FREE $\mathbf{C}_+$-ACTIONS ON $\mathbf{C}^3$ ARE TRANSLATIONS

SHULIM KALIMAN [1]

Department of Mathematics, University of Miami
Coral Gables, Florida 33124, USA
(e-mail: kaliman@math.miami.edu)

ABSTRACT. Let X be a smooth contractible three-dimensional affine algebraic variety with a free algebraic $\mathbf{C}_+$-action on it such that $S = X//\mathbf{C}_+$ is smooth. We prove that X is isomorphic to $S \times \mathbf{C}$ and the action is induced by a translation on the second factor. As a consequence we show that any free algebraic $\mathbf{C}_+$-action on $\mathbf{C}^3$ is a translation in a suitable coordinate system.

1. Introduction. In 1968 Rentschler [Re] proved that every algebraic action of the additive group $\mathbf{C}_+$ of complex numbers on $\mathbf{C}^2$ is triangular in a suitable polynomial coordinate system (which follows also from an earlier theorem of Gutwirth [Gu]). This implies that a free $\mathbf{C}_+$-action on $\mathbf{C}^2$ (i.e. an action without fixed points) can be viewed as a translation. In 1984 Bass [Ba] found a $\mathbf{C}_+$-action on $\mathbf{C}^3$ which is not triangular in any coordinate system, and in 1990 Winkelmann [Wi] constructed a free $\mathbf{C}_+$-action on $\mathbf{C}^4$ which is not a translation. But the question about free $\mathbf{C}_+$-actions on $\mathbf{C}^3$ remained open (e.g., see [Sn] , [Kr], [DaFr], [DeFi]). While working on this problem we consider a more general situation when there is a nontrivial algebraic $\mathbf{C}_+$-action on a complex three-dimensional affine algebraic variety X such that its ring of regular functions is factorial. By a theorem of Zariski [Za] the algebraic quotient $X//\mathbf{C}_+$ is isomorphic to an affine surface S. Let $\pi : X \to S$ be the natural projection. Then there is a curve $\Gamma \subset S$ such that for $E = \pi^{-1}(\Gamma)$ the variety $X \setminus E$ is isomorphic to $(S \setminus \Gamma) \times \mathbf{C}$ over $S \setminus \Gamma$. The study of morphism $\pi|_E : E \to \Gamma$ is central for this paper. As an easy consequence of the Stein factorization one can show that $\pi|_E = \theta \circ \kappa$ where $\kappa : E \to Z$ is a surjective morphism into a curve Z with general fibers isomorphic to $\mathbf{C}$, and $\theta : Z \to \Gamma$ is a quasi-finite morphism. A more delicate fact (Proposition

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3.2) is that in the case of a smooth contractible X morphism θ is, actually, finite and, furthermore, each irreducible component Z^1 of Z, such that $\theta|_{Z^1}$ is not injective, is rational and $\theta(Z^1)$ is a polynomial curve. A geometrical observation (Proposition 4.2) shows that in the case of a free $\mathbf{C}_+$-action and a smooth S such a component Z^1 cannot exist. If the restriction of θ to any irreducible component of Z is injective then Γ can be chosen empty (Theorem 5.2), i.e. X is isomorphic to $S \times \mathbf{C}$ over S. In combination with Miyanishi's theorem [Miy80] this yields the long-expected result.

Theorem 1. *Every free algebraic $\mathbf{C}_+$-action on $\mathbf{C}^3$ is a translation in a suitable polynomial coordinate system.*

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2. The Existence of Decomposition $\pi|_E = \theta \circ \kappa$.

Lemma 2.1. *Let X be a factorial affine algebraic variety of dimension 3 with a nontrivial $\mathbf{C}_+$-action on it and $\pi : X \to S$ be the quotient morphism. Then S is a factorial affine algebraic surface, $\pi^{-1}(s_0)$ is either a curve or empty for any $s_0 \in S$, and $E = \pi^{-1}(\Gamma)$ is a nonempty irreducible surface for every closed irreducible curve $\Gamma \subset S$. Furthermore, if $\Gamma = g^*(0)$ for a regular function g on S then $E = (g \circ \pi)^*(0)$.*

Proof. By [Za] S is an affine algebraic surface. Suppose that E is the union of a surface E' and a curve C where E' and C may be empty. Assume $\pi(E')$ is not dense in Γ. Consider a rational function h on S with poles on Γ only. Let e be the product of h and a regular function that does not vanish at general points of Γ but vanishes on $\pi(E')$ with sufficiently large multiplicity such that $e \circ \pi$ vanishes on $E' \setminus C$. As X is factorial, by deleting singularities in codimension 2 we see that $e \circ \pi$ is a regular function on X. As it is invariant under the action, e must be regular. Contradiction. Thus $\pi(E')$ is dense in Γ.

Since X is factorial E' is the zero fiber of a regular function f on X. As f does not vanish on a general fiber $\pi^{-1}(s) \simeq \mathbf{C}$ it is constant on each general fiber. Hence f is invariant under the action, i.e. $f = g \circ \pi$ where g a regular function on S which implies that g is a defining function for Γ (i.e. S is factorial) and, furthermore, $E = E' \neq \emptyset$.

If for a regular function f_1 on X its zero fiber is an irreducible component of E then f_1 is again invariant under the action, i.e. $f_1 = g_1 \circ \pi$. As g_1 vanishes on Γ only it must be proportional to g. Thus E is irreducible and we get the last statement as well. If we assume that $\pi^{-1}(s_0)$ contains a surface for $s_0 \in S$ the similar argument yields a regular function on S that vanishes at a finite set only which is impossible. $\square$

If we have a nontrivial $\mathbf{C}_+$-action on a three-dimensional affine algebraic variety X with a quotient morphism $\pi : X \to S = X//\mathbf{C}_+$ then it is well-known that there is a closed curve $\Gamma \subset S$ such that for $\breve{S} = S \setminus \Gamma$ and $E = \pi^{-1}(\Gamma)$ the variety $X \setminus E = \pi^{-1}(\breve{S})$ is naturally isomorphic to $\breve{S} \times \mathbf{C}$ over $\breve{S}$. We say that two $\mathbf{C}_+$-actions on X are equivalent if they have the same collection of general orbits (or, equivalently, the associated locally nilpotent derivations [2] have the same kernel) and, therefore, generate the same quotient morphism.

Lemma 2.2. *Let X be a factorial affine algebraic variety of dimension 3 with a nontrivial $\mathbf{C}_+$-action on it, $\pi : X \to S$ be the quotient morphism, E and Γ be as before. Then there exist a curve Z, a quasi-finite morphism $\theta : Z \to \Gamma$, and a surjective morphism $\kappa : E \to Z$ so that $\pi|_E = \theta \circ \kappa$ and Z contains a Zariski dense subset Z^* for which $E^* = \kappa^{-1}(Z^*)$ is isomorphic to $Z^* \times \mathbf{C}$ over Z^*. Furthermore, there exists an equivalent action that is free on E^*.*

Proof. Extend $\pi|_E : E \to \Gamma$ to a proper morphism $\bar{\pi} : \bar{E} \to \bar{\Gamma}$ of complete varieties. Consider a closed curve Γ' in $\bar{E} \setminus E$ that contains singularities of each general fiber of $\bar{\pi}$ provided these singularities exist. One can suppose that $\bar{E}$ is contained in $\mathbf{P}^n$ and blow $\mathbf{P}^n$ up with respect to the ideal sheaf generated by Γ'. Replacing $\bar{E}$ with its strict transform we obtain a new proper morphism $\bar{\pi} : \bar{E} \to \bar{\Gamma}$. Repeating this procedure several times, if necessary, we remove the singularities of general fibers $\bar{\pi}$ and make them smooth. Thus for a general point $s \in \bar{\Gamma}$ the fibers $\pi^{-1}(s)$ and $\bar{\pi}^{-1}(s)$ have the same number of connected components, and we shall see below that the components of $\pi^{-1}(s)$ are isomorphic to $\mathbf{C}$. By the Stein factorization theorem morphism $\bar{\pi}$ is a composition of morphisms $\bar{\kappa} : \bar{E} \to \bar{Z}$ and $\bar{\theta} : \bar{Z} \to \bar{\Gamma}$ where $\bar{Z}$ is a curve, $\bar{\kappa}$ has connected fibers, and the $\bar{\theta}$ is finite. Set $\kappa = \bar{\kappa}|_E$, $Z = \kappa(E)$ and $\theta = \bar{\theta}|_Z$. This is

[2] Recall that there is a natural bijective correspondence between $\mathbf{C}_+$-actions on X and locally nilpotent derivations of the algebra of regular functions on X (e.g., see [Re])

what we need in the first statement of Lemma.

For the second statement we suppose that the action is trivial on an irreducible component E_1 of E, i.e. the associated locally nilpotent derivation ν vanishes (as a vector field) on E_1. By Lemma 2.1 $E_1 = \pi^{-1}(\Gamma_1)$ where Γ_1 is an irreducible component of Γ. Consider a regular function g_1 on S for which $\Gamma_1 = g_1^*(0)$. Dividing ν by a power of $g_1 \circ \pi$ we obtain an equivalent locally nilpotent derivation that does not vanish identically on E_1 which implies the second statement and the fact that for a general point $s \in \Gamma$ the fiber $\pi^{-1}(s)$ is a disjoint union of $\mathbf{C}$-curves which are orbits. $\square$

Notation 2.3. Unless it is stated otherwise for the rest of the paper we suppose that $X, S, E, \Gamma, Z, E^*, Z^*, \pi, \theta$, and κ are as Lemma 2.2. Furthermore, we suppose always that Z^* is of form $\theta^{-1}(\Gamma^*)$ where $\Gamma^* = \Gamma \setminus F$ and F is a finite subset of Γ. There is some freedom in the choice of F and Γ. In particular, taking "bigger" F and Γ we suppose that Γ^*, Z^*, and $S^* = S \setminus F$ are smooth. We set also $X^* = \pi^{-1}(S^*)$, $\breve{S} = S \setminus \Gamma$, and $\breve{X} = \pi^{-1}(\breve{S})$, i.e. $\breve{X} = X \setminus E \simeq \breve{S} \times \mathbf{C}$.

3. Finiteness of θ.

In the case of $S \simeq \mathbf{C}^2$ there is an isomorphism $H_0(F) \simeq H_3(S \setminus F)$: namely we assign to each $x_0 \in F$ the 3-cycle presented by a small sphere with center at x_0. The same remains true when S is normal (and S is even factorial by Lemma 2.1).

Lemma 3.1. (i) *There exists a natural isomorphism* $\chi : H_0(F) \to H_3(S^*)$.

(ii) *Let* $\varphi : H_3(S^*) \to H_1(\Gamma^*)$ *be the composition of the Thom isomorphism* $H_3(S^*, \breve{S}) \to H_1(\Gamma^*)$ *and the natural homomorphism* $H_3(S^*) \to H_3(S^*, \breve{S})$*, let* $s_0 \in F$*,* $\omega = \chi(s_0)$*, and* Γ_j*,* $j = 1, \ldots, k$ *be irreducible analytic branches of* Γ *at* s_0*. Suppose that* γ_j *is a simple loop in* Γ_j *around* s_0 *with positive orientation. Then* $\varphi(\omega) = \sum_{j=1}^k \gamma_j$ *(where we treat* γ_j *as an element of* $H_1(\Gamma^*)$*).*

Proof. As $H_3(S) = H_4(S) = 0$ [Mil, Th. 7.1] the exact homology sequence of pair (S, S^*) shows that $H_3(S^*) \simeq H_4(S, S^*)$. Thus in (i) it suffices to prove that $H_4(S, S^*) \simeq H_0(F)$. As S may not be smooth we cannot apply Thom's isomorphism. But by the excision theorem $H_4(S, S^*) \simeq H_4(U, U \setminus F)$ where U is a small Euclidean neighborhood of F in S, i.e. U is a disjoint union $\bigcup_i U_i$ of neighborhoods U_i of a point s_i running over F. Hence $H_4(S, S^*) = \oplus_i H_4(U_i, U_i \setminus s_i)$. Let B_ϵ be the ball of radius ϵ in $\mathbf{C}^n \supset S$ with center at $s_0 \in F$ and $U_0 = S \cap B_\epsilon$. For small $\epsilon > 0$, $\omega = S \cap \partial B_\epsilon$ is a three-dimensional real manifold and U_0 is a cone over ω [Lo]. Hence $H_4(U_0, U_0 \setminus s_0) \simeq H_3(U_0 \setminus s_0) \simeq \mathbf{Z}$ and U_0 and ω are generators of $H_4(U_0, U_0 \setminus s_0)$ and $H_3(U_0 \setminus s_0)$ respectively. Thus $H_4(S, S^*) \simeq H_0(F)$ which is (i). For small $\epsilon > 0$ the sphere ∂B_ϵ and, thus, ω meet each Γ_j transversally along a simple loop γ_j around s_0 in Γ_j. Hence the description of Thom's isomorphism [Do] yields $\varphi(\omega) = \sum_{j=1}^k \gamma_j$. □

Proposition 3.2. *Let the notation of 2.3 hold, $H_3(X) = 0$, and X^* be smooth.*

(i) *Then θ is a finite morphism.*

(ii) *Let X be also smooth and $H_2(X) = 0$. Let Z^1 be an irreducible component of Z and $\Gamma^1 = \theta(Z^1)$. If $\theta|_{Z^1} : Z^1 \to \Gamma^1$ is not bijective then Γ^1 is a polynomial curve (i.e. its normalization is $\mathbf{C}$) and Z^1 is rational.*

Proof. Recall that $\breve{X} \simeq \breve{S} \times \mathbf{C}$. We have the following commutative diagram of the exact homology sequences of pairs

$$\begin{array}{ccccccccccccc}
\ldots \longrightarrow & H_{j+1}(\breve{X}) & \longrightarrow & H_{j+1}(X^*) & \longrightarrow & H_{j+1}(X^*, \breve{X}) & \longrightarrow & H_j(\breve{X}) & \longrightarrow & H_j(X^*) & \longrightarrow \ldots \\
& \downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \simeq & & \downarrow & \\
\ldots \longrightarrow & H_{j+1}(\breve{S}) & \longrightarrow & H_{j+1}(S^*) & \longrightarrow & H_{j+1}(S^*, \breve{S}) & \longrightarrow & H_j(\breve{S}) & \longrightarrow & H_j(S^*) & \longrightarrow \ldots
\end{array}$$

which we consider for $j = 2$. As Γ^*, S^*, and E^* are smooth, and $X^* \setminus \breve{X} = E^*$, the Thom isomorphism [Do] implies $H_i(X^*, \breve{X}) \simeq H_{i-2}(E^*)$. Similarly, $H_i(S^*, \breve{S}) \simeq H_{i-2}(\Gamma^*)$. As $H_3(\breve{S}) = 0$ [Mil, Th. 7.1] and $H_i(E^*) = H_i(Z^*)$ we have the crucial diagram

$$\begin{array}{ccccccccccccc}
\ldots \longrightarrow & 0 & \longrightarrow & H_3(X^*) & \xrightarrow{\varphi'} & H_1(Z^*) & \xrightarrow{\psi'} & H_2(\breve{X}) & \xrightarrow{\chi'} & H_2(X^*) & \longrightarrow \ldots \\
& \downarrow & & \downarrow \delta_3 & & \downarrow \theta_1 & & \simeq \downarrow \breve{\delta}_2 & & \downarrow \delta_2 & \\
\ldots \longrightarrow & 0 & \longrightarrow & H_3(S^*) & \xrightarrow{\varphi} & H_1(\Gamma^*) & \xrightarrow{\psi} & H_2(\breve{S}) & \xrightarrow{\chi} & H_2(S^*) & \longrightarrow \ldots
\end{array} \qquad (*)$$

The next step is to show that $\operatorname{Im} \varphi'$ is contained in the kernel G' of the homomorphism $H_1(Z^*) \to H_1(Z)$ induced by the natural embedding $Z^* \hookrightarrow Z$ (the similar fact that $\operatorname{Im} \varphi$ is contained in the kernel G of the homomorphism $H_1(\Gamma^*) \to H_1(\Gamma)$ induced by the natural embedding $\Gamma^* \hookrightarrow \Gamma$ is a consequence of Lemma 3.1). As $H_4(X) = 0$ [Mil, Th. 7.1] the exact homology sequence of pair (X, X^*) implies that $H_3(X^*)$ is isomorphic to $H_4(X, X^*)$. By Lemma 2.1 $L = \pi^{-1}(F) = E \setminus E^* = X \setminus X^*$ is a curve. If U is a small Euclidean neighborhood of L in X then $H_4(U, U \setminus L) = H_4(X, X^*)$ by the excision theorem. That is, every 3-cycle $\omega' \in H_3(X^*)$ can be chosen in $U \setminus L$. Thom's isomorphism maps the image of ω' in $H(U, U \setminus L)$ to a 1-cycle $\alpha \in H_1(E^* \cap U)$. As Thom's isomorphisms are functorial under open embeddings [Do, Ch. 8, 11.5] we can treat α as an element of $H_1(E^*)$. Thus the image of α under the homomorphism generated by the natural projection $E^* \to Z^*$ coincides with $\varphi'(\omega') \in H_1(Z^*)$. As α is contained in a small neighborhood $U \cap E$ of L in E we see that $\varphi'(\omega')$ is contained in a small neighborhood of the finite set $Z \setminus Z^*$ which implies that $\operatorname{Im} \varphi' \subset G'$.

Assume that (i) is not true. Let $\bar{\theta} : \bar{Z} \to \bar{\Gamma}$ be the extension of θ to completions of Z and Γ. As θ is not finite the set $F' = (\bar{Z} \setminus Z) \setminus \bar{\theta}^{-1}(\bar{\Gamma} \setminus \Gamma)$ is not empty. Let $z_0 \in F'$. One can suppose that $s_0 = \bar{\theta}(z_0) \in F$. Let Γ_j, $j = 1, \ldots, k$ be irreducible analytic branches of Γ at s_0, and let γ_j be a simple loop in Γ_j around s_0 with positive orientation. We treat $\gamma := \sum_{j=1}^k \gamma_j$ as an element of $H_1(\Gamma^*)$. By Lemma 3.1 $H_3(S^*)$ is generated by elements ω of form $\omega = \chi(s_0)$ and $\varphi(\omega) = \gamma$. Take a simple loop β_0 around z_0 in an irreducible analytic branch Z_1 of $\bar{Z}$ at z_0 such that $\bar{\theta}(Z_1)$ is, say, Γ_1. Take other simple loops in Z^* around the points of $\bar{\theta}^{-1}(s_0)$ whose images under θ are contained in $\bigcup_{j=2}^k \Gamma_j$. Then we can construct an integer $\mathbf{Z}$-linear combination β of β_0 and these other loops so that $\theta_1(\beta) = m\gamma$ where $m > 0$. As $\gamma \in \operatorname{Im} \varphi$ we have $\psi(\gamma) = 0$. As $\breve{\delta}_2$ is an isomorphism we have $\psi'(\beta) = 0$, i.e. $\beta \in \operatorname{Im} \varphi' \subset G'$. On the other hand $\beta = \beta_1 + \beta_2$ where β_1 (resp. β_2) is a $\mathbf{Z}$-linear combination of simple loops around points of $\bar{\theta}^{-1}(s_0) \cap (Z \setminus Z^*)$ (resp. $\bar{\theta}^{-1}(s_0) \cap F'$). That is, $\beta_1 \in G'$ and β_2 is nonzero since it contains β_0 with a nonzero coefficient. Hence β_2 belongs to G' as an element of $H_1(Z^*)$. But this is impossible since each irreducible component of Z^* has punctures in $\bar{\theta}^{-1}(\bar{\Gamma} \setminus \Gamma)$ and, therefore, the simple loops around points of $F' \cup (Z \setminus Z^*)$ are linearly independent. This contradiction implies (i).

The proof of (ii) consists of three steps.

Step 1. Show that for any selfintersection point s_0 of Γ (i.e. the number of irreducible analytic branches of Γ at s_0 is at least 2) the set $\theta^{-1}(s_0)$ is one point.

Assume the contrary. Say, for simplicity, that $\theta^{-1}(s_0)$ consists of two points $z_0, z_1 \in Z \setminus Z^*$. Let $Z \setminus Z^* = \{z_i\}$ and G_i' be the kernel of the homomorphism $H_1(Z \setminus z_i) \to$

$H_1(Z)$ induced by the natural embedding $Z \setminus z_i \hookrightarrow Z$. Let $\beta' \in G_0' \cap \operatorname{Im} \varphi'$ (i.e. $\beta' = \varphi'(\omega')$). Then $\delta_3(\omega') = m\omega$ where $m \in \mathbf{Z}$ and ω is as before. As $\varphi(m\omega) = \theta_1(\beta')$ we see that $\theta_1(\beta') = m\gamma$. Similarly $\theta_1(\beta'')$ is proportional to γ for every $\beta'' \in G_1' \cap \operatorname{Im} \varphi'$. On the other hand since s_0 is a selfintersection point, we can find $\beta_1 \in G_0'$ and $\beta_2 \in G_1'$ such that $\theta_1(\beta_1)$ and $\theta_1(\beta_2)$ are not proportional to γ (i.e. $\beta_1, \beta_2 \notin \operatorname{Im} \varphi'$) but $\theta_1(\beta_1 + \beta_2)$ is. The commutativity of (*) implies that $\beta_1 + \beta_2 \in \operatorname{Ker} \psi' = \operatorname{Im} \varphi'$. We shall get a contradiction now by showing that $\beta_1, \beta_2 \in \operatorname{Im} \varphi'$.

Indeed, G' is naturally isomorphic to $\bigoplus_i G_i'$. Moreover, since $L = \pi^{-1}(F)$ is the disjoint union $\bigcup_i \kappa^{-1}(z_i)$ one can choose a neighborhood U of L in X as a disjoint union of neighborhoods of the curves $\kappa^{-1}(z_i)$. Hence, repeating the argument with the excision theorem, we get the decomposition $\operatorname{Im} \varphi' = \oplus_i (G_i' \cap \operatorname{Im} \varphi')$. Hence $\beta_1, \beta_2 \in \operatorname{Im} \varphi'$ as $\beta_1 + \beta_2 \in \operatorname{Im} \varphi'$ which yields Step 1.

Step 2. Consider a non-canonical isomorphism $H_1(\Gamma^*) = G \oplus H_1(\Gamma^{\text{norm}})$ (resp. $H_1(Z^*) = G' \oplus H_1(Z^{\text{norm}})$) where Γ^{norm} (resp. Z^{norm}) is a normalization of Γ (resp. Z). Let us show that in (ii) the ranks of $H_1(\Gamma^{\text{norm}})$ and $H_1(Z^{\text{norm}})$ (resp. $G/\operatorname{Im} \varphi$ and $G'/\operatorname{Im} \varphi'$) are the same.

Indeed, $X^* = X \setminus L$ where L is a curve. Hence $H_2(X^*) = 0$ since X is smooth and $H_2(X) = 0$. As $\operatorname{Im} \varphi' \subset G'$ diagram (*) implies now that $H_2(\breve{X})$ is isomorphic to $(G'/\operatorname{Im} \varphi') \oplus H_1(Z^{\text{norm}})$. Similarly $H_2(\breve{S})$ contains a subgroup isomorphic to $(G/\operatorname{Im} \varphi) \oplus H_1(\Gamma^{\text{norm}})$ and diagram (*) shows that $\operatorname{Im} \breve{\delta}_2$ is contained in this last subgroup. Thus $H_2(\breve{S})$ is isomorphic to $(G/\operatorname{Im} \varphi) \oplus H_1(\Gamma^{\text{norm}})$ as $\breve{\delta}_2$ is an isomorphism. Note that the rank of $\breve{\delta}_2(G'/\operatorname{Im} \varphi')$ is the same as the rank of $G/\operatorname{Im} \varphi$ as $\theta_1(G')$ is of finite index in G. Hence the rank of $G'/\operatorname{Im} \varphi'$ is greater that or equal to the rank of $G/\operatorname{Im} \varphi$. The rank of $H_1(Z^{\text{norm}})$ is also greater than or equal to the rank of $H_1(\Gamma^{\text{norm}})$ since we have a dominant map of Riemann surfaces $Z^{\text{norm}} \to \Gamma^{\text{norm}}$. As $(G'/\operatorname{Im} \varphi') \oplus H_1(Z^{\text{norm}})$ is isomorphic to $(G/\operatorname{Im} \varphi) \oplus H_1(\Gamma^{\text{norm}})$ via $\breve{\delta}_2$, we see that the ranks of $(G'/\operatorname{Im} \varphi')$ and $(G/\operatorname{Im} \varphi)$ (resp. $H_1(Z^{\text{norm}})$ and $H_1(\Gamma^{\text{norm}})$) coincide.

Step 3. By virtue of (i) and Step 1 it suffices to show that $\theta|_{Z^1} : Z^1 \to \Gamma^1$ is birational. Assume the contrary. Then Step 2 implies that the genus of Z^1 (and, therefore, Γ^1) is zero. Assume that the normalization of Γ^1 is different from $\mathbf{C}$, i.e. $\bar{\Gamma}^1 \setminus \Gamma^1$ contains at least two point $\bar{s}_1$ and $\bar{s}_2$ where $\bar{\Gamma}^1$ is the closure of Γ^1 in $\bar{\Gamma}$. Let $\bar{Z}^1$ be the closure of Z^1 in $\bar{Z}$. Then either (1) $\bar{\theta}$ maps $\bar{Z}^1 \setminus Z^1$ bijectively onto $\bar{\Gamma}^1 \setminus \Gamma^1$ or (2) it does not. In (1) set $\bar{z}_i = \bar{\theta}^{-1}(\bar{s}_i)$ and let β^i (resp γ^i) be a simple loop in Z^* around

$\bar{z}_i$ (resp. in Γ^* around $\bar{s}_i$). If l is the number of points in a general fiber of $\bar{\theta}$ then $\bar{z}_i$ is a branch point of order l for $\bar{\theta}$ (recall that $l \geq 2$ as $\bar{\theta}$ is not birational). Hence $\theta_1(\beta^i) = l\gamma^i$. This implies that γ^i is not in $G + \operatorname{Im}\theta_1$. Since $G \supset \operatorname{Im}\varphi = \operatorname{Ker}\psi$, we see that $\psi(\gamma^i)$ is not in $\operatorname{Im}\breve{\delta}_2 \circ \psi'$. On the other hand $\operatorname{Im}\psi' = H_2(\breve{X})$ as $H_2(X^*) = 0$. This is a contradiction since $\breve{\delta}_2$ is an isomorphism. In (2) we can suppose that there are at least two point $\bar{z}_1'$ and $\bar{z}_1''$ in $\bar{\theta}^{-1}(\bar{s}_1)$. A nonzero $\mathbf{Z}$-linear combination β^0 of simple loops around these points such $\theta_1(\beta^0) = 0$ is an element of $\operatorname{Ker}\theta_1 \setminus G'$. But diagram (*) implies again that such elements cannot exist which concludes (ii). $\square$

Remark 3.3. Proposition 3.2 implies that π is surjective which is a generalization of the result of Bonnet [Bo] who proved this in the case of $X \simeq \mathbf{C}^3$. This implies that when X is Cohen-Macaulay, the algebra of regular functions on X is faithfully flat over the algebra of regular functions on S (this follows from [Ma, Chap. 2, (3.J) and Th. 3] and [Ei, Th. 18.16], see also [Da]).

4. Bijectivity of θ.

Lemma 4.1. *Let X^* be smooth, s be a general point of Γ, and $x \in \pi^{-1}(s)$. Then the linear map $\pi_* : T_xX \to T_sS$ of the tangent spaces generated by π is surjective, i.e. morphism π is smooth at x.*

Proof. Let $z = \kappa(x)$, U be a small coordinate neighborhood of s, $\Gamma_U = \Gamma \cap U$, Z_U be the connected component of $\theta^{-1}(\Gamma_U)$ that contains z, and $E_U = \kappa^{-1}(Z_U)$, i.e. $x \in E_U$. As s is general $\theta|_{Z_U} : Z_U \to \Gamma_U$ is an biholomorphism. By Lemma 2.2 $E_U \simeq Z_U \times \mathbf{C}$ and $\kappa|_{E_U}$ is the natural projection to Z_U. Hence, as $\pi|_E = \theta \circ \kappa$ and Z_U is biholomorphic to Γ_U, there exists a vector in $T_xE_U \subset T_xX$ which is mapped by $\pi_* : T_xX \to T_sS$ to a nonzero vector tangent to Γ_U. Let Γ_U be given by $\zeta = 0$ in a local coordinate system (ζ, η) on U and $g = \zeta \circ \pi$, $f = \eta \circ \pi$ (i.e. in a small neighborhood $V \subset X$ of x morphism π is given by (g, f)). By Lemma 2.1 E_U coincides with $g^*(0)$ in V. As E_U and V are smooth g may be viewed as an element of a local coordinate system on V. Hence π_* sends any vector from T_xX transversal to E_U to a vector from T_sS transversal to Γ_U which implies the desired conclusion. $\square$

Proposition 4.2. *Let the assumption of Proposition 3.2 (ii) hold, S be smooth, and $\pi : X \to S$ be the quotient morphism of a $\mathbf{C}_+$-action Φ on X.*

(i) *Let $z \in Z$, $s = \theta(z)$, U be a small Euclidean neighborhood of s in S, $\Gamma_U = \Gamma \cap U$,* *and Z_U be the connected component of $\theta^{-1}(\Gamma_U)$ that contains z. Suppose that $\theta|_{Z_U}$ is injective. Then $W = \pi^{-1}(U \setminus \Gamma_U) \cup \kappa^{-1}(Z_U)$ is biholomorphic to $U \times \mathbf{C}$ over U.*

(ii) *Let the action be free. Then θ is bijective and, therefore, π makes X is a locally trivial analytic $\mathbf{C}$-fibration over S by (i).*

Proof. Consider a general point x in $\kappa^{-1}(z)$ and a germ $\mathcal{P}$ of a smooth analytic surface at x transversal to $\kappa^{-1}(z)$. We can assume that $\varrho = \pi|_{\mathcal{P}} : \mathcal{P} \to U$ is surjective and it is finite as $\mathcal{P}$ does not contain fibers of π. Suppose first that the action is free at x (i.e., the irreducible component L_0 of $\kappa^{-1}(z)$, that contains x, is an orbit of Φ) and show that ϱ is biholomorphic.

Assume the contrary. Then as S is smooth there must be a branch curve $R \subset \mathcal{P}$ of ϱ. For any point $x_1 \in R$ and $s_1 = \pi(x_1)$ the linear map $\varrho_* : T_{x_1}\mathcal{P} \to T_{s_1}S$ is not surjective. On the other hand, as Φ is free, all orbits close to L_0 must be transversal to $\mathcal{P}$. For any point $x_2 \in \mathcal{P}$, such that $\mathcal{P}$ is transversal to the fiber of π through x_2 and morphism π is smooth at x_2, the linear map $\varrho_* : T_{x_2}\mathcal{P} \to T_{s_2}S$ (where $s_2 = \pi(x_2)$) is surjective. Morphism π is obviously smooth at any point $x_2 \in \breve{X}$ as $\breve{X} \simeq \breve{S} \times \mathbf{C}$ over $\breve{S}$. Thus $R \subset \mathcal{P} \cap E$. But Lemma 4.1 implies that π is smooth at general points of this curve $\mathcal{P} \cap E$. Contradiction.

Let the assumption of (i) hold. If the action is free on L_0 then each orbit meets $\mathcal{P}$ at most at one point (by injectivity of ϱ) and the union V of orbits through $\mathcal{P}$ is biholomorphic to $\mathcal{P} \times \mathbf{C} \simeq U \times \mathbf{C}$. Note that V is contained in W and $W \setminus V$ is of codimension 2 in W. As U can be chosen Stein, removing singularities in codimension 2 we get a holomorphic map $W \to V$ which is identical on V. This implies that W and V are biholomorphic. Suppose now that Φ acts trivially on L_0, i.e. ϱ is finite but not a priori biholomorphic. Let $X' = \mathcal{P} \times_U \pi^{-1}(U)$ and $\tau : X' \to \pi^{-1}(U)$ be the natural projection. Set $W' = \tau^{-1}(W)$ and $W_0' = \tau^{-1}(W \setminus \kappa^{-1}(z))$. There is an analytic surface $\mathcal{P}'$ in W' which is mapped biholomorphically on $\mathcal{P}$ by τ. Let $x' \in \mathcal{P}'$ be the preimage of x and $\mathcal{P}_0' = \mathcal{P}' \setminus x'$. Note that Φ generates an analytic $\mathbf{C}_+$-action Φ' on X'. By the second statement of Lemma 2.2 we can suppose that that Φ acts freely on $W \setminus \kappa^{-1}(z)$ and, therefore, Φ' acts freely on W_0'. By construction, each orbit meets $\mathcal{P}_0'$ at most at one point. Let $V_0' \simeq \mathcal{P}_0' \times \mathbf{C}$ be the union of orbits through $\mathcal{P}_0'$. Note that V_0' can be embedded naturally in the manifold $V' = \mathcal{P}' \times \mathbf{C}$ which can be chosen Stein. As U and, therefore, X' can be also chosen Stein, the theorem about removing singularities in codimension 2 shows, as before, that V' is biholomorphic to

W'. It remains to note now that as Φ' acts freely on $V_0' \simeq \mathcal{P}_0' \times \mathbf{C}$ it must act freely on $W' \simeq \mathcal{P} \times \mathbf{C}$ contrary to the assumption that Φ acts trivially on L_0.

In (ii) assume that there is a component Z^1 of the Z such that $\theta|_{Z^1}$ is not bijective. Step 1 in the proof of Proposition 3.2 implies that $\theta|_{Z^1}$ is not birational. Furthermore, Proposition 3.2 (ii) implies that Z^1 contains a branch point z of θ whose order is $m \geq 2$. As $\pi|_E = \theta \circ \kappa$ and θ has a branch point of order m we see that $\varrho = \pi|_{\mathcal{P}}$ is at least m-sheeted contrary to the fact that ϱ must be bijective. □

5. Main theorem.

We present a proof of the Main Theorem based on the following general result, due to J. Kollár (the original proof relied on more topological considerations some of which are outlined in 5.4(3)).

Proposition 5.1 (Kollár) *Let $\pi : X \to S$ be a morphism of smooth (not necessarily affine) algebraic varieties (of any dimension) which is also a locally trivial analytic $\mathbf{C}$-fibration. Then X is the complement to a section of an algebraic $\mathbf{P}^1$-bundle over S. Furthermore, if S is affine then X is the total space of an algebraic line bundle, and if, in addition, $\pi : X \to S$ is a quotient morphism of a free $\mathbf{C}_+$-action, X is isomorphic to $S \times \mathbf{C}$ over S.*

Proof. Extend π to a proper morphism $\bar{\pi} : \bar{X} \to S$ with a smooth $\bar{X}$. Choose an irreducible divisor $D \subset S$ above which $\bar{\pi}$ is not smooth. Localize at the generic point of D. We obtain a morphism of a smooth surface to a curve (over a field extension k of $\mathbf{C}$) whose general fibers are projective lines. The special fiber $\mathcal{F}_0$ is a tree of rational curves, one of which $\mathcal{F}_0^1$ meets X (over k) and, therefore, has multiplicity 1 in $\mathcal{F}_0$. As the intersection number of the canonical class with each fiber is -2, $\mathcal{F}_0^1$ can not be the only (-1)-curve in $\mathcal{F}_0$. Contracting these other (-1)-curves repeatedly we obtain a projective line bundle over the whole curve including point D. Returning to complex numbers we see that there is a codimension 2 subset $T \subset S$ and an algebraic $\mathbf{P}^1$-bundle $\bar{\pi}_T : \bar{X}_T \to S \setminus T$ with a section $R_T := \bar{X}_T \setminus X$.

Thus $(\bar{\pi}_T)_* \mathcal{O}_{\bar{X}_T}(R_T)$ is a rank 2 bundle on $S \setminus T$ (where $\mathcal{O}_{\bar{X}_T}$ is, of course, the structure sheaf). Its push forward to S is a unique reflexive sheaf $\mathcal{E}$ on S which extends $(\bar{\pi}_T)_* \mathcal{O}_{\bar{X}_T}(R_T)$ (this holds both analytically and algebraically).

On the other hand, since every automorphism of the affine line extends to $\mathbf{P}^1$, there is an analytic $\mathbf{P}^1$-bundle over S containing X, and the similar push forward to

S gives a rank 2 analytic bundle extending $(\bar{\pi}_T)_* \mathcal{O}_{\bar{X}_T}(R_T)$. This implies that $\mathcal{E}$ is locally free and, therefore, generates an algebraic $\mathbf{P}^1$-bundle $\tilde{X} \to S$. The closure of R_T gives a rational holomorphic section $\tilde{R} \subset \tilde{X}$ such that $\tilde{X} \setminus \tilde{R}$ agrees with X over $S \setminus T$. Furthermore, by construction this embedding $\pi^{-1}(S \setminus T) \hookrightarrow \tilde{X} \setminus \tilde{R}$ extends to a biholomorphic map $X \to \tilde{X} \setminus \tilde{R}$. As rational holomorphic functions are regular the last map is an isomorphism and $\tilde{R}$ is a regular section.

A $\mathbf{P}^1$-bundle with a section comes from an extension of line bundles. Any such extension splits if S is affine. This gives us a second section in the $\mathbf{P}^1$-bundle disjoint from the first one, which makes X a line bundle over S. Applying the free action to the zero section of this line bundle we see that X is isomorphic to $S \times \mathbf{C}$ over S. □

Propositions 3.2, 4.2, and 5.1 imply immediately our main result.

Theorem 5.2. *Let the assumption of Lemma 2.2 hold, X be smooth, and $H_2(X) = H_3(X) = 0$.*

(i) *Then Γ can be chosen so that each irreducible component Z^0 of Z (resp. $\Gamma^0 = \theta(Z^0)$ of Γ) is rational (resp. a polynomial curve), and $\theta|_{Z^0} : Z^0 \to \Gamma^0$ is not bijective.*

(ii) *If S is smooth and the $\mathbf{C}_+$-action Φ on X has no fixed points then X is isomorphic to $S \times \mathbf{C}$ over S and the action is generated by a translation on the second factor.*

Proof of Theorem 1. By [Miy80] $\mathbf{C}^3//\mathbf{C}_+ \simeq \mathbf{C}^2$ for every nontrivial $\mathbf{C}_+$-action on $\mathbf{C}^3$. Hence Theorem 5.2 (ii) implies the desired conclusion. □

Remark 5.4. (1) Theorem 1 has the following reformulation in the language of locally nilpotent derivations. Every locally nilpotent derivation of $\mathbf{C}^{[3]}$ that vanishes nowhere (as a vector field on $\mathbf{C}^3$) is a partial derivative in a suitable coordinate system.

(2) In the case of a smooth contractible X the assumption on smoothness of S in Theorem 5.2 (ii) will be removed in the coming paper of the author and N. Saveliev.

(3) For contractible X Theorem 5.2 (ii) and, therefore, Theorem 1 follow immediately from injectivity of θ without using Propositions 4.2 (i) and 5.1. Indeed, by Fujita's result (e.g., see [Ka94, Prop. 3.2]) a smooth affine contractible X is factorial and its invertible functions are nonzero constants. Hence the same is true for S. Injectivity of θ is the same as the geometric irreducibility of π in codimension 1

in terminology of [Miy87]. Thus the assumption of [Miy87, Theorem 2] holds which implies that X a vector line bundle over S. As S is affine factorial its Picard group is trivial [Fu, Proposition 1.17], i.e. X is isomorphic to $S \times \mathbf{C}$ over S.

(4) Consider an n-dimensional smooth contractible affine algebraic variety X and a non-degenerate algebraic action of a unipotent group U on it. Suppose that U is of dimension $n-2$ (i.e. U is isomorphic to $\mathbf{C}^{n-2}$ as an affine algebraic variety). As we mentioned, X is factorial and by Zariski's theorem [Za] $S = X//U$ is still an affine algebraic surface. Now one can adjust the proofs of Proposition 3.2 and the injectivity statement in Proposition 4.2 for this situation with one additional assumption: beginning from Lemma 2.2 we suppose that there are at most finitely many orbits non-isomorphic to $\mathbf{C}^{n-2}$ (this will provide the Zariski dense cylinder $E^* \simeq Z^* \times \mathbf{C}^{n-2}$ in E). As Miyanishi's theorem [Miy80] says that $\mathbf{C}^n//U \simeq \mathbf{C}^2$ in the case of a free U-action, we see that for such actions the analogue of morphism θ remains bijective.

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