX, i.e. the set of q-dimensional subspaces of the tangent bundle. For every sheaf $\mathcal{F}$ on X denote by $\Gamma^0(X, \mathcal{F})$ the space of global sections. That is, $\Gamma^0(X, \Omega^{n-2}_1(X))$ is the subset of $\Gamma^0(X, \Omega^{n-2}(X))$, that consists of forms divisible by p, and $\Gamma^0(X, \Omega^{n-1}_0(X))$ is the set of algebraic $(n-1)$-forms on X whose restriction to the zero fiber C of p yields a trivial form on C.

As a consequence of Lemma 3.6 we have the following fact.

3.7. **Lemma.** *Let* $\kappa^f_i = \nu^i_1(fp)\nu^i_2 - \nu^i_2(fp)\nu^i_1$ *and let the following condition hold:*
(B) *there exists a collection* $\{\nu^i_1, \nu^i_2\}^m_{i=1}$ *of pairs of commuting vector fields from* $\mathrm{IVF}^\omega_{\mathrm{alg}}(X)$ *such that the set* $\{\iota_{\nu^i_1} \circ \iota_{\nu^i_2}(\omega)\}^m_{i=1}$ *generates the space of algebraic* $(n-2)$*-forms* $\Gamma^0(X, \Omega^{n-2}(X))$ *on* X *as* $\mathbb{C}[X]$*-module.*

Then the image of $\Gamma^0(X, \Omega^{n-2}_1(X))$ *under the outer differentiation* $\mathrm{d} : \Gamma^0(X, \Omega^{n-2}(X)) \to \Gamma^0(X, \Omega^{n-1}(X))$ *is generated as a vector space by* $(n-1)$*-forms* $\{\iota_{\kappa^f_i}(\omega)\}^n_{i=1}, f \in \mathbb{C}[X]$.

3.8. **Application of Grothendieck's theorem.** Let $\mathcal{Z}^0(X, \Omega_0^{n-1}(X))$ be the subspace of closed algebraic $(n-1)$-forms in $\Gamma^0(X, \Omega_0^{n-1})$. Clearly, for every algebraic vector field $\nu \in \mathrm{IVF}_\omega(X)$ tangent to C we have $\iota_\nu(\omega) \in \mathcal{Z}^0(X, \Omega_0^{n-1}(X))$. Our aim now is to show that under mild assumption the homomorphism

$$\mathrm{d} : \Gamma^0(X, \Omega_1^{n-2}(X)) \to \mathcal{Z}^0(X, \Omega_0^{n-1}(X))$$

is surjective and, therefore, condition (A1) from Proposition 3.3 follows from condition (B) from Lemma 3.7. Denote by $\mathcal{F}_i'$ (resp. $\mathcal{F}_i$) the space of algebraic sections of Ω^i_{n-1-i} (resp. Ω^i) over X. Note that the outer differentiation d makes

$$\mathcal{F}'(*) := \ldots \to \mathcal{F}_i' \to \mathcal{F}_{i+1}' \to \ldots \text{ and } \mathcal{F}(*) := \ldots \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \ldots$$

complexes, and that the surjectivity we need would follow from $H^{n-1}(\mathcal{F}'(*)) = 0$.

3.9. **Proposition.** *Let $H^{n-1}(X, \mathbb{C}) = 0$ and let the homomorphism $H^{n-2}(X, \mathbb{C}) \to H^{n-2}(C, \mathbb{C})$ generated by the natural embedding $C \hookrightarrow X$ be surjective. Then $H^{n-1}(\mathcal{F}'(*)) = 0$.*

Proof. Consider the following short exact sequence of complexes $0 \to \mathcal{F}'(*) \to \mathcal{F}(*) \to \mathcal{F}''(*) \to 0$ where $\mathcal{F}_i'' = \mathcal{F}_i / \mathcal{F}_i'$ in complex $\mathcal{F}''(*)$. This implies the following long exact sequence in cohomology

$$\ldots \to H^{n-2}(\mathcal{F}(*)) \to H^{n-2}(\mathcal{F}''(*)) \to H^{n-1}(\mathcal{F}'(*)) \to H^{n-1}(\mathcal{F}(*)) \to \ldots,$$

i.e. we need to show (i) that the homomorphism $H^{n-2}(\mathcal{F}(*)) \to H^{n-2}(\mathcal{F}''(*))$ is surjective and (ii) that $H^{n-1}(\mathcal{F}(*)) = 0$. By the Grothendieck theorem [9] De Rham cohomology on smooth affine algebraic varieties can be computed via the complex of algebraic differential forms, i.e. $H^{n-1}(\mathcal{F}(*)) = H^{n-1}(X, \mathbb{C})$ which implies (ii). Similarly, $H^{n-2}(\mathcal{F}(*)) = H^{n-2}(X, \mathbb{C})$. Note that $\mathcal{F}_i'' = \mathcal{F}_i / (p^{n-1-i}\mathcal{F}_i)$ for $i \leq n-2$. In particular, modulo the space $\mathcal{S}$ of (the restrictions to C of) algebraic $(n-2)$-form that vanish on $\Lambda^{n-2}TC$ the term $\mathcal{F}_{n-2}''$ coincides with the space $\mathcal{T}$ of algebraic $(n-2)$-forms on C (more precisely, we have the following exact sequence $0 \to \mathcal{S} \to \mathcal{F}_{n-2}'' \to \mathcal{T} \to 0$). One can see that each closed $\tau \in \mathcal{S}$ is of form $\mathrm{d}p \wedge \tau_0$ where τ_0 is a closed $(n-3)$-form on C. Hence $\tau = \mathrm{d}(p\tau_0)$ (where $p\tau_0$ can be viewed as an element of $\mathcal{F}_{n-3}'' = \mathcal{F}_{n-3}/(p^2 \mathcal{F}_{n-3})$) is an exact form. Thus the $(n-2)$-cohomology of complex $\mathcal{F}''(*)$ coincides with the $(n-2)$-cohomology of the algebraic De Rham complex on C and, therefore, is equal to $H^{n-2}(C, \mathbb{C})$ by the Grothendieck theorem. Now homomorphism from (i) becomes $H^{n-2}(X, \mathbb{C}) \to H^{n-2}(C, \mathbb{C})$ which implies the desired conclusion.

□

Thus we have Theorem 1 from Introduction as a consequence of Remark 3.4 and the following more general fact (which gives, in particular, an affirmative answer to an open question of Varolin ([24], section 7) who asked whether the hypersurface $\{(a, b, c, d) \in \mathbb{C}^4 : a^2 c - bd = 1\}$ in $\mathbb{C}^4$ has the volume density property).

Theorem 3. *Suppose $n \geq 2$ and let X be an n-dimensional smooth affine algebraic variety with $H^{n-1}(X, \mathbb{C}) = 0$ and a volume form ω satisfying conditions*

(B) *there exists a collection* $\{\nu_1^i, \nu_2^i\}_{i=1}^m$ *of pairs of commuting vector fields from* $\mathrm{IVF}_{\mathrm{alg}}^{\omega}(X)$ *such that the set* $\{\iota_{\nu_1^i} \circ \iota_{\nu_2^i}(\omega)\}_{i=1}^m$ *generates the space of algebraic* $(n-2)$*-forms* $\Gamma^0(X, \Omega^{n-2}(X))$ *on* X *as* $\mathbb{C}[X]$*-module;*

(A2) $\mathrm{VF}_{\mathrm{alg}}(X)$ *is generated by* $\mathrm{IVF}_{\mathrm{alg}}^{\omega}(X)$ *as a module over* $\mathbb{C}[X]$.[2]

Suppose also that p *is a regular function on* X *with a smooth reduced zero fiber* C *such that the homomorphism* $H^{n-2}(X, \mathbb{C}) \to H^{n-2}(C, \mathbb{C})$ *generated by the natural embedding* $C \hookrightarrow X$ *is surjective. Let* $X' \subset X \times \mathbb{C}^2_{u,v}$ *be the hypersurface given by* $uv = p$ *and let* ω' *be the pullback of the form* $\omega \wedge \mathrm{d}u/u$ *on* $Z = X \times \mathbb{C}^*_u$ *under the natural projection* $X' \to Z$ [3]. *Then* X' *has the algebraic* ω'*-density property.*

3.10. **Algebraic volume density for** $SL_2(\mathbb{C})$ **and** $PSL_2(\mathbb{C})$**.** Since $X' = SL_2(\mathbb{C})$ is isomorphic to the hypersurface $uv = x_1x_2 + 1 =: p(\bar{x})$ in $Y = \mathbb{C}^4_{\bar{x},u,v} = X \times \mathbb{C}^2_{u,v}$ with $\bar{x} = (x_1, x_2)$ and $X = \mathbb{C}^2_{\bar{x}}$, Theorem 1 implies that $SL_2(\mathbb{C})$ has the algebraic volume density property with respect to the volume form ω' on X' such that $\omega' \wedge \mathrm{d}P = \Omega$ where $P = uv - p(\bar{x})$ and $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}u \wedge \mathrm{d}v$ is the standard volume form on $\mathbb{C}^4$. On the other hand by Remark 2.2 (1) we can consider forms $(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}u)/u$, $(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}v)/v$, etc.. Each of these forms coincides with ω' up to a sign because their wedge-products with $\mathrm{d}P$ are $\pm\Omega$. Note that $(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}u)/u$ is invariant with respect to the $\mathbb{C}_+$-action on $SL_2(\mathbb{C})$ given by $(x_1, x_2, u, v) \to (x_1, x_2 + tx_1, u, v + tu)$, $t \in \mathbb{C}_+$ which is generated by multiplications of a $\mathbb{C}_+$-subgroup of $SL_2(\mathbb{C})$. Thus ω' is invariant with respect to such multiplications. Similarly, consideration of $(\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}v)/v$ yields invariance with respect to the $\mathbb{C}_+$-action $(x_1, x_2, u, v) \to (x_1 + tx_2, x_2, u + tv, v)$, etc.. This implies that ω' is invariant with respect to multiplication by any element of $SL_2(\mathbb{C})$ and we proved the following result, which is originally due to Varolin ([24], Theorem 2).

3.11**. Proposition.** *Group* $SL_2(\mathbb{C})$ *has the algebraic volume density property with respect to the invariant volume form.*

Furthermore, since the vector fields $\nu_1 = \partial/\partial x_1$ and $\nu_2 = \partial/\partial x_2$ on $X = \mathbb{C}^2_{\bar{x}}$ commute and satisfy condition (B) of Lemma 3.7 we see that any vector field μ_0 tangent to the zero fiber C of p is of form $\nu_1(fp)\nu_2 - \nu_2(fp)\nu_1$ where f is a polynomial on X. This fact will used in the next unpleasant computation which is similar to the argument in Proposition 3.3.

3.12**. Proposition.** *Group* $PSL_2(\mathbb{C})$ *has the algebraic volume density property with respect to the invariant volume form.*

Proof. Consider now $X'' = X'/\mathbb{Z}_2 \simeq PSL_2(\mathbb{C})$ where the $\mathbb{Z}_2$-action on X' given by $(u, v, \bar{x}) \to (-u, -v, -\bar{x})$. Note that $\mathbb{C}[X'']$ can be viewed as the subring of $\mathbb{C}[X']$ generated by monomials of even degrees. Hence completely integrable vector fields of form

$$\nu_1' = u^{i+1}\partial/\partial x_k + u^i \frac{\partial p}{\partial x_k}\partial/\partial v \text{ and } \nu_2' = v^{i+1}\partial/\partial x_j + v^i \frac{\partial p}{\partial x_j}\partial/\partial u$$

[2]Clearly, the standard volume form on $\mathbb{C}^n$ satisfies both these conditions.

[3]One can check that $\omega' \wedge dP = \omega \wedge du \wedge dv|_{X'}$ where $P = uv - p$.

$$\text{(resp. } \nu_1'' = u^{i+1}x_j\partial/\partial x_k + u^i x_j \frac{\partial p}{\partial x_k}\partial/\partial v \text{ and } \nu_2'' = v^{i+1}x_k\partial/\partial x_j + v^i x_k \frac{\partial p}{\partial x_j}\partial/\partial u)$$

on X' with even (resp. odd) i can viewed as fields on X''. The same is true for μ_f from Lemma 2.5 provided f is a linear combination of monomials of even degrees. Fields ν_1', ν_2', μ_f are of zero divergence. If $j \neq k$ the same holds for ν_1'' and ν_2'' . Any algebraic vector field λ on X'' can be viewed as a vector field on X' and, therefore, it is given by formula (8). Since this field on X' came from X'' each $\tilde{\mu}_i^k$ (resp. $\tilde{\mu}_0$) in that formula consists of summands of form $q(\bar{x})\partial/\partial x_k$ where polynomial $u^i q(\bar{x})$ (resp. $v^i q(\bar{x})$) is a linear combination of monomials of odd degrees. Our plan is to simplify the form of a vector field λ with divergence zero on X'' by adding elements of the Lie algebra generated by fields like $\mu_f, \nu_1', \nu_2', \nu_1'', \nu_2''$.

Recall that $\tilde{\mu}_0$ is generated by a field μ_0 on X and it was shown in the proof of Proposition 3.3 that μ_0 is tangent to C. Hence, as we mentioned before $\mu_0 = \nu_1(fp)\nu_2 - \nu_2(fp)\nu_1$. Furthermore, if λ comes from a field on X'' polynomial $f \in \mathbb{C}[x_1, x_2]$ must contain monomials of even degrees only. Thus by virtue of Lemma 2.5 (v) adding to λ vector fields of form $[[\mu_f, \nu_1'], \nu_2']$ we can suppose that $\tilde{\mu}_0 = 0$ without changing the divergence of λ.

Following the pattern of the proof of Proposition 3.3 let us add to λ the zero divergence fields of form $[\mu_f, \nu_l']$ and $[\mu_f, \nu_l'']$. Since we have to require that $j \neq k$ in the definition of ν_1'' and ν_2'' we cannot eliminate summands $u^{i+1}\tilde{\mu}_{i+1}^1$ and $v^{i+1}\tilde{\mu}_{i+1}^2$ completely. However, Lemma 2.5 (iv) shows that after such addition one can suppose that $u^{i+1}\tilde{\mu}_{i+1}^1$ vanishes for even i, and for odd i it is a linear combination of terms of form $u^{i+1}x_k^m\partial/\partial x_k$ where m is odd (and similarly for $v^{i+1}\tilde{\mu}_{i+1}^2$).

Consider the semi-simple vector field $\nu = x_1\partial/\partial x_1 - x_2\partial/\partial x_2$ on X. Then $\nu' = u^{i+1}\tilde{\nu} + u^i\nu(p)\partial/\partial v$ is a completely integrable zero divergence vector field on X' and for odd i it can be viewed as a field on X''. Set $f = x_1^{m-1}$. By Lemma 2.5 (iv)

$$[\mu_f, \nu'] = (i+1)u^{i+1}x_1^{m-1}\tilde{\nu} + \alpha\partial/\partial u + \beta\partial/\partial v.$$

Thus adding a multiple of $[\mu_f, \nu']$ to λ we can replace terms $u^{i+1}x_1^m\partial/\partial x_1$ in $u^{i+1}\tilde{\mu}_{i+1}^1$ by $u^{i+1}x_1^{m-1}x_2\partial/\partial x_2$. If $m \geq 2$ the latter can be taken care of by adding fields of form $[\mu_f, \nu_i'']$. If $m = 1$ we cannot eliminate immediately terms like $u^i x_1\partial/\partial x_1$ or $u^i x_2\partial/\partial x_2$, but adding fields of form $cu^i\tilde{\nu}$ where c is a constant we can suppose that only one of these terms is present. The same is true for similar terms with u replaced by v. Thus adding elements from $\mathrm{Lie}_{alg}^{\omega}(X'')$ we can reduce λ to a zero divergence field of the following form

$$\lambda = \sum_{i \geq 1}(c_i u^i x_1\partial/\partial x_1 + d_i v^i x_2\partial/\partial x_2) + g_1\partial/\partial u + g_2\partial/\partial v$$

where constants c_i and d_i may be different from zero only for even indices i and by formula (7) $g_k = \sum_{i\geq 1}(a_i^k(\bar{x})u^i + b_i^k(\bar{x})v^i) + a_0^k(\bar{x})$ with a_i^k and b_i^k being polynomials on X. Since divergence $\mathrm{div}_\omega \lambda = 0$ we immediately have $a_{i+1}^1 = -c_i/(i+1)$ and $b_{i+1}^2 = -d_i/(i+1)$, i.e. these polynomials are constants.

Consider now an automorphism of X'' (and, therefore, of X') given by $(u, v, x_1, x_2) \to (-x_1, x_2, -u, v)$, i.e. it exchanges the role of pairs (u, v) and (x_1, x_2). It transforms λ into a field

$$\sum_{i \geq 1} (a^1_{i+1} x_1^{i+1} \partial/\partial x_1 + b^2_{i+1} x_2^{i+1} \partial/\partial x_2) + \lambda_0$$

where λ_0 does not contain nonzero summands of form $a x_1 \partial/\partial x_1$ (resp. $b x_2 \partial/\partial x_2$) with a (resp. b) being a regular function on X'' non-divisible by x_2 (resp. x_1). Hence adding fields of form $[[\mu_f, \nu_1'], \nu_2']$, $[\mu_f, \nu_k']$, and $[\mu_f, \nu_k'']$ as before we can suppose that $\tilde{\mu}_0$ and each $\tilde{\mu}_i^1$ and $\tilde{\mu}_i^2$ are equal to zero, i.e. $\lambda = e\partial/\partial u + g\partial/\partial v$. Furthermore, arguing as in Proposition 3.3 we see $\lambda = h(u\partial/\partial u - v\partial/\partial v)$ where h is a polynomial on X, i.e. λ is completely integrable. Since by construction it is a vector field on X'', we have proved that X'' possesses the algebraic volume density property.

□

4. Two basic facts about the algebraic volume density property

By considering differential forms and vector fields in local coordinate systems one can see that the map $\nu \to \iota_\nu(\omega)$ is bijective and, therefore, establishes a duality between algebraic (resp. holomorphic) vector fields and the similar $(n-1)$-forms on X. This duality in combination with the Grothendieck theorem [9] enables us to prove another important fact.

4.1**. Proposition.** *For an affine algebraic manifold X equipped with an algebraic volume form ω the algebraic volume density property implies the volume density property (in the holomorphic sense).*

Proof. We need to show that any holomorphic vector field μ such that $\mu(\omega) = 0$ can be approximated by an algebraic vector field ν with $L_\nu(\omega) = 0$. Since the form $\iota_\mu(\omega)$ is closed, by the Grothendieck theorem one can find a closed algebraic $(n-1)$-form τ_{n-1} such that $\iota_\mu(\omega) - \tau_{n-1}$ is exact, i.e. $\iota_\mu(\omega) - \tau_{n-1} = \mathrm{d}\,\tau_{n-2}$ for some holomorphic $(n-2)$-form τ_{n-2}. Then we can approximate τ_{n-2} by an algebraic $(n-2)$-form τ_{n-2}'. Hence the closed algebraic $(n-1)$-form $\tau_{n-1} + \mathrm{d}\,\tau_{n-2}'$ yields an approximation of $\iota_\mu(\omega)$. By duality $\tau_{n-1} + \mathrm{d}\,\tau_{n-2}'$ is of form $\iota_\nu(\omega)$ for some algebraic vector field ν (approximating μ) and by Lemma 3.6 (1) ν is of zero ω-divergence which is the desired conclusion.

□

4.2**. Lemma.** *If X has the algebraic volume density property, then there exist finitely many algebraic vector fields $\sigma_1, \ldots, \sigma_m \in \mathrm{Lie}^\omega_{alg}(X)$ that generate $\mathrm{VF}_{\mathrm{alg}}(X)$ as a $\mathbb{C}[X]$-module.*

Proof. Let $n = \dim X$. We start with the following.

Claim. The space of algebraic fields of zero divergence generates the tangent space of X at each point.

Let $x \in X$ and U be a Runge neighborhood of x such that $H^{n-1}(U, \mathbb{C}) = 0$ (for example take a small sublevel set of a strictly plurisubharmonic exhaustion function on X with minimum at x). Shrinking U we can assume that in some holomorphic coordinate system $z_1, \ldots, z_n$ on U the form $\omega|_U$ is the standard volume $\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$. Thus the holomorphic vector fields $\partial/\partial z_i$ on U are of zero divergence and they span the tangent space at x. We need to approximate them by global algebraic fields of zero divergence on X which would yield our claim. For that let $\nu \in \mathrm{VF}^{\omega}_{hol}(U)$. The inner product $\iota_\nu(\omega) =: \alpha$ is by Lemma 3.6 (1) a closed $(n-1)$-form on U and since $H^{n-1}(U, \mathbb{C}) = 0$ we can find an $(n-2)$-form β on U with $\mathrm{d}\beta = \alpha$. Since U is Runge in X we can also approximate β by a global algebraic $(n-2)$-form $\tilde{\beta}$ (uniformly on compacts in U). Then the closed algebraic $(n-1)$-form $\mathrm{d}\tilde{\beta}$ approximates α and the unique algebraic vector field θ defined by $\iota_\theta(\omega) = \mathrm{d}\tilde{\beta}$ approximates ν. Since $\mathrm{d}\tilde{\beta}$ is closed, the field θ is of zero divergence which concludes the proof of the Claim.

Now it follows from the Claim and the algebraic volume density property that there are n vector fields in $\mathrm{Lie}^{\omega}_{alg}(X)$ which span the tangent space at a given point $x \in X$. By standard induction on the dimension, adding more fields to span the tangent spaces at points where it was not spanned yet, we get the assertion of the lemma.

□

Let us suppose that X and Y are affine algebraic manifolds equipped with volume forms ω_X and ω_Y respectively.

4.3. **Proposition.** *Suppose that X (resp. Y) has the algebraic ω_X (resp. ω_Y) volume density property. Let $\omega = \omega_X \times \omega_Y$. Then $X \times Y$ has the algebraic volume density property relative to ω.*

Proof. By Lemma 4.2 we can suppose that $\sigma_1, \ldots, \sigma_m \in \mathrm{Lie}^{\omega_X}_{alg}(X)$ (resp. $\delta_1, \ldots, \delta_n \in \mathrm{Lie}^{\omega_Y}_{alg}(Y)$) generate $\mathrm{VF}_{\mathrm{alg}}(X)$ as a $\mathbb{C}[X]$-module (resp. $\mathrm{VF}_{\mathrm{alg}}(Y)$ as a $\mathbb{C}[Y]$-module).

Denote by F_Y the vector subspace (over $\mathbb{C}$) of $\mathbb{C}[Y]$ generated by $\mathrm{Im}\,\delta_1, \ldots, \mathrm{Im}\,\delta_n$. Then $\mathbb{C}[Y] = F_Y \oplus V$ where V is another subspace whose basis is $v_1, v_2, \ldots$. Set $F_Y' = \mathbb{C}[X] \otimes F_Y$ and $V' = \mathbb{C}[X] \otimes V$, i.e. the algebra of regular functions on $X \times Y$ is $A = \mathbb{C}[X] \otimes \mathbb{C}[Y] = F_Y' \oplus V'$. Let $f_i \in \mathbb{C}[X]$ and $g_j \in \mathbb{C}[Y]$. Note that f_i is in the kernel of all completely integrable fields used in the Lie combination for δ_i and thus $f_i\delta_i \in \mathrm{Lie}^{\omega_Y}_{alg}(Y)$, analogously $g_i\sigma_i \in \mathrm{Lie}^{\omega_X}_{alg}(X)$. The fields δ_i and σ_j generate (vertical and horizontal) vector fields on $X \times Y$ that are denoted by the same symbols. Consider

$$[f_i\delta_i, g_j\sigma_j] = \delta_i(f_ig_j)\sigma_j - \sigma_j(f_ig_j)\delta_i.$$

By construction δ_i and σ_j commute and moreover $\mathrm{Span}\, f_i \cdot g_i = \mathbb{C}[X \times Y]$. Hence the coefficient before σ_j runs over $\mathrm{Im}\,\delta_i$ and, therefore, for any $\alpha_1', \ldots, \alpha_n' \in F_Y'$ there are $\beta_1', \ldots, \beta_m' \in A$ such that the vector field

$$\sum_j \alpha_j'\sigma_j - \sum_i \beta_i'\delta_i$$

belongs to $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X \times Y)$. Thus adding vector fields of this form to a given vector field

$$\nu = \sum_j \alpha_j\sigma_j - \sum_i \beta_i\delta_i$$

from $\mathrm{VF}^{\omega}_{\mathrm{alg}}(X \times Y)$ we can suppose that each $\alpha_j \in V'$. Hence one can rewrite ν in the following form

$$\nu = \sum_l \sum_j (h_{jl} \otimes v_l)\sigma_j - \sum_i \beta_i \delta_i$$

where each $h_{jl} \in \mathbb{C}[X]$. Then one has

$$0 = \operatorname{div} \nu = \sum_l (\sum_j \sigma_j(h_{jl})) \otimes v_l - \sum_i \delta_i(\beta_i).$$

Since the first summand is in V' and the last is in F'_Y we see that $\sum_j \sigma_j(h_{jl}) = 0$, i.e. each vector field $\sum_j h_{jl}\sigma_j$ belong to $\mathrm{VF}^{\omega_X}_{alg}(X)$ and by the assumption to $\mathrm{Lie}^{\omega_X}_{alg}(X)$. Hence it suffices to prove the following

Claim. Consider the subspace $B \subset \mathrm{VF}^{\omega}_{\mathrm{alg}}(X \times Y)$ that consists of vector fields of form

$$\nu = \sum_i \beta_i \delta_i \, .$$

Then B is contained in $\mathrm{Lie}^{\omega}_{alg}(X \times Y)$.

Indeed, consider a closed embedding of Y into a Euclidean space. Then it generates filtration on $\mathbb{C}[Y]$ by minimal degrees of extensions of regular functions to polynomials. In turn this generates filtrations $B = \bigcup B_i$ and $\mathrm{Lie}^{\omega_Y}_{alg}(X \times Y) = \bigcup L_i$. Note that each B_i or L_i is a finitely generated $\mathbb{C}[X]$-module, i.e. they generate coherent sheaves on X. Furthermore, since Y has algebraic ω_Y-density property we see that the quotients of B_i and L_i with respect to the maximal ideal corresponding to any point $x \in X$ coincide. Thus $B_i = L_i$ which implies the desired conclusion.

$\square$

Note that up to a constant factor the completely integrable vector field $z\partial/\partial z$ on the group $X = \mathbb{C}^*$ is the only field of zero divergence with respect to the invariant volume form $\omega = \frac{\mathrm{d}z}{z}$, i.e., X has the algebraic volume density property. Hence we have the following (see also Corollary 4.5 in [22]).

4.4. **Proposition.** *For every $n \geq 1$ the torus $(\mathbb{C}^*)^n$ has the algebraic volume density property with respect to the invariant form.*

5. Algebraic volume density for locally trivial fibrations

5.1. Let Y be an affine algebraic manifold with a volume form ω and F_Y be the subspace of $\mathbb{C}[Y]$ that consists of images of vector fields from $\mathrm{Lie}^{\omega}_{alg}(Y)$, i.e. $V \simeq \mathbb{C}[Y]/F_Y$ is the subspace that appeared in the proof of Proposition 4.3. If V were trivial so would be the proof, but in the general case $V \neq 0$. We shall see later that Proposition 4.3 can be extended to some locally trivial fibrations with fiber Y for which, in particular, V is at most one-dimensional. More precisely, we shall need manifolds Y satisfying the following property

$$(C) \qquad \text{either } \mathbb{C}[Y] = F_Y \text{ or } \mathbb{C}[Y] \simeq F_Y \oplus \mathbb{C}$$

where the isomorphism is natural and the second summand denotes constant functions on Y.

5.2. **Lemma.** *Let Y be the smooth hypersurface in $\mathbb{C}^{n+2}_{u,v\bar{x}}$ given by $P = uv + q(\bar{x}) - 1 = 0$ where $q(\bar{x}) = \sum_{i=1}^n x_i^2$ (i.e. after a coordinate change $uv + q(\bar{x})$ can be replaced by any non-degenerate quadratic form). Suppose that Y is equipped with a volume ω_Y such that $\mathrm{d}P \wedge \omega_Y = \Omega|_Y$ where Ω is the standard volume form on $\mathbb{C}^n$. Then*

(1) *Y has property (C) and*

(2) *$Y/\mathbb{Z}_2$ has property (C) where the $\mathbb{Z}_2$-action is given by $(u, v, \bar{x}) \to (-u, -v, -\bar{x})$.*

Proof. Consider the semi-simple vector field $\mu = u\partial/\partial u - v\partial/\partial v$ on Y. It generates $\mathbb{Z}$-grading of $\mathbb{C}[Y] = \oplus_{i\in\mathbb{Z}} A_i$ such that $\operatorname{Ker}\mu = A_0$ and $\operatorname{Im}\mu = \oplus_{i\in\mathbb{Z}, i\neq 0} A_i \subset F_Y$. Note that $A_0 \simeq \mathbb{C}[x_1, \ldots, x_n]$ since $uv = 1 - q(x) \in A_0$. Assume for simplicity that $n \geq 2$ and replace x_1 and x_2 by $u' = x_1 + \sqrt{-1}x_2$ and $v' = x_1 - \sqrt{-1}x_2$ in our coordinate system. Consider the semi-simple vector field $\mu' = u'\partial/\partial u' - v'\partial/\partial v'$ whose kernel is $A_0' = \mathbb{C}[u, v, x_3, \ldots, x_n]$. Thus monomials containing u' and v' (or, equivalently, x_1 or x_2 in the original coordinate system) are in $\operatorname{Im}\mu' \subset F_Y$. Repeating this procedure with other x_i and x_j instead of x_1 and x_2 we see that F_Y contains every nonconstant monomial which is (1).

For (2) note that $\mathbb{C}[Y/\mathbb{Z}_2]$ is the subring of $\mathbb{C}[Y]$ generated by monomials of even degrees and that the semi-simple vector fields that we used preserve the standard degree function. That is, if a monomial M_1 of even degree belongs, say, to $\operatorname{Im}\mu$ then $M_1 = \mu(M_2)$ where M_2 is also a monomial of even degree. This yields (2).

□

Since $SL_2(\mathbb{C})$ is isomorphic to the hypersurface $uv - x_1x_2 = 1$ in $\mathbb{C}^4_{u,v,x_1,x_2}$ we have the following.

5.3. **Corollary.** *Both $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ have property (C).*

5.4. **Remark.** In fact for Y equal to $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ we have $\mathbb{C}[Y]/F_Y \simeq \mathbb{C}$. More precisely, set $F = \operatorname{Span}\{\nu(f) : \nu \in \operatorname{VF}^{\omega}_{\mathrm{alg}}(Y),\ f \in \mathbb{C}[Y]\}$. Note that vector fields of form $f\nu$ span all algebraic vector fields because of Claim in Lemma 4.2 (and Lemma 5.8 below). Therefore, $(n-1)$-forms $\iota_{f\nu}(\omega)$ generate all algebraic $(n-1)$-forms on Y where $n = \dim Y$. By Lemma 3.6 (2), $\mathrm{d}(\iota_{f\nu}(\omega)) = \nu(f)\omega$ which implies that the image of $\Omega^{n-1}(Y)$ in $\Omega^n(Y)$ under outer differentiation coincides with $F\omega$. Since $\mathrm{d}(\Omega^n(Y)) = 0$ we have $\mathbb{C}[Y]/F \simeq H^n(Y, \mathbb{C})$ by the Grothendieck theorem. By Proposition 4.1 in [14] for a smooth hypersurface $Y \subset \mathbb{C}^{m+2}$ given by $uv = p(x)$ we have $H_*(Y) = H_{*-2}(C)$ where C is the zero fiber of p. Thus the universal coefficient formula implies that $\dim \mathbb{C}[Y]/F = \operatorname{rank} H^{m-1}(C, \mathbb{C})$. For $SL_2(\mathbb{C})$ presented as such a hypersurface we have $p(x_1, x_2) = x_1x_2 - 1$, i.e. C is a hyperbola and $H^1(C, \mathbb{C}) = \mathbb{C}$ which yields the desired conclusion because $F = F_Y$ for manifolds with the algebraic volume density property.

5.5. **Notation.** Further in this section X, Y, and W are smooth affine algebraic varieties and $p : W \to X$ is a locally trivial fibration with fiber Y in the étale topology. We suppose also that Y is equipped with a unique (up to a constant factor) algebraic volume form ω_Y, and $\operatorname{VF}_{\mathrm{alg}}(W, p)$ (resp. $\operatorname{VF}^{\omega_Y}_{alg}(W, p)$) is the space of algebraic vector fields tangent to the fibers of p (resp. and such that the restriction to each fiber has zero divergence relative to ω_Y.) Similarly $\mathrm{Lie}_{alg}^{\omega_Y}(W,p)$ will be the Lie algebra generated by completely integrable vector fields from $\mathrm{VF}_{alg}^{\omega_Y}(W,p)$. We denote the subspace of $\mathbb{C}[W]$ generated by functions of form $\{\mathrm{Im}\,\nu|\nu \in \mathrm{Lie}_{alg}^{\omega_Y}(W,p)\}$ by $F(W,p)$.

5.6**. Definition.** We say that a family $\delta_1,\ldots,\delta_n,\ldots \in \mathrm{Lie}_{alg}^{\omega_Y}(Y)$ satisfies condition (D) if

(D1) it generates $\mathrm{Lie}_{alg}^{\omega_Y}(Y)$ as a Lie algebra and

(D2) $\mathrm{VF}_{\mathrm{alg}}(Y)$ as a $\mathbb{C}[Y]$-module.

5.7**. Remark.** (i) Note that (D1) implies the sets $\{\delta_i(\mathbb{C}[Y])\}$ generate the vector space F_Y.

(ii) For (D2) it suffices to require that the set of vector fields $\delta_1,\ldots,\delta_n,\ldots$ generates the tangent space at each point of Y. This is a consequence of the next simple fact (e.g., see Exercise 5.8 in [10]) which is essentially the Nakayama lemma.

5.8**. Lemma.** *Let $A \subset B$ be a finitely generated $\mathbb{C}[X]$-module and its submodule. Suppose that for every point $x \in X$ one has $A/M_x = B/M_x$ where M_x is the maximal ideal in $\mathbb{C}[X]$ associated with x. Then $A = B$*

5.9**. Example.** Let $\sigma_1,\sigma_2,\ldots$ (resp. $\delta_1,\delta_2,\ldots$) be a family on X with respect to volume ω_X (resp. on Y with respect to volume ω_Y) satisfying Condition (D) from 5.6. Denote their natural lifts to $X \times Y$ by the same symbols. Consider the set $\mathcal{S}$ of "horizontal" and "vertical" fields of form $f\sigma_i$ and $g\delta_j$ where f (resp. g) is a lift of a function on Y (resp. X) to $X \times Y$. It follows from the explicit construction in the proof of Proposition 4.3 that $\mathcal{S}$ generates the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}^{\omega}(X \times Y)$ for $\omega = \omega_X \times \omega_Y$, i.e. it satisfies condition (D1) of Definition 5.6. Remark 5.7 (2) implies that condition (D2) also holds and, therefore, the family $\mathcal{S}$ satisfies Condition (D) on $X \times Y$.

In particular, consider a torus $\mathbb{T} = (\mathbb{C}^*)^n$ with coordinates $z_1,\ldots,z_n$. One can see that the vector field $\nu_j = z_j\partial/\partial z_j$ is a family on the j-th factor with respect to the invariant volume on $\mathbb{C}^*$ such that it satisfies Condition (D). Thus fields of form $f_j\nu_j$ $(j = 1,\ldots,n)$ with f_j being independent of z_j generate a family on $\mathbb{T}$ with respect to the invariant volume for Condition (D) is also valid.

5.10**. Convention.** Furthermore, we suppose that vector fields $\delta_1,\ldots,\delta_n,\ldots$ form a family $\mathcal{S}$ in $\mathrm{Lie}_{alg}^{\omega_Y}(Y)$ satisfying (D) and there are vector fields $\delta_1',\ldots,\delta_l',\ldots \in \mathrm{VF}_{alg}^{\omega_Y}(W,p)$ such that up to nonzero constant factors the set of their restrictions to any fiber of p contains $\delta_1,\ldots,\delta_n,\ldots$ under some isomorphism between this fiber and Y. (Note that if $p : W \to X$ is a Zariski locally trivial fibration this Convention is automatically true.)

5.11**. Lemma.** *Suppose that $p : W \to X$ is one of the following*

(i) *a principal SL_n-bundle;*

(ii) *a quotient of a semi-simple group W with respect to a subgroup $Y \simeq PSL_2$ that corresponds to a root of the Dynkin diagram for W.*

Then Convention 5.10 *holds.*

Proof. Recall that SL_n is a special group in terminology of [19, Section 4] which means that every principal SL_n-bundle is Zariski trivial and therefore yields the validity of Convention 5.10 and statement (i).

Though PSL_n is not a special group, let us show that under the assumption of (ii) we have a Zariski locally trivial fibration as well. Indeed, $W = W'/F'$ where F' is a finite subgroup of the center of a simply connected semi-simple group W'. The preimage Y' of Y in W' is isomorphic to SL_2, i.e. $Y = Y'/F$ where the order of the subgroup F of $Y' \cap F'$ is 2. If Y' is contained in an SL_3-subgroup of W then the generator a of F cannot be presented as $a = b^k$, $k \geq 2$ with b in the center of W'. In combination with the classification of centers of simple Lie groups [18] this fact implies that the same remains true for every Y' corresponding to a node of the Dynkin diagram of W.

In the additive form $F' \simeq \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_m}$ where $d_i | d_{i+1}$. Then a corresponds to an element $(a_1, \ldots, a_m)$ such that the $GCD(a_1, \ldots, a_m) = 1$ because of the above description of a. On the other hand each a_i is either zero of $d_i/2$ since the order of a is 2. Hence the greatest common divisor is 1 if only if the smallest nonzero $a_i = 1$ and $d_i = 2$. In this case F is a summand in F', i.e. $F' = F \oplus \Gamma$. Set $\tilde{W} = W'/\Gamma$, that is $W = \tilde{W}/F$. Then $\tilde{p} : \tilde{W} \to X$ is a principal SL_2-bundle induced by $p : W \to X$. As we mentioned this bundle is Zariski locally trivial and therefore it has a section over a Zariski neighborhood of each point in X. Hence $p : W \to X$ has similar sections which implies that this PSL_2-bundle is also locally trivial. This yields the desired conclusion. □

In the rest of the section we suppose that Convention 5.10 is valid. Then we have the following.

5.12**. Lemma.**

(1) *A function $g \in \mathbb{C}[W]$ is contained in $F(W,p)$ if and only if its restriction to each general fiber of p belongs to F_Y. Furthermore, if Y has property (C) from* 5.1 *then* $\mathbb{C}[W] \simeq F(W,p) \oplus \mathbb{C}[X]$[4].

(2) *Suppose that Y has the algebraic volume density property. Then* $\mathrm{VF}_{alg}^{\omega_Y}(W,p) = \mathrm{Lie}_{alg}^{\omega_Y}(W,p)$.

Proof. There exists a cover $X = \bigcup_i X_i$ such that for each i one can find an étale surjective morphism $X_i' \to X_i$ for which the variety $W_i' := W_i \times_{X_i} X_i'$ is naturally isomorphic to $X_i' \times Y$ where $W_i = p^{-1}(X_i)$. Lifting functions on W to some W_{i_0}' which is the direct product we can introduce filtration $\mathbb{C}[W] = \bigcup_{i \geq 0} G_i$ as we did in the Claim in Proposition 4.3 (i.e. take a closed embedding $Y \hookrightarrow \mathbb{C}^m$ and consider the minimal degrees of extensions of function on W_{i_0}' to $X_{i_0}' \times \mathbb{C}^m$ with respect to the second factor).

Consider now the set S of functions $g \in \mathbb{C}[W]$ such that for a general $x \in X$ the restriction $g|_{p^{-1}(x)}$ is in F_Y. Since the degree (generated by the embedding $Y \hookrightarrow \mathbb{C}^m$) of the restriction of g to each (not necessarily general) fiber $p^{-1}(x_0) \simeq Y$ is bounded by

[4] In fact for (1) one needs only that the sets $\{\delta_i(\mathbb{C}[Y])\}$ generate the vector space F_Y with δ_i running over the family $\mathcal{S}$ from Convention 5.10 .

the same constant we see that $g|_{p^{-1}(x_0)}$ belongs to F_Y (because the finite-dimensional subspace of F_Y that consists its elements, whose degrees are bounded by this constant, is closed). Set $S_i = S \cap G_i$ and $F_i = F(W,p) \cap G_i$. If suffices to show that $S_i = F_i$ for every i. Note that both S_i and F_i are finitely generated $\mathbb{C}[X]$-modules and because of the existence of fields $\delta_1', \ldots, \delta_l', \ldots$ in Convention 5.10 we see that for every $x \in X$ there is an equality $S_i/M_x = F_i/M_x$ where M_x is the maximal ideal in $\mathbb{C}[X]$ associated with x. Hence the first statement of (1) follows from Lemma 5.8.

For the second statement of (1) note that $G_i/M_x = (F_i \oplus \mathbb{C}[X])/M_x$ since Y has property (C) from 5.1 and $\mathbb{C}[X]/M_x = \mathbb{C}$. Thus another application of Lemma 5.8 implies the desired conclusion.

The filtration of functions that we introduced, yields a filtrations of vector fields $\mathrm{VF}_{alg}^{\omega_Y}(W,p) = \bigcup B_i$ and $\mathrm{Lie}_{alg}^{\omega_Y}(W,p) = \bigcup L_i$ as where B_i and L_i are again finitely generated $\mathbb{C}[X]$-modules. Because $\mathcal{S}$ in Convention 5.10 satisfies Condition (D) from 5.6 and Y has the algebraic volume density property we have $B_i/M_x = L_i/M_x$. Thus by Lemma 5.8 $B_i = L_i$ which yields (2). □

5.13**. Definition.** (1) Suppose that ω_X is a volume form on X, $X = \bigcup X_i$, X_i', W_i, and W_i' are as in the proof of Lemma 5.12, $\varphi_i : W_i' \to X_i' \times Y$ is the natural isomorphism and ω is an algebraic volume form on W such that up to a constant factor $\varphi_i^*(\omega)$ coincides with $(\omega_X \times \omega_Y)|_{X_i \times Y}$ for each i. Then we call $p : W \to X$ a volume fibration (with respect to the volume forms ω_X, ω_Y, and ω).

(2) We call a derivation $\sigma' \in \mathrm{Lie}_{\mathrm{alg}}^{\omega}(W)$ a lift of a derivation $\sigma \in \mathrm{Lie}_{alg}^{\omega_X}(X)$ if for every $w \in W$ and $x = p(w)$ one has $p_*(\sigma'(w)) = \sigma(x)$. (Note that the Lie bracket of two lifts is a lift.) We say that σ' is p-compatible if for any $\delta' \in \mathrm{VF}_{alg}^{\omega}(W,p)$ we have $[\sigma', \delta'] \in \mathrm{VF}_{alg}^{\omega}(W,p)$ and the span $\mathrm{Span}\,(\mathrm{Ker}\,\sigma' \cdot \mathrm{Ker}\,\delta')$ coincides with $\mathbb{C}[W]$.

We shall see later (Lemma 6.11) that for a reductive group G and its Levi semi-simple subgroup L the natural morphism $p : G \to G/L$ can be viewed as a volume fibration with respect to appropriate volume forms such that the base possesses a family of algebraic vector fields satisfying condition (D) and admitting p-compatible lifts.

Since any algebraic vector field tangent to fiber of $p : W \to X$ has zero ω-divergence if and only if its restriction on each fiber has zero ω_Y-divergence we have the following consequence of Lemma 5.12 (2).

5.14**. Corollary.** *Let $p : W \to X$ be a volume fibration whose fiber has the algebraic volume density property and $\mathrm{VF}_{alg}^{\omega}(W,p)$ (resp. $\mathrm{Lie}_{alg}^{\omega}(W,p)$) be the space of zero ω-divergence algebraic vector fields tangent to the fibers of p (resp. the Lie algebra generated by completely integrable algebraic vector fields tangent to the fibers of p and of zero ω-divergence). Then $\mathrm{VF}_{alg}^{\omega}(W,p) = \mathrm{Lie}_{alg}^{\omega}(W,p)$.*

The next fact will not be used further but it is interesting by itself.

5.15**. Proposition.** *Let $p : W \to X$ be a a volume fibration with both fiber Y and base X having property (C) from* 5.1*. Suppose also that X possesses a family of algebraic* *vector fields satisfying condition (D) and admitting p-compatible lifts. Then W has property (C) as well.*

Proof. Indeed, the existence of lifts for a family of vector fields satisfying condition (D) from 5.6 makes F_X and, therefore, $F(W,p) \oplus F_X$ a natural subspace of F_W. It remains to note that $\mathbb{C}[W] = F(W,p) \oplus F_X \oplus \mathbb{C}$ by the assumption and by Lemma 5.12.

$\square$

5.16**. Proposition.** *Let $p : W \to X$ be a volume fibration such that its fibers have property (C) and $\mathrm{Lie}_{alg}^{\omega_X}(X)$ contains a family of vector fields satisfying condition (D) and admitting p-compatible lifts. Let Θ be the set of p-compatible lifts of this family. Consider the space L generated by $\mathrm{Lie}_{alg}^{\omega}(W,p)$ and vector fields of form $\nu := [f\sigma', \delta']$ where $\sigma' \in \Theta, \delta' \in \mathrm{Lie}_{alg}^{\omega}(W,p)$, and $f \in \mathrm{Ker}\,\sigma'$. Suppose that $T = p^*(TX)$ is the pull-back of the tangent bundle TX to W, $\varrho : TW \to T$ is the natural projection, and $\mathcal{L} = \varrho(L)$. Then*

(1) *$\mathcal{L}$ is a $\mathbb{C}[X]$-module;*

(2) *$\mathcal{L}$ consists of all finite sums $\sum_{\sigma' \in \Theta} h_{\sigma'} \varrho(\sigma')$ where $h_{\sigma'} \in F(W,p)$.*

Proof. The space $\mathrm{Lie}_{\mathrm{alg}}^{\omega}(W,p)$ is, of course, a $\mathbb{C}[X]$-module. Thus it suffices to consider fields like $\nu = [\sigma', \delta']$ only. Since σ' is a lift of $\sigma \in \mathrm{Lie}_{alg}^{\omega_X}(X)$ we see that $\sigma'(\mathbb{C}[X]) \subset \mathbb{C}[X]$ where we treat $\mathbb{C}[X]$ in this formula as a subring of $\mathbb{C}[W]$. Then for every $\alpha \in \mathbb{C}[X]$ we have $\alpha\nu = [\sigma', \alpha\delta'] - \sigma'(\alpha)\delta'$ which implies (1).

Let $f_1 \in \mathrm{Ker}\,\sigma'$, $f_2 \in Ker\delta'$, and $h = \delta'(f_1 f_2)$. Then by the p-compatibility assumption $[f_1\sigma', f_2\delta'] = h\sigma' + a$ with $a \in \mathrm{VF}_{\mathrm{alg}}(W,p)$ and, furthermore, the span of functions like h coincides with $\delta'(\mathbb{C}[W])$. Thus $\mathcal{L}$ contains $F(W,p)\varrho(\sigma')$ which is (2).

$\square$

Theorem 4. *Let $p : W \to X$ and Θ be as in Proposition* 5.16 *and let Convention* 5.10 *hold. Suppose also that the fibers and the base of p have the algebraic volume density property. Then W has the algebraic volume density property.*

Proof. Suppose that δ_i' is as in Convention 5.10, $\{\sigma_i\}$ is a family on X satisfying Condition (D), and $\sigma_i' \in \mathrm{Lie}_{\mathrm{alg}}^{\omega}(W)$ is a p-compatible lift of σ_i. Let $\kappa \in \mathrm{VF}_{alg}^{\omega}(W)$. Then $\kappa = \sum_i h_i \sigma_i' + \theta$ where $h_i \in \mathbb{C}[W]$ and $\theta \in \mathrm{VF}_{\mathrm{alg}}(W,p)$. By Proposition 5.16 and Lemma 5.12(1) adding an element of L to κ one can suppose that each $h_i \in \mathbb{C}[X]$. Since θ is a $\mathbb{C}[W]$-combination of $\delta_1', \ldots, \delta_l', \ldots$ and $\mathrm{div}_{\omega}(f\delta_i') = \delta_i'(f)$, we see that $\mathrm{div}_{\omega}\,\theta \in F(W,p)$. On the other hand $\mathrm{div}_{\omega} \sum_i h_i \sigma_i' = \sum_i \sigma_i(h_i) = \mathrm{div}_{\omega_X} \sum_i h_i \sigma_i \in \mathbb{C}[X]$. Hence $\mathrm{div}_{\omega_X} \sum_i h_i \sigma_i = 0$ and $\theta \in \mathrm{VF}_{alg}^{\omega}(W,p)$. By assumption $\sum_i h_i \sigma_i \in Lie_{alg}^{\omega_X}(X)$. In combination with the existence of lifts for σ_i and Corollary 5.14 this implies the desired conclusion.

$\square$

6. Volume forms on homogeneous spaces

6.1**. Definition.** We say that an affine algebraic variety X is (weakly) rationally connected if for any (resp. general) points $x, y \in X$ there are a sequence of points

$x_0 = x, x_1, x_2, \ldots, x_n = y$ and a sequence of polynomial curves $C_1, \ldots, C_n$ in X such that $x_{i-1}, x_i \in C_i$.

6.2. **Remark.** (1) This notion of rational connectness is not, of course, new. For projective varieties it was introduced independently in [5] and in [16] where it means that any two general points can be connected by a chain of rational curves.

(2) Since finite morphisms transform polynomial curves into polynomial curves we have the following: if X is an affine (weakly) rationally connected variety and $f : X \to Y$ is a finite morphism then Y is also an affine (weakly) rationally connected variety.

6.3. **Example.** It is easy to see that $SL_2(\mathbb{C})$ is affine rationally connected. (Indeed, $SL_2(\mathbb{C})$ can be presented as an algebraic locally trivial $\mathbb{C}$-fibration over $\mathbb{C}^2$ without the origin o. Over any line in $\mathbb{C}^2$ that does not contain o this fibration is trivial and, therefore, admits sections which implies the desired conclusion.) Hence any semi-simple group is rationally connected since its simply connected covering is generated by $SL_2(\mathbb{C})$-subgroups.

6.4. **Proposition.** *Let X be an affine manifold and ω, ω_1 be algebraic volume forms on X.*

(1) *If X is weakly rationally connected then $\omega = c\omega_1$ for nonzero constant c.*

(2) *If G is a rationally connected linear algebraic group (say, unipotent or semi-simple) acting on X then $\omega \circ \Phi_g = \omega$ for the action $\Phi_g : X \to X$ of any element $g \in G$.*

Proof. (1) Note that $\omega = h\omega_1$ where h is an invertible regular function on X. Let x, y, x_i, C_i be as in Definition 6.1. By the fundamental theorem of algebra h must be constant on each C_i. Hence $h(x) = h(y)$ which implies the first statement.

(2) Let e be the identity in G. Then we have a sequence $g_0 = e, g_1, g_2, \ldots, g_m = g$ in g such that for any $i \geq 1$ there is a polynomial curve C_i in G joining g_{i-1} and g_i. Again for every $a \in C_i$ we have $\omega \circ \Phi_a = h(a)\omega$ where h is a nonvanishing regular function on C_i, i.e. a constant. This implies $\omega \circ \Phi_g = \omega \circ \Phi_e = \omega$ which concludes the proof. □

For a Lie group G one can construct a left-invariant (resp. right-invariant) algebraic volume form by spreading the volume element at identity by left (resp. right) multiplication (one of these forms can be transformed into the other by the automorphism $\varphi : G \to G$ given by $\varphi(g) = g^{-1}$). Proposition 6.4 yields now the following well-known facts.

6.5. **Corollary.** *For a semi-simple Lie group G its left-invariant volume form is automatically also right-invariant.*

6.6. **Remark.** Since up to a finite covering any reductive group G is a product of a torus and a semi-simple group we see that the left-invariant volume form on this group is also right-invariant.

6.7. **Proposition.** *Let W be a linear algebraic group group, Y be its rationally connected subgroup, and $X = W/Y$ be the homogeneous space of left cosets. Then there* *exists an algebraic volume form ω_X on X invariant under the action of W generated by left multiplication.*

Proof. Consider a left-invariant volume form ω on W and left-invariant vector fields $\nu_1, \ldots, \nu_m$ on the coset $eY \simeq Y$, where e is the identity of W, so that they generate basis of the tangent space at any point of this coset. Extend these vector fields to W using left multiplication. Since eY is a fiber of the natural projection $p : W \to X$ and left multiplication preserves the fibers of p we see that the extended fields are tangent to all fibers of p. Consider the left-invariant form $\omega_X = \iota_{\nu_1} \circ \ldots \circ \iota_{\nu_m}(\omega)$ (where ι_{ν_i} is the inner product of vector fields and differential forms). By construction it can be viewed as a non-vanishing form on vectors from the pull-back of the tangent bundle TX to W. To see that it is actually a volume form on X we have to show that it is invariant under right multiplication by any element $y \in Y$. Such multiplication generates an automorphism of TW that sends vectors tangent (and, therefore, transversal) to fibers of p to similar vectors. Hence it transforms ω_X into $f_y \omega_X$ where $y \to f_y$ is an algebraic homomorphism from Y into the group of non-vanishing regular functions of W. Since the rationally connected group Y has no nontrivial algebraic homomorphisms into $\mathbb{C}^*$ we have $f_y \equiv 1$ which yields the desired conclusion.

□

By Mostow's theorem [17] a linear algebraic group W contains a Levi reductive subgroup X such that as an affine algebraic variety W is isomorphic to $X \times Y$ where Y is the unipotent radical of W. More precisely, each element $w \in W$ can be uniquely presented as $w = g \cdot r$ where $g \in X$ and $r \in Y$. This presentation allows us to choose this isomorphism $W \to X \times Y$ uniquely.

6.8. **Corollary.** *For the isomorphism $W \to X \times Y$ as before the left invariant volume form ω on W coincides with $\omega_X \times \omega_Y$ where ω_X is a left-invariant volume form on X and ω_Y is an invariant form on Y.*

Proof. Note that ω is invariant by left multiplications (in particular by elements of X) and also by right multiplication by elements of Y (see, Lemma 6.4 (2)). This determines ω uniquely up to a constant factor. Similarly, by construction $\omega_X \times \omega_Y$ is invariant by left multiplications by elements of X and by right multiplication by elements of Y.

□

6.9. **Example.** Consider the group W of affine automorphisms $z \to az + b$ of the complex line $\mathbb{C}$ with coordinate z. Then $Y \simeq \mathbb{C}_+$ is the group of translations $z \to z + b$ and we can choose $X \simeq \mathbb{C}^*$ so that its elements are automorphisms of form $z \to az$. One can check that the left-invariant volume ω on W coincides with $\frac{\mathrm{d}a}{a^2} \wedge \mathrm{d}b$ while $\omega_X = \mathrm{d}a/a$ and $\omega_Y = \mathrm{d}b$. The isomorphism $W \to X \times Y$ we were talking about presents $az + b$ as a composition of az and $z + b/a$. Thus in this case Corollary 6.8 boils down to the equality

$$\frac{\mathrm{d}a}{a^2} \wedge \mathrm{d}b = \frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}b}{a}.$$

This example admits natural extensions to groups of higher dimensional affine transformations. For the sake of notation we consider such an extension only for the group of 3×3 invertible upper triangular matrices

$$W = \begin{bmatrix} x_1 & y & z \\ 0 & x_2 & w \\ 0 & 0 & x_3 \end{bmatrix}$$

where $x_1, x_2, x_3 \in \mathbb{C}^*, y, z, w \in \mathbb{C}$. Its Haar form is given by

$$\frac{\mathrm{d}x_1}{x_1^3} \wedge \frac{\mathrm{d}x_2}{x_2^2} \wedge \frac{\mathrm{d}x_1}{x_1} \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w.$$

Then $Y \simeq \mathbb{C}^3$ is the group of unipotent upper triangular matrices and $X \simeq (\mathbb{C}^*)^3$ is the group of the invertible diagonal matrices. The isomorphism $W \to X \times Y$ is given by

$$\begin{bmatrix} x_1 & y & z \\ 0 & x_2 & w \\ 0 & 0 & x_3 \end{bmatrix} \mapsto \left(\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, \begin{bmatrix} 1 & \frac{y}{x_1} & \frac{z}{x_1} \\ 0 & 1 & \frac{w}{x_2} \\ 0 & 0 & 1 \end{bmatrix} \right)$$

In this case Corollary 6.8 yields the equality

$$\frac{\mathrm{d}x_1}{x_1^3} \wedge \frac{\mathrm{d}x_2}{x_2^2} \wedge \frac{\mathrm{d}x_3}{x_3} \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w = (\frac{\mathrm{d}x_1}{x_1} \wedge \frac{\mathrm{d}x_2}{x_2} \wedge \frac{\mathrm{d}x_3}{x_3}) \wedge (\frac{\mathrm{d}y}{x_1} \wedge \frac{\mathrm{d}z}{x_1} \wedge \frac{\mathrm{d}w}{x_2}).$$

6.10**. Proposition.** *Let W be a linear algebraic group group, Y be its rationally connected subgroup, and $X = W/Y$ be the homogeneous space of left cosets. Suppose that ω is the left-invariant volume form on W, ω_Y the invariant volume form on Y and ω_X the volume form on the quotient constructed in Proposition* 6.7*. Then the natural projection $p : W \to X$ is a volume fibration with respect to the volume forms ω, ω_X, and ω_Y.*

Proof. Choose locally nilpotent derivations $\sigma_1', \ldots, \sigma_k'$ and semi-simple derivations σ_{k+1}', $\ldots, \sigma_n'$ on W generated by the left multiplication of W by elements of its $\mathbb{C}_+$ and $\mathbb{C}^*$-subgroups and such that they generate tangent space at each point of W. Since they commute with morphism p they yield locally nilpotent and semi-simple derivations $\sigma_1, \ldots, \sigma_n$ on X with the same property. Take any point $x \in X$ and suppose that $\sigma_l, \ldots, \sigma_m$ generate the tangent space $T_x X$ (where $l \leq k \leq m$). Then we have the dominant natural morphism $\psi : G \to X$ from the group $G := \mathbb{C}^{k-l+1} \times (\mathbb{C}^*)^{m-k}$ given by the formula $\bar{t} = (t_l, \ldots, t_m) \to h_{t_l} \circ \cdots \circ h_{t_m}(x)$ where $\bar{t} \in G$ and h_{t_j} is the action of the element t_j from the $\mathbb{C}_+$ or $\mathbb{C}^*$-group corresponding to the j-th factor in G. This morphism is étale at the identical element $o = (0, \ldots, 0, 1, \ldots, 1)$ of G and $\psi(o) = x$. The restriction of ψ to an open Zariski dense subvariety Z of $\mathbb{C}^{m-l+1}$ may be viewed as an étale neighborhood of $x \in X$. Suppose that ω_Z (resp. $\tilde{\sigma}_i$) is the lift of the form ω_X (resp. vector field σ_i) to Z. By Proposition 6.7 ω_Z is invariant under the local phase flow generated by $\tilde{\sigma}_i$. Set $W' = W \times_X Z$. Then by construction, W' is naturally isomorphic to $Z \times Y$ and under this isomorphism each field σ_i' corresponds to the horizontal lift of $\tilde{\sigma}_i$ to $Z \times Y$. Hence $\omega_Z \times \omega_Y$ is invariant under the local phase flow generated by this lift of $\tilde{\sigma}_i$. It is also invariant under right multiplication by elements of Y by Proposition 6.4 (2) and, therefore, determined uniquely by its value at one point. But the form ω is also invariant under the local phase flow generated by σ_i' and under right multiplication by elements of Y again by Proposition 6.4 (2). Therefore, the preimage of ω on W coincides with $\omega_Z \times \omega_Y$ since one can choose ω_Y so that both forms coincide at one point.

□

We finish this section with the following useful observation.

6.11. **Lemma.** *Let W be a reductive group and Y be its Levi semi-simple subgroup. Then the base of the volume fibration $p : W \to X := W/Y$ possesses a family of algebraic vector fields satisfying condition (D) and such that every element of this family admits a p-compatible lift.*

Proof. Let T be the connected component of the center of G. That is, T is a torus $(\mathbb{C}^*)^n$ and $X \simeq T/(T \cap Y)$ is also a torus $(\mathbb{C}^*)^n$ since the group $T \cap Y$ is finite. Let us start with the case when $T \cap Y$ is trivial, i.e. $X = T$. Then $W = X \times Y$ and p is the projection to the first factor. In particular, any "vertical" field $\delta' \in \mathrm{AVF}(W,p)$ contains $\mathbb{C}[X] \subset \mathbb{C}[W]$ in its kernel. Every vector field $\sigma \in \mathrm{Lie}_{alg}^{\omega_X}(X)$ has similarly a lift $\sigma' \in \mathrm{Lie}_{\mathrm{alg}}^{\omega}(W)$ such that this lift contains $\mathbb{C}[Y]$ in its kernel. In particular, $\operatorname{Span}\operatorname{Ker}\sigma' \cdot \operatorname{Ker}\delta' = \mathbb{C}[W]$. Furthermore, any vertical field δ' is of form $\sum_j f_j \delta_j$ where $f_j \in \mathbb{C}[W]$ and δ_j is the natural lift of vector field on Y to W. Since $[\sigma', \delta_j] = 0$ we have $[\sigma', \delta'] \in \mathrm{AVF}(W,p)$ which shows that any family of vector fields on X satisfying Condition (D) has the desired p-compatible lifts.

In the general case when $T \cap Y$ is not trivial we have a commutative diagram

$$\begin{array}{ccc} T \times Y & \xrightarrow{\varphi} & W \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{\psi} & X \end{array}$$

where the horizontal arrows are unramified finite coverings. Let $z_1, \ldots, z_n$ be natural coordinates on $T \simeq (\mathbb{C}^*)^n$ and $w_1, \ldots, w_n$ be natural coordinates on $X \simeq (\mathbb{C}^*)^n$. Then up to constant factors we have $w_j = \prod_{i=1}^n z^{k_{ij}}$. By Example 5.9 a family on X satisfying Condition (D) consists of vector fields of form $\nu = f_j w_j \partial/\partial w_j$ where f_j is a function on X independent of w_j. Note that $w_j \partial/\partial w_j = \sum_{i=1}^n k_{ij} z_i \partial/\partial z_i$. Since $z_i \partial/\partial z_i$ is associated with multiplication by elements of a $\mathbb{C}^*$-subgroup of T it may be viewed as a field on X and we can find its lift to W. Thus the fields $w_j \partial/\partial w_j$ and also ν have lifts to W and we need to check that they are p-compatible.

Let σ' be the lift of one of these fields and σ'' be its preimage on $T \times Y$. Each vertical vector field δ' on W (i.e. it is from the kernel of p_*) generates a vertical vector field δ'' on $T \times Y$ (i.e. it is from the kernel of q_*). As we showed before $[\sigma'', \delta''] \in \mathrm{AVF}(T \times Y, q)$ and therefore $[\sigma', \delta'] \in \mathrm{AVF}(W, p)$. Furthermore, since $\operatorname{Span}\operatorname{Ker}\sigma'' \cdot \operatorname{Ker}\delta'' = \mathbb{C}[T \times Y]$

we have still the equality $\operatorname{Span}\operatorname{Ker}\sigma' \cdot \operatorname{Ker}\delta' = \mathbb{C}[W]$ by virtue of Lemma 9.4. This concludes the proof of p-compatibility and the Lemma.

□

6.12. **Remark.** Convention 5.10 holds under the assumption of Lemma 6.11. This is clear in the case of W isomorphic to the direct product of Y and a torus $T = (\mathbb{C}^*)^k$. In the general case when $W = (Y \times T)/F$ (where F is a finite subgroup of the center of $Y \times T$) one can note that up to factors the vector fields on Y, that will be used in the construction in Section 8, are associated with multiplications by elements of $\mathbb{C}^*$ and $\mathbb{C}_+$-subgroups. Therefore, their natural extensions to $Y \times T$ commute with multiplications by elements of F and can be pushed down to W.

7. Compatibility

7.1. **Notation.** Let G be a semi-simple Lie group, S_0 and S be its SL_2 or PSL_2-subgroups, and $p : G \to X := G/S_0$ be the natural projection into the set of left cosets. Suppose that δ is a completely integrable algebraic vector field on S_0 generated by right multiplications. Then it generates $\delta' \in \operatorname{Lie}^{\omega}_{\text{alg}}(G, p)$. Let $H \simeq \mathbb{C}_+$ be a subgroup of S. Left multiplication by elements of H generate a locally nilpotent derivation σ' on G. Note that $[\sigma', \delta'] = 0$ (i.e. we have an $(S \times S_0)$-action on G) and σ' generates a locally nilpotent derivation σ on X associated with the corresponding H-action on X.

We think of S_0 being fixed and our aim is to find "many" S such that σ' is p-compatible for S, i.e. the vector space generated by $\operatorname{Ker}\sigma' \cdot \operatorname{Ker}\delta'$ coincides with $\mathbb{C}[G]$. From now on we use the (seemingly overloaded) notation of strictly semi-compatibility for pairs of vector fields (for Definition see the Appendix) since it was introduced in the work of Donzelli, Dvorsky and the first author [6] and we like to stick to this earlier introduced notation. We apologize for any inconvenience to the reader.

7.2. **Lemma.** *Suppose $g_0 \in G$ and $S \cap g_0 S_0 g_0^{-1} = \Gamma$. Then the isotropy group of the point $g_0 S_0 \in X$ under the S-action is Γ. In particular, if the S-orbit of $g_0 S_0$ is closed then Γ is reductive by the Matsushima theorem.*

Proof. The coset $g_0 S_0$ is fixed under the action of $s \in S$ if and only if $s g_0 S_0 \subset g_0 S_0$ which implies that $g_0^{-1} s g_0 \in S_0$ and we have the desired conclusion.

□

In the proof of Proposition 7.3 we use slightly modified results from [12]. For the reader's convenience we discuss these results in the Appendix (Lemmas 9.4 and 9.5).

7.3. **Proposition.** *Let $\Gamma_g = S \cap g S_0 g^{-1}$ be finite for every $g \in G$. Then σ' is p-compatible.*

Proof. Consider the quotient morphism $r : G \to Z := G//(S \times S_0)$. Since Γ_g is always finite all orbits are equidimensional and, therefore, closed (indeed, for a reductive group $S \times S_0$ the closure of a non-closed orbit must contain a closed orbit, automatically of smaller dimension, which is impossible because all orbits are of the same dimension).

By Luna's slice theorem for every point $z \in Z$ there exists a Zariski neighborhood $U \subset Z$, a Γ_g-invariant slice $V \subset G$ through a point of $r^{-1}(z)$ such that $r|_V : V \to U$ is a surjective quasi-finite morphism, and a surjective étale morphism $W \to r^{-1}(U)$ where $W = V \times_{\Gamma_g} (S \times S_0)$. In particular, we have a natural surjective quasi-finite morphism $W'' := V \times (S \times S_0) \to r^{-1}(U)$. Clearly, the algebraic vector fields σ'' and δ'' on W'' induced by σ' and δ' are strictly semi-compatible, i.e. the span of $\operatorname{Ker} \sigma'' \cdot \operatorname{Ker} \delta''$ coincides with $\mathbb{C}[W'']$. Note also that for any $\mathbb{C}_+ \simeq H < S$ the quotient $G//H$ is smooth and the quotient morphism $G \to G//H$ is a holomorphic $\mathbb{C}$-fibration over its image. By Lemmas 9.4 and 9.5 in Appendix the restrictions of σ' and δ' to $r^{-1}(U)$ are also strictly semi-compatible. Thus there is a cover $Z = \bigcup U_i$ such that each U_i is of form $U_i = Z \setminus g_i^{-1}(0)$ with $g_i \in \mathbb{C}[Z]$ and the restrictions of of σ' and δ' are strictly semi-compatible on each $W_i = r^{-1}(U_i)$. For any function $h \in \mathbb{C}[Z]$ its restriction $h|_{W_i}$ is contained in $\operatorname{Ker} \sigma'|_{W_i} \cap \operatorname{Ker} \delta'|_{W_i}$. Since for any function $\varphi \in \mathbb{C}[W_i]$ there exists $m > 0$ such that $\varphi g_i^m \in \mathbb{C}[G]$ and since $g_i \in \operatorname{Ker} \sigma' \cap \operatorname{Ker} \delta'$, for an appropriate m the function hg_i^m belongs to the span of $\operatorname{Ker} \sigma' \cdot \operatorname{Ker} \delta'$. Now the desired conclusion follows from the standard application of the Nullstellensatz.

□

7.4. **Lemma.** *Let G, S_0, X, and S be as in Lemma 7.2 and $\Gamma_g = S \cap gS_0g^{-1}$ where $g \in G$. Suppose that Γ_g does not contain a torus $\mathbb{C}^*$ for every $g \in G$. Then every Γ_g is finite.*

Proof. Assume that Γ_{g_0} is not finite for some $g_0 \in G$. Then Γ_{g_0} cannot be reductive (without a torus) and the S-orbit O of $g_0S_0 \in X$ is not closed by the second statement of Lemma 7.2. Furthermore, since any two-dimensional subgroup of $SL_2(\mathbb{C})$ contains $\mathbb{C}^*$ we see that Γ_{g_0} is one-dimensional, i.e. O is two-dimensional. Since S is reductive the closure of O must contain a closed orbit O_1 of some point $g_1S_0 \in X$. Thus $\dim O_1 \leq 1$ and $\dim \Gamma_{g_1} \geq 2$. But in this case as we mentioned Γ_{g_1} contains a torus which yields a contradiction.

□

In order to find S such that $\Gamma_g = g^{-1}S_0g \cap S$ does not contain a torus for every $g \in G$ we need to remind the notion of a principal SL_2 or PSL_2-subgroup of a semi-simple group G (resp. principal $\mathfrak{sl}_2$-subalgebra in the Lie algebra $\mathfrak{g}$ of G) from [4]. Recall that a semi-simple element h of $\mathfrak{g}$ is called regular if the dimension of its centralizer is equal to the rank of $\mathfrak{g}$ (more precisely, this centralizer coincides with a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$). An $\mathfrak{sl}_2$ subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is called principal if it contains a regular semi-simple element h such that every eigenvalue of its adjoint operator is an even integer. The SL_2 (or PSL_2) subgroup generated by such subalgebra is also called principal. For instance, in SL_n up to conjugation every regular element is a diagonal matrix with distinct eigenvalues and any principal SL_2-subgroup acts irreducibly on the natural n-space. Any two principal SL_2-subgroups are conjugated in G and any SL_2-subgroup corresponding to a root is not principal (unless $\mathfrak{g} = \mathfrak{sl}_2$) since its semi-simple elements are not regular.

7.5. **Lemma.** *If S is a principal SL_2 (resp. PSL_2) subgroup of a semi-simple group G and S_0 be any subgroup of G that does not contain regular semi-simple elements. Then $\Gamma_g = g^{-1}S_0 g \cap S$ is finite for every $g \in G$.*

Proof. Note that Γ_g cannot contain a torus since otherwise S_0 contains a regular semi-simple element. Lemma 7.4 implies now the desired conclusion.

□

7.6. **Proposition.** *Let G be a semi-simple Lie group different from $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$. Suppose that $S_0, Z = G/S_0$, $p : G \to Z$, and σ' are is in Notation 7.1. Let S_0 correspond to a root in the Dynkin diagram. Then σ' can be chosen that it is p-compatible (for any S_0 corresponding to a root in the Dynkin diagram!!). Furthermore, there are enough of these p-compatible completely integrable algebraic vector fields σ', so that the Lie algebra L generated by them generates $\mathrm{VF}_{\mathrm{alg}}(Z)$ as a $\mathbb{C}[Z]$-module.*

Proof. Let an SL_2 (or PSL_2) subgroup S_0 correspond to a root and S be a principal SL_2 (or PSL_2) subgroup. By Proposition 7.3 and Lemma 7.5 σ' is p-compatible and we are left with the second statement. Suppose that X, Y, H is a standard triple in the $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}$ of S, i.e. $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$ [5]. In particular, the locally nilpotent vector fields generated by X and Y are of form σ' and they are p-compatible. Suppose that the centralizer of H is the Cartan subalgebra $\mathfrak{h}$ associated with the choice of a root system and X_0, Y_0, H_0 is an $\mathfrak{sl}_2$-triple corresponding to one of the roots. Conjugate S by $x_0 = e^{\varepsilon X_0}$ where ε is a small parameter. Up to terms of order 2 element H goes to $H + \varepsilon[H, X_0]$ after such conjugation, i.e. $[H, X_0]$ belongs (up to second order) to the Lie algebra generated by X, Y, and the nilpotent elements of the Lie algebra of principal SL_2-subgroup $x_0^{-1} S x_0$. Since each X_0 is an eigenvector of the adjoint action of H we have $[H, X_0] = aX_0$. Furthermore, $a \neq 0$ since otherwise X_0 belongs to the centralizer $\mathfrak{h}$ of the regular element H. Thus X_0 and similarly Y_0 are (up to second order) in the Lie algebra L generated by fields of form σ'. The same is true for $H_0 = [X_0, Y_0]$. Thus values of L at any point $z \in Z$ generate the tangent space $T_z Z$ which implies L generates $\mathrm{VF}_{\mathrm{alg}}(Z)$ as a $\mathbb{C}[Z]$-module. □

8. Main Theorem

8.1. **Notation.** In this section G is a semi-simple Lie group except for the proof of Theorem 2. By S_i we denote an SL_2 or PSL_2-subgroup of G (for each index $i \geq 0$) and by $p_i : G \to X_i = G/S_i$ the natural projection. In the case of $S_i \simeq PSL_2$ we assume additionally the validity of Convention 5.10 (which is true under the assumption of Lemma 5.11). By abusing notation we treat $\mathbb{C}[X_i]$ as the subring $p_i^*(\mathbb{C}[X_i])$ in $\mathbb{C}[G]$. Note that Lemma 5.12 implies that $\mathbb{C}[G] \simeq F(G, p_i) \oplus \mathbb{C}[X_i]$ and denote by $\mathrm{pr}_i : \mathbb{C}[G] \to \mathbb{C}[X_i]$ the natural projection. For any semi-simple complex Lie group B denote by $B^{\mathbb{R}}$ its maximal compact subgroup whose complexification coincides

[5] It is unfortunate, but we have to use the classical notation X, Y, H for a standard triple of an $\mathfrak{sl}_2$-algebra while in the rest of the text these symbols denote affine algebraic varieties and groups.

with B (it is unique up to conjugation). Let $K_i = S_i^{\mathbb{R}}$. Define a linear operator $\mathrm{av}_i : \mathbb{C}[G] \to \mathbb{C}[G]$ by

$$\mathrm{av}_i(f) = \int_{K_i} f(wk)\,\mathrm{d}\,\mu_{K_i}(k)$$

for any function $f \in \mathbb{C}[G]$ where $\mu_{K_i}(k)$ is the bi-invariant normalized Haar measure on K_i.

8.2**. Lemma.** *In Notation* 8.1 *we have*

(i) *the right multiplication by an element $k \in K_i$ generates a map $\Psi : \mathbb{C}[G] \to \mathbb{C}[G]$ (given by $f(w) \to f(wk)$) whose restriction to $F(G, p_i)$ is an isomorphism;*

(ii) $\mathrm{Ker}\,\mathrm{av}_i = F(G, p_i)$, *i.e.* $\mathrm{av}_i = \mathrm{pr}_i$ *and* $f - \mathrm{av}_i(f) \in F(G, p_i)$ *for every* $f \in \mathbb{C}[G]$.

Proof. The right multiplication transforms every fiber $Y := p_i^{-1}(x)$ into itself and each completely integrable algebraic vector field on it into a similar field. Hence for every $f \in F(G, p_i)$ we have $\Psi(f)|_{p_i^{-1}(x)} \in F_Y$. Now (i) follows from Lemma 5.12. Thus operator av_i respects the direct sum $\mathbb{C}[G] \simeq F(G, p_i) \oplus \mathbb{C}[X_i]$ and sends $\mathbb{C}[G]$ onto $\mathbb{C}[X_i]$ so that its restriction to $\mathbb{C}[X_i]$ is identical map. This implies (ii).

□

8.3**. Lemma.** *Let S_0 and K_0 be as before and let $L = G^{\mathbb{R}}$ contain K_0. Consider the natural inner product on $\mathbb{C}[G]$ given by*

$$h_1 \cdot h_2 = \int_{l \in L} h_1(l)\bar{h}_2(l)\mathrm{d}\mu_L(l)$$

where μ_L is the bi-invariant measure on L. Then $\mathbb{C}[X_0]$ is the orthogonal complement of $F(G, p_0)$.

Proof. Consider $h_1 \in \mathbb{C}[G]$ and $h_2 \in \mathbb{C}[X_0]$. Show that $\mathrm{av}_0(h_1) \cdot h_2 = h_1 \cdot h_2$. We have

$$I := \mathrm{av}_0(h_1) \cdot h_2 = \int_L \int_{K_0} h_1(lk_0)\bar{h}_2(l)\mathrm{d}\mu_{K_0}(k_0)\mathrm{d}\mu_L(l).$$

By Fubini's theorem

$$I = \int_{K_0} \int_L h_1(lk_0)\bar{h}_2(l)\mathrm{d}\mu_L(l)\mathrm{d}\mu_{K_0}(k_0).$$

Set $l' = lk_0$. Then $h_1(lk_0) = h_1(l')$ and $h_2(l) = h_2(l'k_0^{-1}) = h_2(l')$ since h_2 is right K_0-invariant. Using the fact that measures are invariant we see that I coincides with

$$\int_{K_0} \int_L h_1(l')\bar{h}_2(l')\mathrm{d}\mu_L(l')\mathrm{d}\mu_{K_0}(k_0) =$$

$$\int_L h_1(l')\bar{h}_2(l')\mathrm{d}\mu_L(l')$$

where the last equality holds since measure μ_{K_0} is normalized. Thus $\mathrm{av}(h_1) \cdot h_2 = h_1 \cdot h_2$. Now the desired conclusion follows from Lemma 8.2 and the fact that $C[G] \simeq F(G, p_0) \oplus \mathbb{C}[X_0]$. □

8.4. **Corollary.** *Let $S_0, \ldots, S_m$ be as in Notation 8.1 with each $K_i = S_i^{\mathbb{R}} \subset L$. Set $F = \sum_{i=0}^m F(G, p_i)$. Then the orthogonal complement of F in $\mathbb{C}[G]$ coincides with the subspace V of functions that are invariant with respect to any S_i-action generated by the right multiplication. In particular, if this set $S_0, \ldots, S_m$ contains all SL_2 or PSL_2-subgroups corresponding to simple positive roots for the Lie algebra of G then this orthogonal complement V consists of constants only and $\mathbb{C}[G] \simeq F \oplus \mathbb{C}$.*

Proof. Indeed, treating $\mathbb{C}[X_i]$ as $p_i^*(\mathbb{C}[X_i])$ we see that by Lemma 8.3 the orthogonal complement of F in $\mathbb{C}[G]$ is $V = \bigcap_{i=0}^m \mathbb{C}[X_i]$ which is exactly the space of functions invariant under each S_i-action. For the second statement note that if the sequence $\{S_i\}$ of subgroups generate the whole group G then these invariant functions must be constants. $\square$

8.5. **Lemma.** *Let $\{S_i\}_{i=0}^m$, F, and V be as in Corollary 8.4 and $f_0 \in \mathbb{C}[G] \backslash F$. Consider the smallest subspace $U \subset \mathbb{C}[G]$ that contains f_0 and such that for every i and every $f \in U$ function $\mathrm{av}_i(f)$ is also in U. Then*

(1) *U is of some finite dimension N;*
(2) *$\dim U \cap F = N - 1$ and $\dim U \cap V = 1$.*

Proof. Consider a closed embedding $\rho : G \hookrightarrow \mathbb{C}^n$ such that the induced action of G on $\mathbb{C}^n$ is linear. This yields a filtration on $\mathbb{C}[G]$ defined by minimal degrees of polynomial extensions of regular functions on G to $\mathbb{C}^n$. Let W_k be the subspace of $\mathbb{C}[G]$ that consists of functions of degree at most k and $\Phi_l : \mathbb{C}[G] \to \mathbb{C}[G]$ be the automorphism given by $f(w) \to f(wl)$ for $l \in L$. Since the G-action on $\mathbb{C}^n$ is linear each automorphism Φ_l sends W_k into itself. Hence the definition of av_i implies that $\mathrm{av}_i(W_k) \subset W_k$. Thus $U \subset W_k$ as soon as $f_0 \in W_k$ which yields (1).

Denote the orthogonal projection onto V by $\mathrm{pr} : \mathbb{C}[G] \to V$ and let $f_0' = \mathrm{pr}(f_0)$. Since $f_0 \notin F$ we have $f_0' \neq 0$. Let P be the hyperplane in $\mathbb{C}[G]$ that consists of vectors of form $f_0' + P_0$ where P_0 is the hyperspace orthogonal to f_0'. In particular P contains f_0. Since $f_0' \in \mathbb{C}[X_i]$ for every i we see that P is orthogonal to each $\mathbb{C}[X_i]$. Recall that the operator $\mathrm{av}_i = \mathrm{pr}_i$ is just the orthogonal projection to $\mathbb{C}[X_i]$, i.e. P is invariant with respect to these operators. In particular, if we set $f_J = \mathrm{av}_{j_1} \circ \cdots \circ \mathrm{av}_{j_s}(f_0)$ for a multi-index $J = (j_1, \ldots, j_s)$ with $j_t \in \{0, \ldots, m\}$ then $f_J \in P \cap U$.

We want to show that for some sequence of such multi-indices f_J is convergent to a nonzero element of V or, equivalently, f_J' is convergent to an element of $V \cap P_0$ for $f_J' = f_J - f_0'$. Consider the subspace U' generated by vectors of form f_J'. Let $U_i' = U' \cap \mathbb{C}[X_i]$ and $I = (J, i)$, i.e. $f_I' = \mathrm{av}_i(f_J')$. By construction the operator $\mathrm{pr}_i|_{U'} = \mathrm{av}_i|_{U'}$ is just the orthogonal projection to U_i'. Hence if $f_J' \notin U_i'$ we have $||f_I'|| < ||f_J'||$. Since U' is finite-dimensional this implies that one can choose $\{f_J\}$ convergent to an element $v \in U'$ and we can suppose that v has the smallest possible norm. Then $\mathrm{pr}(v) = v$ because of the last inequality. On the other hand $\mathrm{pr}(f_J') = \mathrm{pr}(f_J) - \mathrm{pr}(f_0') = \mathrm{pr} \circ \mathrm{av}_{j_1} \circ \cdots \circ \mathrm{av}_{j_s} \circ (f_0) - \mathrm{pr}(f_0) = f_0' - f_0' = 0$. Thus $v = 0$. This shows $f_0' \in V \cap U$ and therefore $\dim U \cap V \geq 1$.

On the other hand U contains a subspace U_0 generated by vectors of form $f_0 - f_J$. One can see that U_0 is of codimension 1 in U. Furthermore, $f_0 - f_I = (f_0 - f_J) + (f_J - f_I)$. Note that $f_J - f_I = f_J - \mathrm{av}_i(f_J) \in F(G, p_i) \subset F$ by Lemma 8.2. Thus, using induction by the length of the multi-index J one can show that $f_0 - f_J \in F$. That is, $\dim U \cap F \geq N - 1$ which concludes the proof.

□

8.6. **Proposition.** *Any semi-simple group G has the algebraic volume density with respect to the invariant volume.*

Proof. Choose $S_0, \ldots, S_m$ as in Corollary 8.4 and such that they correspond to simple nodes in the Dynkin diagram (it is possible since every semi-simple group G has a compact real form, i.e. we can suppose that $S_i^{\mathbb{R}} = K_i \subset L = G^{\mathbb{R}}$). Consider the natural projections $p_i : G \to X_i := G/S_i$ to the sets of left cosets. Choose p_i-compatible completely integrable algebraic vector fields σ' as in Proposition 7.6 and denote their collection by Θ. That is, vector fields from Θ are of zero divergence, they commute with any $\delta \in \mathrm{VF}_{alg}^{\omega}(G, p_i)$, and they are independent from index i. Furthermore, these fields from Θ can be viewed also as zero divergence vector fields on X_i that generate $\mathrm{VF}_{\mathrm{alg}}(X_i)$ as a $\mathbb{C}[X_i]$-module.

Let us fix an index i. Any algebraic vector field on G is of form

$$\nu = \sum_{\theta \in \Theta} h_\theta \theta + \delta$$

where the sum contains only finite number of nonzero terms, $h_\theta \in \mathbb{C}[G]$ and $\delta \in \mathrm{VF}_{\mathrm{alg}}(G, p_i)$. Since $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ have property (C) from 5.1, $\mathbb{C}[G] = F(G, p_i) \oplus \mathbb{C}[X_i]$ by Lemma 5.12. Thus by virtue of Proposition 5.16 adding fields from $\mathrm{Lie}_{alg}^{\omega}(G)$ to ν we get a field

$$\nu_i = \sum_{\theta \in \Theta} h_\theta^i \theta + \delta_i$$

where $\delta_i \in \mathrm{VF}_{\mathrm{alg}}^{\omega}(G, p_i)$, $h_\theta^i = \mathrm{av}_i(h_\theta)$, and $\mathrm{av}_i = \mathrm{pr}_i$ is as Notation 8.1. That is, $h_\theta^i \in \mathbb{C}[X_i]$. Suppose that $\mathrm{div}_\omega \, \nu = 0$ and, therefore, $\mathrm{div}_\omega(\nu_i) = 0$. Note that

$$\mathrm{div}_\omega(\sum_{\theta \in \Theta} h_\theta^i \theta) = \sum_{\theta \in \Theta} \theta(h_\theta^i) \in \mathbb{C}[X_i]$$

while $\mathrm{div}_\omega \, \delta_i \in F(G, p_i)$. Hence $\mathrm{div}_\omega(\sum_{\theta \in \Theta} h_\theta^i \theta) = \mathrm{div}_\omega(\delta_i) = 0$. Since $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ have the algebraic volume density property, $\delta_i \in \mathrm{Lie}_{alg}^{\omega}(G, p)$ by Corollary 5.14. Thus $\nu - \tilde{\nu}_i \in \mathrm{Lie}_{alg}^{\omega}(G)$ where

$$\tilde{\nu}_i = \sum_{\theta \in \Theta} h_\theta^i \theta.$$

In particular it suffices to show that $\tilde{\nu}_i \in \mathrm{Lie}_{alg}^{\omega}(G)$ and, therefore, we can suppose that $\delta = 0$ in the original formula for ν. Repeating this procedure we see that $\nu - \tilde{\nu}_J \in$

$\mathrm{Lie}^{\omega}_{alg}(G)$ where

$$\tilde{\nu}_J = \sum_{\theta \in \Theta} h^J_\theta \theta$$

for a multi-index $J = (j_1, \ldots, j_s)$ with $j_t \in \{0, \ldots, m\}$ and $h^J_\theta = \mathrm{av}_{j_1} \circ \cdots \circ \mathrm{av}_{j_s}(h_\theta)$.

By Corollary 8.4 and Lemma 8.5 the vector space generated by h_θ and functions of form h^J_θ is also generated by constants and functions of form $h_\theta - h^J_\theta$. Thus adding to $\nu = \sum_{\theta\in\Theta} h_\theta \theta$ vector fields of form $\nu - \tilde{\nu}^J_\theta$ and $c\theta$ (where $c \in \mathbb{C}$) we can reduce the number of nonzero terms in this sum. Hence $\nu \in \mathrm{Lie}^{\omega}_{alg}(G)$ which implies the desired conclusion.

□

8.7. **Proof of Theorem 2.** Let us start with the case when G is reductive. Suppose that Y is its Levi semi-simple subgroup. Then by Proposition 6.11 $p : G \to X := G/Y$ is a volume fibration whose base possesses a family of algebraic vector fields with property (C) admitting p-compatible lifts. Furthermore, the base (which is a torus) and the fiber of this fibration have algebraic volume density property by Proposition 8.6. Thus G has the algebraic volume density property by Theorem 4.

Now consider an arbitrary linear algebraic group G and let Y be its unipotent ideal and X be a Levi reductive subgroup of G. By Corollary 6.8 the Mostow isomorphism $G \to X \times Y$ makes the left invariant volume ω on G equal to $\omega_X \times \omega_Y$ where ω_X is left invariant on X and ω_Y is invariant on Y. Now by Proposition 4.3 G has the algebraic volume density property with respect to ω which concludes the proof of our Main Theorem.

8.8. **Remark.** Theorem 2 remains, of course, valid if instead of the left invariant volume form we consider the right invariant one, because the affine automorphism $G \to G$, $g \to g^{-1}$ transforms the left invariant volume form into the right one while preserving the complete integrability of the algebraic vector fields.

9. Appendix: Strictly semi-compatible fields

9.1. **Notation.** In this section H_i is isomorphic to $\mathbb{C}_+$ for $i = 1, 2$. We suppose also that X is a normal affine algebraic variety equipped with nontrivial algebraic H_i-actions (in particular, each H_i generates an algebraic vector field δ_i on X). The categorical quotients will be denoted $X_i = X//H_i$ and the quotient morphisms by $\rho_i : X \to X_i$.

9.2. **Definition.** A pair (δ_1, δ_2) of algebraic vector fields (as in Notation 9.1) is called strictly semi-compatible if the span of $\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2$ coincides with $\mathbb{C}[X]$.

We shall need the following obvious geometric reformulation of Definition.

9.3. **Proposition.** *Let δ_1 and δ_2 be as in Notation 9.1. Set $\rho = (\rho_1, \rho_2) : X \to Y := X_1 \times X_2$ and Z equal to the closure of $\rho(X)$ in Y. Then δ_1 and δ_2 are strictly semi-compatible if and only if $\rho : X \to Z$ is an isomorphism.*

9.4. **Lemma.** *Let X, H_i, X_i, δ_i, and ρ_i be as in Notation 9.1 with δ_1 and δ_2 being strictly semi-compatible and $[\delta_1, \delta_2] = 0$. Set $\Gamma = H_1 \times H_2$. Let X' be a normal affine algebraic variety equipped with a non-degenerate Γ-action and $p : X \to X'$ be a finite Γ-equivariant morphism (for each $i = 1, 2$), i.e. we have commutative diagrams*

$$\begin{array}{ccc} X & \stackrel{\rho_i}{\to} & X_i \\ \downarrow p & & \downarrow q_i \\ X' & \stackrel{\rho_i'}{\to} & X_i' \end{array}$$

where $\rho_i' : X' \to X_i' = X'//H_i$ is the quotient morphism of the H_i-action on X'. Suppose also that ρ_1' makes X' an étale locally trivial $\mathbb{C}$-fibration over $\rho_1'(X')$, and X_1, X_2 are affine[6]. *Then* $\operatorname{Span}(\mathbb{C}[X_1'] \cdot \mathbb{C}[X_2']) = \mathbb{C}[X']$.

Proof. Since p is finite, every $f \in \mathbb{C}[X_i] \subset \mathbb{C}[X]$ is a root of a minimal monic polynomial with coefficients in $\mathbb{C}[X']$ that are constant on H_i-orbits (since otherwise f is not constant on these orbits). By the universal property these coefficients are regular on X_i', i.e. f is integral over $\mathbb{C}[X_i']$ and q_i is finite. Consider the commutative diagram

$$\begin{array}{ccc} X & \stackrel{\rho}{\to} & X_1 \times X_2 \\ \downarrow p & & \downarrow q \\ X' & \stackrel{\rho'}{\to} & X_1' \times X_2' \end{array}$$

where $\rho = (\rho_1, \rho_2)$, $\rho' = (\rho_1', \rho_2')$, and $q = (q_1, q_2)$. Let Z (resp. Z') be the closure of $\rho(X)$ in $X_1 \times X_2$ (resp. $\rho'(X')$ in $X_1' \times X_2'$). By Proposition 9.3 $\rho : X \to Z$ is an isomorphism and, therefore, by Lemma 3.6 in [12] $\rho' : X' \to Z'$ is birational finite. Since the statement of this Lemma is equivalent to the fact that ρ' is an isomorphism, it suffices to prove ρ' is a holomorphic embedding.

Consider an orbit $O \subset X$ of H_1 and set $O' = p(O)$, $O_2' = \rho_2'(O')$. Each of these orbits is isomorphic to $\mathbb{C}_+$ and, therefore, the H_1-equivariant finite morphisms $p|_O : O \to O'$ and $\rho_2'|_{O'} : O' \to O_2'$ must be isomorphisms. Thus one has a regular function on X_2' whose restriction yields a coordinate on $O' \simeq O_2' \subset X_2'$. Since locally X' is biholomorphic to $U \times O'$ where U is an open subset of $\rho_1'(X') \subset X_1'$ we see that $\rho' : X' \to X_1' \times X_2'$ is a local holomorphic embedding, i.e. it remains to show that ρ' is injective. For any $x \in X$ set $x' = p(x)$, and $x_i' = \rho_i'(x')$. Assume that x and $y \in X$ are such that $(x_1', x_2') = (y_1', y_2')$. Arguing as in Lemma 3.6[7] in [12] we can suppose that x and y belong to the same fiber of ρ_1 that is, by assumption, an H_1-orbit O. Since $\rho_2'|_{O'} : O' \to O_2'$ is an isomorphism we have $x' = y'$ which implies the desired conclusion.

□

[6] In all cases we apply this Lemma the $\mathbb{C}_+$-action generated by H_i extends to an algebraic $SL_2(\mathbb{C})$-action and, therefore, X_i is affine automatically by the Hadziev theorem [11].

[7] It is shown in that Lemma that $\rho_j(p^{-1}(x')) = q_j^{-1}(x_j')$ for a general point $x \in X$ and to adjust the argument to the present situation one needs it to be true for every point in X, but this follows, of course, from continuity and finiteness of q_j.

9.5. **Lemma.** *Let the assumption of Lemma* 9.4 *hold with one exception: instead of the finiteness of* p *we suppose that there are a surjective quasi-finite morphism* $r : S \to S'$ *of normal affine algebraic varieties equipped with trivial* Γ*-actions and a surjective* Γ*-equivariant morphism* $\varrho' : X' \to S'$ *such that* X *is isomorphic to fibred product* $X' \times_{S'} S$ *with* $p : X \to X'$ *being the natural projection (i.e.* p *is surjective quasi-finite). Then the conclusion of Lemma* 9.4 *remains valid.*

Proof. By construction, $X_i = X_i' \times_{S'} S$. Thus we have the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\rho} & (X_1' \times X_2') \times_{(S' \times S')} (S \times S) & \xrightarrow{(\tau,\tau)} & S \times S \\ \downarrow p & & \downarrow q & & \downarrow (r,r) \\ X' & \xrightarrow{\rho'} & X_1' \times X_2' & \xrightarrow{(\tau',\tau')} & S' \times S'. \end{array}$$

Set Z (resp. Z') equal to the closure of $\rho(X)$ in $X_1 \times X_2$ (resp. $\rho'(X')$ in $X_1' \times X_2'$) and let $D \simeq S$ (resp. $D' \simeq S'$) be the diagonal subset in $S \times S$ (resp. $S' \times S'$). Since $X = X' \times_{S'} S$ we see that $Z = Z' \times_{D'} D$.

Assume that $\rho'(x') = \rho'(y') =: z'$ for some $x', y' \in X'$. Then by the commutativity of the diagram we have also $\varrho'(x') = \varrho'(y') =: s'$. Since r is surjective $r(s) = s'$ for some $s \in S$. Thus the elements (x', s) and (y', s) of $X' \times_{S'} S$ go to the same element (z', s') of $Z' \times_{D'} D$ under morphism ρ. By Lemma 9.3 $\rho : X \to Z$ is an isomorphism and therefore $x' = y'$. Hence $\rho' : X' \to Z'$ is bijective[8]. It was shown in the proof of Lemma 9.4 that ρ' is locally biholomorphic, i.e. it is an isomorphism which implies the desired conclusion. □

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