# ON A THEOREM OF AX

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In 1969 J. Ax proved a remarkable theorem [A] which implies that every injective endomorphism of an algebraic variety $X$ (over an algebraically closed field $k$ of characteristics 0) is surjective. A. Borel [B] gave another proof of this result based on a cohomology-theoretic argument. A similar approach was used by Kawamata who rediscovered this theorem later [I]. M. Miyanishi [OP] asked whether the following generalization of this result is true.

**Conjecture.** *Let $\varphi : X \to X$ an endomorphism of an algebraic variety $X$ over $k$, and let $E$ be a closed subvariety of $X$ of codimension at least 2. Suppose that the restriction of $\varphi$ to $X \setminus E$ is injective. Then $\varphi$ is an autormorphism.*

Unlike the Ax theorem this conjecture cannot be extended to schemes.

*Example.* Let $X$ be the union of two samples of $\mathbf{C}^n$ glued along $\mathbf{C}^n \setminus o$ where $o$ is the origin, i.e. the preimage $E$ of $o$ under the natural projection $X \to \mathbf{C}^n$ consists of two points $x_1$ and $x_2$. Consider the endomorphism $\varphi : X \to X$ which is identical on $X \setminus E$ and sends $E$ to $x_1$. This is the desired counterexample.

Nevertheless, following the idea of Borel and Kawamata, we shall show that the answer is positive in the case of affine and complete algebraic varieties.

**Lemma** 1. *Let $\varphi : X \to X$ be a bijective endomorphism of an algebraic variety over a field $k$ of characteristics zero. Then $\varphi$ is an automorphism.*

*Proof.* Consider a normalization $\nu : Y \to X$ of $X$. Then $\varphi$ generates a bijective endomorphism $\psi : Y \to Y$. Since $k$ is of zero characteristics this means that $\psi$ is birational and, therefore, an automorphism by the Zariski Main theorem. Let $\mathcal{R}$ and $\mathcal{S}$ be the structure sheaves on $Y$ and $X$ respectively. Treat $\mathcal{S}$ as a subsheaf of $\mathcal{R}$ (by identifying $\mathcal{S}$ with $\nu^*(\mathcal{S})$). Put $\mathcal{S}_1 = (\psi^*)^{-1}(\mathcal{S})$ and $\mathcal{S}_i = (\psi^*)^{-1}(\mathcal{S}_{i-1})$. Assume

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that $\mathcal{S}_1 \neq \mathcal{S}$. Then $\mathcal{S}_i \neq \mathcal{S}_{i-1}$ since $\psi^*$ is bijective, i.e. we get a sequence of strict inclusions $\mathcal{S} \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots$. On the other hand choose a finite affine open covering $\{X_j\}$ of $X$. Then $\{Y_j\}$ is an open affine covering of $Y$ where $Y_j = \nu^{-1}(X_i)$. The restriction $\mathcal{R}|_{Y_j}$ (resp. $\mathcal{S}|_{Y_j}$, $\mathcal{S}_i|_{Y_j}$) is generated by its global sections which form an affine domain $R^j$ (resp. $S^j$, $S_i^j$). Note that $R^j$ is a finitely generated module over $S^j$ and each $S_i^j$ is an $S^j$-submodule of $R^j$. Hence the ascending chain condition implies that sequence $\mathcal{S}|_{Y_j} \subset \mathcal{S}_1|_{Y_j} \subset \mathcal{S}_2|_{Y_j} \subset \ldots$ must be stationary. As our covering is finite, sequence $\mathcal{S} \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots$ must be also stationary which is a contradiction. Thus $\mathcal{S}_1 = \mathcal{S}$. Since $\varphi^* = \psi^*|_{\mathcal{S}}$ we see that $\varphi^*$ is an automorphism. □

**Lemma** 2. *The Conjecture is true provided it is true for normal algebraic varieties.*

*Proof.* Consider a normalization $\nu : Y \to X$ of $X$. Then $\varphi : X \to X$ generates an endomorphism $\psi : Y \to Y$. By the hypothesis $\psi$ is surjective. Let $\ell(x)$ be the number of points of $\nu^{-1}(x)$ for $x \in X$, $n = \max_{x \in X} \ell(x)$, and $X_n$ be the subvariety of $X$ that consists of all points $x$ such that $\ell(x) = n$. As $\psi(\nu^{-1}(x)) \subset \nu^{-1}(\varphi(x))$ for every $x \in X$ and $\psi$ is bijective, $\ell(\varphi(x)) \geq \ell(x)$. Hence $\varphi(x) \in X_n$ for every $x \in X_n$, and by the same reason there is no $x_1 \neq x$ such that $\varphi(x_1) = \varphi(x)$. That is, $\varphi|_{X_n} : X_n \to X_n$ is injective and, therefore, surjective by the Ax theorem. Thus $\varphi(X \setminus X_n) \subset X \setminus X_n$. Replacing $X$ by $X \setminus X_n$ we see that $\varphi(X_{n-1}) \subset X_{n-1}$ and $\varphi|_{X_{n-1}} : X_{n-1} \to X_{n-1}$ is injective. Induction implies that $\varphi : X \to X$ is injective; hence it is an automorphism by the Ax theorem and Lemma 1. □

**Lemma** 3. *Let $E$ be a closed subvariety of $X$ of codimension at least 2, and $\varphi : X \setminus E \to X$ be an injective morphism. Let $F = X \setminus \varphi(X \setminus E)$. Then* $\dim E = \dim F$.

*Proof.* Assume the contrary. Let, say $m = \dim E < \dim F = n$ (the case of $\dim E > \dim F$ is similar). Without loss of generality we can suppose that $X$ is normal. Then by the Zariski Main theorem, $\varphi$ is an embedding. Hence $\varphi(X \setminus E)$ is open in $X$ and $F$ is closed. Let $F_1 = F \cup \varphi(F \setminus E)$. Note that $F_1$ is closed and it contains more irreducible components of dimension $n$ than $F$ does. Let $E_1 = E \cup \varphi^{-1}(E)$. Note that $E_1$ is closed, $\dim E_1 = m$, and $\varphi^2$ generates an isomorphism between $X \setminus E_1$ and $X \setminus F_1$. Replacing $E$ (resp. $F$, $\varphi$) by $E_1$ (resp. $F_1$, $\varphi^2$) we increase the number of irreducible components of $F$ of dimension $n$; hence we can make it as large as we wish while keeping the dimension of $E$ the same.

Following the proof of Kawamata we shall use De Rham homology from [H]. For every $l$-dimensional variety $Y$ we have $H_i(Y) = 0$ for $i > 2l$, and $H_i(Y)$ is a finite-dimensional $k$-vector space which is nonzero for $i = 2l$. Hence the Mayer-Viertoris sequence implies that the dimension of $H_{2l}(Y)$ is at least the number of irreducible components of $Y$. Furthermore, we have the following exact sequences [H, Th. 1.2].

$$\ldots \to H_i(E) \to H_i(X) \to H_i(X \setminus E) \to H_{i-1}(E) \to \ldots,$$

$$\ldots \to H_i(F) \to H_i(X) \to H_i(X \setminus F) \to H_{i-1}(F) \to \ldots.$$

As $m < n < j = \dim X$ we get from the first sequence $H_i(X) \simeq H_i(X \setminus E)$ for $2j \geq i \geq 2m+2$. As $X \setminus E$ is isomorphic to $X \setminus F$ we have $H_i(X) \simeq H_i(X \setminus F)$ for $2j \geq i \geq 2m+2$. Hence from the second sequence we have

$$0 \to H_{2n+1}(X) \to H_{2n+1}(X) \to H_{2n}(F) \to H_{2n}(X) \to \ldots.$$

That is, we have a monomorphism $H_{2n}(F) \to H_{2n}(X)$. As we mentioned the dimension of the last $k$-vector space is bounded while we can make the dimension of $H_{2n}(F)$ as large as we wish by increasing the number of components of $F$. This is the desired contradiction. □

**Theorem.** *The Miyanishi Conjecture is true for affine and complete varieties.*

*Proof.* Let us suppose first that $X$ is affine. By Lemma 2 we can also suppose that $X$ is normal. Note that $F$ does not contain irreducible components of codimension at least 2 since $\varphi^{-1}$ can be extended to these components by the theorem about deleting singularities of regular functions on normal varieties. Lemma 3 implies that $F$ and $E$ must be empty.

When $X$ is complete then $F = \varphi(E)$ as $\varphi(X) = X$. Since we can assume normality, $\dim F < \dim E$ by the Zariski Main theorem whence $F$ and $E$ are empty by Lemma 3. Now Lemma 1 implies the desired conclusion. □

*Remark.* Lemmas above remain true for an integral scheme $X$ of finite type over $k$ (in the case of algebraic varieties Lemma 3 can be also extracted from the "conservation property" proven in [G] by metamathematical argument). The reason why the Miyanishi conjecture is not valid for such (non-separated) schemes is that the image $\varphi(X)$ may not be closed in $X$ as in the counterexample in the beginning of the paper. However, if we require additionally in the Miyanishi conjecture that morphism $\varphi$ is closed than the conjecture holds for complete schemes of finite type and the proof remains valid without any change.

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