# COMPLETIONS OF $\mathbb{C}^*$-SURFACES

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*Dedicated to Masayoshi Miyanishi*

ABSTRACT. Following an approach of Dolgachev, Pinkham and Demazure, we classified in [FlZa$_1$] normal affine surfaces with hyperbolic $\mathbb{C}^*$-actions in terms of pairs of $\mathbb{Q}$-divisors $(D_+, D_-)$ on a smooth affine curve. In the present paper we show how to obtain from this description a natural equivariant completion of these $\mathbb{C}^*$-surfaces. Using elementary transformations we deduce also natural completions for which the boundary divisor is a standard graph in the sense of [FKZ] and show in certain cases their uniqueness. This description is especially precise in the case of normal affine surfaces completable by a zigzag i.e., by a linear chain of smooth rational curves. As an application we classify all zigzags that appear as boundaries of smooth or normal $\mathbb{C}^*$-surfaces.

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**Acknowledgements:** This research was started during a visit of the first two authors at the Fourier Institute, Grenoble, and continued during a visit of two of us at the Max Planck Institute of Mathematics in Bonn. We thank these institutions for their generous support and excellent working conditions. The research was also partially supported by NSA Grant no. MDA904-00-1-0016 and the DFG-Schwerpunkt Komplexe Geometrie.

*1991 Mathematics Subject Classification*: 14R05, 14R20, 14J50.

*Key words*: $\mathbb{C}^*$-action, $\mathbb{C}_+$-action, affine surface.

## 1. Introduction

An irreducible normal affine surface $X = \operatorname{Spec} A$ endowed with an effective $\mathbb{C}^*$-action will be called a $\mathbb{C}^*$*-surface*. In the *elliptic case* the action possesses an attractive or repulsive fixed point and in the *parabolic case* an attractive or repulsive curve consisting of fixed points. A simple and convenient description for these surfaces, based on the fact that the $\mathbb{C}^*$-action corresponds to a grading of the coordinate ring $A$ of $X$, was elaborated by Dolgachev, Pinkham and Demazure, so it was called in [FlZa$_1$, I] a *DPD-presentation*. Namely, in the elliptic case our surface is represented as

$$X = \operatorname{Spec} A \qquad \text{with} \quad A = \bigoplus_{k\ge 0} H^0(C, \mathcal{O}_C(\lfloor kD \rfloor)) \cdot u^k \,,$$

where $u$ is an indeterminate, $D$ is an ample $\mathbb{Q}$-divisor on a smooth projective curve $C$ and $\lfloor kD \rfloor$ denotes the integral part. The curve $C = \operatorname{Proj} A$ is then the orbit space of the $\mathbb{C}^*$-action on the complement of its unique fixed point in $X$. Likewise, in the parabolic case

$$X = \operatorname{Spec} A_0[D] \qquad \text{with} \quad A_0[D] = \bigoplus_{k\ge 0} H^0(C, \mathcal{O}(\lfloor kD \rfloor)) \cdot u^k \,,$$

where now $D$ is a $\mathbb{Q}$-divisor on a smooth affine curve $C = \operatorname{Spec} A_0$, which again is the orbit space of our $\mathbb{C}^*$-action on the complement of its fixed point set in $X$.

All other $\mathbb{C}^*$-surfaces $X$ are *hyperbolic*. Their fixed points are all isolated, attractive in one and repulsive in the other direction. Any such surface is isomorphic to

$$\operatorname{Spec} A_0[D_+, D_-] \qquad \text{with} \quad A_0[D_+, D_-] := A_0[D_+] \oplus_{A_0} A_0[D_-]$$

where $D_\pm$ is a pair of $\mathbb{Q}$-divisors on a normal affine curve $C = \operatorname{Spec} A_0$ with $D_+ + D_- \le 0$ [FlZa$_1$, I].

In this paper we are mainly interested in an explicit description of the completions of such $\mathbb{C}^*$-surfaces. One of the main results is contained in section 3, where we describe a canonical equivariant completion of a hyperbolic $\mathbb{C}^*$-surface in terms of the divisors $D_\pm$, see for instance Corollary 3.18 for the dual graph of its boundary divisor. We also treat in brief the case of elliptic and parabolic surfaces, see Section 3.4.

In [FKZ], Corollary 3.36 we have shown that any normal affine surface $V$ admits a completion for which the dual graph of the boundary is standard (see 2.8). Given a DPD presentation of a $\mathbb{C}^*$-surface $V$, the results of Section 3 provide an explicit equivariant standard completion $\bar V_{\rm st}$ of $V$. More generally, in Section 2 we investigate the question as to when such equivariant standard completions can be found for actions of an arbitrary algebraic group $G$. We show that this is indeed possible for normal affine $G$-surfaces $V$ except for

$$\mathbb{P}^2 \backslash Q, \quad \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta, \quad V_{d,1} \,,$$

where $Q$ is a non-singular quadric in $\mathbb{P}^2$, $\Delta$ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ and $V_{d,1}$, $d \ge 1$, are the Veronese surfaces, see Theorem 2.9. Moreover, equivariant standard completions always exist if $G$ is a torus. We also deduce their uniqueness in certain cases, see Theorem 2.13.

In this paper we study mostly $\mathbb{C}^*$-actions on Gizatullin surfaces. By a *Gizatullin surface* we mean a normal affine surface completable by a *zigzag* that is, a simple normal crossing divisor $D$ with rational components and a linear dual graph $\Gamma_D$. These surfaces are remarkable by a variety of reasons. By a theorem of Gizatullin [Gi, Theorems 2 and

3] (see also [Be, BML], and [$\mathrm{Du}_1$] for the non-smooth case), the automorphism group $\mathrm{Aut}(X)$ of a normal affine surface $X$ has an open orbit with a finite complement in $X$ if and only if either $X \cong \mathbb{C}^* \times \mathbb{C}^*$ or $X$ is a Gizatullin surface. The automorphism groups of Gizatullin surfaces were further studied in [DaGi]. Like in the case of $X = \mathbb{A}^2_{\mathbb{C}}$, such a group has a natural structure of an amalgamated free product.

These surfaces can also be characterized by the Makar-Limanov invariant: a normal affine surface $X = \operatorname{Spec} A$ different from $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$ is Gizatullin if and only if its Makar-Limanov invariant is trivial that is, $\mathrm{ML}(X) := \bigcap \ker \partial = \mathbb{C}$, where the intersection is taken over all locally nilpotent derivations of $A$. Among the hyperbolic $\mathbb{C}^*$-surfaces $X = \operatorname{Spec} A_0[D_+, D_-]$ the Gizatullin ones are characterized by the property that each of the fractional parts $\{D_\pm\} = D_\pm - \lfloor D_\pm \rfloor$ is either zero or supported at one point $\{p_\pm\}$, see [$\mathrm{FlZa}_1$, II].

In Theorem 4.4 we show that an arbitrary ample zigzag can be realized as a boundary divisor of a Gizatullin $\mathbb{C}^*$-surface and even a toric one. However, not every such zigzag appears as the boundary divisor of a *smooth* $\mathbb{C}^*$-surface. More precisely we give in 4.4-4.6 a numerical criterion as to when a zigzag can be the boundary divisor of a smooth Gizatullin $\mathbb{C}^*$-surface. Using this criterion we can exhibit many smooth Gizatullin surfaces which do not admit any $\mathbb{C}^*$-action, see Corollary 4.8. We note that every $\mathbb{Q}$-acyclic Gizatullin surface[1] is a $\mathbb{C}^*$-surface [$\mathrm{Du}_2$, II.5.10]. The latter class was studied e.g., in [DaiRu, $\mathrm{MaMi}_1$, $\mathrm{Du}_2$].

Finally, in 5.13 we investigate $\mathbb{C}^*$-actions on Danilov-Gizatullin surfaces, by which we mean complements $\Sigma_n \setminus S$ of an ample section $S$ in a Hirzebruch surface $\Sigma_n$. By a theorem of Danilov-Gizatullin [DaGi], the isomorphism class of such a surface $V_{k+1}$ depends only on the self-intersection number $S^2 = k + 1 > n$. In particular it does not depend on $n$ and is stable under deformations of $S$ inside $\Sigma_n$. According to Peter Russell[2], given any natural $k$ there are exactly $k$ pairwise non-conjugated $\mathbb{C}^*$-actions on $V_{k+1}$. We give another proof of this result using our DPD-presentations. In a forthcoming paper we will show that a Gizatullin surface which possesses at least 2 non-conjugated $\mathbb{C}^*$-actions is isomorphic to a Danilov-Gizatullin surface.

## 2. Equivariant completions of affine $G$-surfaces

### 2.1. **Equivariant completions.**

**2.1.** By the Kambayashi-Mumford-Sumihiro theorem (see [Su]), any algebraic variety $X$ equipped with an action of a connected algebraic group $G$ admits an equivariant completion. For normal affine varieties this is true even without the connectedness assumption. Indeed, if $X = \operatorname{Spec} A$ is an affine $G$-variety then any $\mathbb{C}$-linear subspace of finite dimension of $A$ is contained in a $G$-invariant one. Choosing an initial $\mathbb{C}$-linear subspace which contains a set of algebra generators of $A$ yields a $G$-invariant finite dimensional subspace $E \subseteq A$ such that the induced map gives an equivariant embedding $X \hookrightarrow \mathbb{A}^N_{\mathbb{C}}$. Letting

$$\mathbb{A}^N_{\mathbb{C}} \xrightarrow{\simeq} \mathbb{A}^N_{\mathbb{C}} \times \{1\} \subseteq \mathbb{A}^N_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$$

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[1] That is $H_i(X, \mathbb{Q}) = 0 \quad \forall i > 0$.

[2] An oral communication. We are grateful to Peter Russell for generously sharing results from unpublished notes [CNR].

be a natural embedding, where $G$ act on the second factor trivially, we get a $G$-action on $\mathbb{P}^N$. The closure $\bar{X}$ of $X$ in $\mathbb{P}^N$ is then an equivariant completion.

If dim $X = 2$ then an equivariant resolution of singularities of such a completion can be obtained as follows. By a theorem of Zariski [Zar], a resolution of singularities of $\bar{X}$ can be achieved via a sequence of normalizations and blowups of points i.e., of maximal ideals. Since both these operations are equivariant, this yields an equivariant resolution. Moreover, the minimal resolution dominated by this equivariant one is equivariant too, provided that $G$ is connected and so stabilizes every component of the exceptional divisor. This is based on the following well known lemma, see e.g., Lemma 7 in [DaGi, I, §7].

**Lemma 2.2.** *Let $X$ be a normal algebraic surface with an action of an algebraic group $G$.*

*(a) Given a contractible $G$-invariant complete curve $C$ in $X$, the action of $G$ descends to the contraction $X/C$.*
*(b) The action of $G$ lifts to the blowup of $X$ in any fixed point of $G$.*

In the following, by an *NC completion* of a normal algebraic surface $V$ we mean a pair $(X, D)$ such that $X$ is a normal complete algebraic surface, $D$ is a normal crossing divisor contained in the regular part $X_{\text{reg}}$ and $V = X \backslash D$. We call this an *SNC completion* if moreover $D$ has only simple normal crossings.

The considerations above lead to the following well known result.

**Proposition 2.3.** *(a) A normal affine algebraic surface $V$ with an action of an algebraic group admits an equivariant SNC completion $(X, D)$.*

*(b) An arbitrary normal algebraic surface $V$ with an action of a connected algebraic group admits an equivariant SNC completion $(X, D)$.*

*(c) Any two equivariant SNC completions $(X_i, D_i)$ of $V$, $i = 1, 2$, are equivariantly dominated by a third one $(X, D)$.*

**2.4.** Let $\Gamma$ be a weighted graph. We recall (see Definitions 2.3 and 2.8 in [FKZ]) that an *inner* blowup $\Gamma' \to \Gamma$ is one performed in an edge of $\Gamma$, and that an *admissible* blowup is one that is inner or performed in an end vertex of $\Gamma$. Moreover a blowdown $\Gamma \to \Gamma''$ is said to be admissible if its inverse is so. A birational transformation of graphs is a sequence of blowups and blowdowns. Given such a sequence

$$\gamma: \quad \Gamma = \Gamma_0 \overset{\gamma_1}{\dashrightarrow} \Gamma_1 \overset{\gamma_2}{\dashrightarrow} \cdots \overset{\gamma_n}{\dashrightarrow} \Gamma_n = \Gamma', \tag{1}$$

we call it *admissible* if every $\gamma_i$ is so, and *inner* if every step is an admissible blowdown or an inner blowup.

**Definition 2.5.** Given two NC completions $(X, D)$, $(X', D')$ of a normal algebraic surface $V = X \backslash D = X' \backslash D'$, by a *birational map* $\psi : (X, D) \dashrightarrow (X', D')$ we mean a birational map $X \dashrightarrow X'$ inducing the identity on $V$. Such a map can be decomposed into a sequence of blowups and blowdowns

$$\tilde{\gamma}: \quad (X, D) = (X_0, D_0) \overset{\tilde{\gamma}_1}{\dashrightarrow} (X_1, D_1) \overset{\tilde{\gamma}_2}{\dashrightarrow} \cdots \overset{\tilde{\gamma}_n}{\dashrightarrow} (X_n, D_n) = (X', D'), \tag{2}$$

where (i) $X_{i+1}$ is a blowdown or a blowup of $X_i$ taking place in the total transform $D_i$ of $D$ in $X_i$ and (ii) $D'$ is the total transform of $D$. Clearly $\tilde{\gamma}$ will induce a birational map $\gamma$ as in (1) of the dual graphs $\Gamma_i$ of $D_i$.

A birational map $\psi : (X, D) \to (X', D')$ will be called *inner* or *admissible* if $\gamma$ has the respective property for a suitable factorization $\tilde{\gamma}$ as above. If $X$ is equipped with an action of an algebraic group $G$ leaving $D$ invariant, then we call $\psi$ or the sequence $\tilde{\gamma}$ *$G$-equivariant* if they are compatible with the action of $G$.

The following observation will be useful.

**Proposition 2.6.** *Let $G$ be a connected algebraic group acting on a normal algebraic surface $V$ and let $(X, D)$ be an equivariant NC completion of $V$. Assume that $\gamma : \Gamma \dashrightarrow \Gamma'$ is a birational transformation of the dual graph $\Gamma$ of $D$ as in (1) that blows down at most vertices of $\Gamma$ corresponding to rational components of $D$. Then there is a sequence of equivariant birational maps $\tilde{\gamma} : (X, D) \dashrightarrow (X', D')$ as in (2) inducing $\gamma$ on the dual graphs of $D$, $D'$ in each of the following cases.*

*(i) $\gamma$ is inner.*
*(ii) $G = \mathbb{T} = (\mathbb{C}^*)^n$ is a torus and $\gamma$ is admissible.*

*Proof.* (i) is immediate from Lemma 2.2, and (ii) follows as well since an action of a torus on the projective line has at least 2 fix points. □

From this Proposition we can deduce the following corollaries.

**Corollary 2.7.** *For a normal surface $V$ with an action of a connected algebraic group $G$ the following hold.*

*(a) $V$ admits a minimal equivariant NC completion $(X, D)$, i.e. $D$ contains no at most linear*[3] *rational $(-1)$-curve.*

*(b) If moreover $G = \mathbb{T}$ is a torus and $(X, D)$ and $(X', D')$ are two minimal equivariant NC completions of $V$ then there is an equivariant admissible birational map $\psi : (X, D) \dashrightarrow (X', D')$.*

*Proof.* (a) is an immediate consequence of Proposition 2.6. If all irreducible components of $D$ (and then also of $D'$) are rational curves then (b) follows from Propositions 2.9 in [FKZ] and 2.6. In the general case we proceed as follows. If $v$ is a vertex of the dual graph $\Gamma$ of $D$ corresponding to a non-rational curve then we add a simple loop at $v$. This procedure results in a new minimal graph $\tilde{\Gamma}$ in which the vertices corresponding to non-rational curves become branching points. In the same way we obtain from the dual graph $\Gamma'$ of $D'$ a graph $\tilde{\Gamma}'$ that is birationally equivalent to $\tilde{\Gamma}$. According to Proposition 2.9 in [FKZ] $\tilde{\Gamma}'$ can be obtained from $\tilde{\Gamma}$ by an admissible birational transformation. Omitting at each step the simple loops just added results in an admissible birational transformation of $\Gamma$ into $\Gamma'$. Applying Proposition 2.6 the assertion follows. □

2.2. **Standard and semistandard completions.** We use below the notions of standard and semistandard graphs as introduced in [FKZ, Definition 2.13]. For the convenience of the reader we recall some of the notations from [FKZ].

**2.8.** Since the weighted dual graph of a divisor on an algebraic surface satisfies the Hodge index theorem we restrict in the sequel to graphs whose intersection form has at most one positive eigenvalue. Following the notations in [FKZ] we use the abbreviation

$$[[w_1, \ldots, w_n]] \quad := \quad \overset{w_1}{\circ}\!\!-\!\!\!-\!\!\!-\overset{w_2}{\circ}\!\!-\!\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\!-\overset{w_n}{\circ}\ , \tag{3}$$

[3] i.e. such that the degree of the corresponding vertex in the dual graph of $D$ is $\leq 2$.

and $((w_1,\ldots,w_n))$ denotes the circular standard graphs obtained from this by connecting the first and the last vertices by an additional edge.

A graph $[[w_1,\ldots,w_n]]$ (or $((w_1,\ldots,w_n))$ ) will be called a (circular) *zigzag* if its intersection form has at most one positive eigenvalue. According to [FKZ, Lemma 2.17 and Proposition 4.13] the standard zigzags are

$$[[0]]\,,\quad [[0,0,0]] \quad \text{and} \quad [[0,0,w_1,\ldots,w_n]], \quad \text{where} \quad n\ge 0,\ w_j\le -2\ \forall j, \tag{4}$$

and the circular standard zigzags

$$((0_a,w)),\quad ((0_b,-1,-1)) \quad \text{and} \quad ((0_b,w_1,\ldots,w_n))\,,^4 \tag{5}$$

where $0\le a\le 3$, $w\le 0$, $b\in\{0,2\}$ and $w_i\le -2\ \forall i$. In geometry there also appear naturally semistandard zigzags, where we have additionally the possibilities

$$[[0,w_1,\ldots,w_n]]\,,\ [[0,w_1,0]], \qquad \text{where } n\ge 0 \quad \text{and} \quad w_j\le -2\ \forall j\,, \tag{6}$$

see [FKZ, Lemma 2.17].

We notice that a standard zigzag $[[0,0,w_1,\ldots,w_n]]$ is unique in its birational class up to reversion

$$[[0,0,w_1,\ldots,w_n]] \rightsquigarrow [[0,0,w_n,\ldots,w_1]]\,, \tag{7}$$

and the circular standard zigzag $((0_b,w_1,\ldots,w_n))$ is unique up to reversion and a cyclic permutation

$$((0_b,w_1,\ldots,w_n)) \rightsquigarrow ((0_b,w_{q-1},\ldots,w_n,\ w_1,\ldots,w_q))\,.$$

The other standard zigzags are unique, see Corollary 3.33 in [FKZ].

In the following an NC divisor $D$ with dual graph $\Gamma$ on an algebraic surface will be called *standard* or *semistandard* if all connected components of $\Gamma\ominus(B(\Gamma)\cup S)$ have this property, where $B(\Gamma)$ is the set of all branching points of $\Gamma$ and $S$ is the set of vertices corresponding to non-rational curves. Similarly, a completion $(X,D)$ of an open algebraic surface is said to be (semi-)standard if $D$ is so.

The next result is an analogue of Theorem 7 in [DaGi, I] which says that any algebraic group action on an affine surface admitting a standard completion (in the sense of [DaGi]), admits also an equivariant standard completion. However note that our standard zigzags form a narrow subclass of those in [DaGi, I].

**Theorem 2.9.** *(a) Every normal affine surface $V$ with an action of a connected algebraic group $G$ admits an equivariant semistandard NC completion $(X,D)$ unless $V$ is one of the surfaces*

$$\mathbb{P}^2\backslash Q,\quad \mathbb{P}^1\times\mathbb{P}^1\backslash\Delta,\quad V_{d,1}\,,$$

*where $Q$ is a non-singular quadric in $\mathbb{P}^2$, $\Delta$ is the diagonal in $\mathbb{P}^1\times\mathbb{P}^1$ and $V_{d,1}$, $d\ge 1$, are the Veronese surfaces*[5].

*(b) If $G=\mathbb{T}$ is a torus and $V$ is an arbitrary normal surface with a $G$-action then there is an equivariant standard completion $(X,D)$.*

[4]Hereafter $0_n$ stands for a sequence of $n$ zeros.

[5]See e.g. Lemma 4.2(a) below.

*Proof.* Let $(Y, E)$ be an equivariant NC completion of $V$. Let us first suppose that $E$ is not an irreducible smooth rational curve so that the dual graph $\Gamma$ of $E$ is not reduced to a point. If all components of $E$ are rational then by Theorem 2.15 in [FKZ] $\Gamma$ can be transformed into a semistandard graph by an inner birational transformation and even into a standard one by an admissible transformation. Thus both claims follow now from Proposition 2.6. If some of the components are not rational, then as in the proof of Corollary 2.7 we can add to $\Gamma$ simple loops so that the vertices corresponding to non-rational curves become branching points. Arguing as before the result also follows in this case.

Assume further that $E$ is a smooth irreducible rational curve. If the group $G$ is solvable then there is a fixed point of $G$ on $E$, and blowing it up successively we can transform $E$ into a chain $[[0, -1, -2, \ldots, -2]]$, see [FKZ, Remark 2.14(1)]. Since this chain can be transformed into a semistandard (standard) one by an equivariant inner (admissible) elementary transformation the result follows also in this case.

Finally, if $G$ is not solvable then it contains a subgroup isomorphic to $\mathbf{SL}_2(\mathbb{C})$ or $\mathbf{PGL}_2(\mathbb{C})$. Using the theorem of Gizatullin and Popov (see Proposition 4.14 in [FlZa$_2$] and the references therein) our surface is one of the list above. $\square$

**Remarks 2.10.** 1. As the proof shows, (a) holds for an arbitrary normal algebraic surface $V$ provided that $G$ is solvable or $V$ admits an equivariant NC completion $(Y, E)$ such that the dual graph of $E$ is not reduced to a point.

2. We cannot expect in general to obtain an equivariant standard completion for a solvable group, because there could be not enough fixed points to perform outer equivariant elementary transformations as required to get a standard form. For instance, the group $G$ of all projective transformations of $\mathbb{P}^2$ which stabilize a line $D$ and a point $A \in D$ is solvable and has the only fixed point $A$. There exists an equivariant completion of $\mathbb{A}^2_{\mathbb{C}} = \mathbb{P}^2 \setminus D$ with semistandard dual graph $[[0, -2]]$, but it is impossible to get such a completion with standard dual graph $[[0, 0]]$.

Next we address the question of uniqueness of (semi-)standard completions. We recall shortly the notion of elementary transformations. Given a linear 0-vertex $v$ of $\Gamma$, so that $\Gamma$ contains $L = [[w, 0, w']]$ we consider the birational map of $\Gamma$ given by

$$[[w-1, 0, w'+1]] \dashrightarrow [[w-1, -1, -1, w']] \longrightarrow [[w, 0, w']]. \tag{8}$$

on $L$, which is the identity on $\Gamma \ominus L$. Similarly, if $v \in \Gamma$ is an end vertex so that $\Gamma$ contains $L' = [[w, 0]]$, we consider the birational map of $\Gamma$ given on $L'$ by

$$[[w-1, 0]] \dashrightarrow [[w-1, -1, -1]] \longrightarrow [[w, 0]]\,. \tag{9}$$

These transformations as well as their inverses are called *elementary transformations* of $\Gamma$.

Similarly, given a completion $(X, D)$ of a normal surface $V$ we can define elementary transformations at any point of a component $C_i \cong \mathbb{P}^1$ of $D$ of selfintersection 0 that corresponds to an at most linear vertex of the dual graph of $D$.

**Proposition 2.11.** *Let $G$ be a connected algebraic group acting on a normal algebraic surface $V$. If $(X_1, D_1)$ and $(X_2, D_2)$ are equivariant semistandard NC completions of $V$, then $(X_2, D_2)$ can be obtained from $(X_1, D_1)$ by a sequence of equivariant elementary transformations of the boundary.*

*Proof.* Let us first assume that the irreducible components of $D_1$ and $D_2$ are all rational. By Proposition 2.3 there is an equivariant NC completion $(X, D)$ of $V$ dominating $(X_i, D_i)$ for $i = 1, 2$. If $\Gamma$, $\Gamma_1$ and $\Gamma_2$ are the respective dual graphs of $D$, $D_1$ and $D_2$ then $\Gamma$ dominates $\Gamma_1$ and $\Gamma_2$. By Theorem 3.1 in [FKZ] we can transform $\Gamma_1$ into $\Gamma_2$ by a sequence of elementary transformations such that every step is dominated by some inner blowup of $\Gamma$. Using Proposition 3.34 from [FKZ] this gives a unique sequence of elementary transformations transforming $(X_1, D_1)$ into $(X_2, D_2)$ such that every step is dominated by an inner blowup , say $(X', D')$, of $(X, D)$. Since by Lemma 2.2 the action of $G$ lifts naturally to $(X', D')$ and $G$ also acts on any blowdown of the boundary $D'$, the result follows in this case.

In the general case we can again add simple loops at the vertices of $\Gamma_1$, $\Gamma_2$ and $\Gamma$ as in the proof Corollary 2.7. Arguing as before the result follows also in this case. □

2.3. **Uniqueness of standard completions.** In general, standard equivariant completions even of $\mathbb{C}^*$-surfaces are by no means unique. Let us give two examples.

**Example 2.12.** 1. Given a Gizatullin $\mathbb{C}^*$-surface $V$ and an equivariant standard completion $(V, D)$ we can reverse the boundary zigzag $D$ as in (7) by a sequence of inner elementary transformations. This leads to another equivariant standard completion, which usually is not isomorphic to the given one.

2. The affine plane $\mathbb{A}^2$ endowed with the $\mathbb{C}^*$-action $t.(x, y) = (tx, ty)$ can be equivariantly completed by $\mathbb{P}^1 \times \mathbb{P}^1$. The dual graph of the boundary divisor is the standard zigzag $[[0, 0]]$ consisting of the curves, say $C_0$ and $C_1$. Blowing up the intersection point $C_0 \cap C_1$ and blowing down $C_1$ gives a component, say $E$ that is pointwise fixed by $\mathbb{C}^*$. Performing an outer blowup of $E$ in a point different from the contraction of $C_1$, and then blowing down $E$, we arrive at a new equivariant completion of $\mathbb{A}^2$ by a standard zigzag as before. However, the equivariant completions of $\mathbb{A}^2$ obtained in this way are not equivariantly isomorphic, although both of them are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the boundary zigzags are the same.

However, the theorem below says in particular that for a torus action on $\mathbb{A}^2$ (which is unique up to conjugation in the automorphism group), the equivariant toric completion by the zigzag $[[0, 0]]$ is a unique such completion, up to an equivariant isomorphism of complete models.

The main result of this section is the following uniqueness theorem.

**Theorem 2.13.** *(a) A non-toric Gizatullin $\mathbb{C}^*$-surface $V$ has a unique equivariant standard completion up to reversing the boundary zigzag. More precisely, any two such completions $(\bar{V}_{\mathrm{st}}, D_{\mathrm{st}})$ and $(\bar{V}'_{\mathrm{st}}, D'_{\mathrm{st}})$ are isomorphic or obtained from each other by reversing the boundary zigzag.*

*(b) A normal affine toric surface $V$ has a unique $\mathbb{T} := (\mathbb{C}^*)^2$-equivariant standard completion up to reversing the boundary zigzag, unless $V$ is one of the surfaces $\mathbb{A}^1 \times \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$.*

Let us note that the toric surfaces $\mathbb{A}^1 \times \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$ are exceptional indeed: for each of them, the standard $\mathbb{T}$-equivariant completion $\bar{V} = \mathbb{P}^1 \times \mathbb{P}^1$ by the zigzag $D = [[0, 0, 0]]$ or by the cycle $D = ((0, 0, 0, 0))$, respectively, admits non-regular equivariant birational transformations (moves or turns and shifts, respectively) extending the identity on $V$, cf. Proposition 3.7 and Theorem 3.18 in [FKZ].

The assertions (a), (b) of the theorem will be shown in 2.22 and 2.16 below, respectively. We need a few preparations.

**Definition 2.14.** Let $V$ be a normal surface with an action of an algebraic group $G$. A curve of fixed points of $G$ in $V$ will be called *$G$-parabolic*, or simply *parabolic* if $G$ is clear from the context.

The following lemma is well known. For the sake of completeness we provide a simple argument.

**Lemma 2.15.** *Let the 2-torus $\mathbb{T}$ act on $V_0 \cong \mathbb{C}^* \times \mathbb{C}^*$ with an open orbit, and let $(\bar{V}, D_0)$ be an equivariant smooth completion of $V_0$ by an SNC divisor $D_0$. Then $D_0$ is a cycle of rational curves without $\mathbb{T}$-parabolic components.*

*Proof.* As follows e.g., from Luna's Étale Slice Theorem, for any regular action of an algebraic reductive group with an open orbit, the fixed point set is finite. (In the toric case there is an easy direct argument; cf. [Su].) Hence $\bar{V}$ cannot contain $\mathbb{T}$-parabolic curves.

The surface $V_0 \simeq \mathbb{C}^* \times \mathbb{C}^*$ admits an equivariant completion $(\mathbb{P}^1 \times \mathbb{P}^1, Z_0)$ by a cycle $Z_0$ consisting of 4 rational curves. Thus there is an equivariant birational transformation $\gamma : D_0 \dashrightarrow Z_0$. We claim that $\gamma$ is inner, so at each step of this transformation the boundary divisor remains a cycle of rational curves, as required. Indeed, this follows by induction on the length of $\gamma$, using the fact that $\gamma$ can blow up only isolated fixed points of $\mathbb{T}$ on the boundary, which are double points of the boundary cycle by the inductive hypothesis. $\square$

**2.16.** *Proof of Theorem 2.13(b).* If $V$ is not one of the surfaces $\mathbb{A} \times \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$, then the boundary zigzag of a standard completion $(\bar{V}_{\rm st}, D_{\rm st})$ is not equal to $[[0,0,0]]$ and is not circular. Comparing with the list in (4) the dual graph of $D_{\rm st}$ is of the form $[[0,0,w_2,\ldots,w_n]]$ with $w_i \le -2$ for all $i$ and $n \ge 1$. Given another standard completion $(\bar{V}'_{\rm st}, D'_{\rm st})$ there is an equivariant domination $(Y, E)$ of these completions. By Lemma 2.15, $E$ being a proper part of a cycle of rational curves, it must be again a zigzag and so by Proposition 3.4 in [FKZ], either $\bar{V}_{\rm st} = \bar{V}'_{\rm st}$, or $D'_{\rm st}$ is obtained by reversing the zigzag $D_{\rm st}$.

We now embark on the proof of the more difficult part (a) of Theorem 2.13. Let us first fix some notations.

**2.17.** Let $V$ be a non-toric Gizatullin surface and $(\bar{V}_{\rm st}, D_{\rm st})$ a completion of $V$ by a standard zigzag $[[0,0,w_2,\ldots,w_n]]$ with $w_i \le -2\ \forall i$ and $n \ge 2$. We let

$$D_{\rm st} = C_0 + \ldots + C_n \,,$$

where the components are numbered according to the weights in the sequence $[[0,0,w_2,\ldots,w_n]]$. We also consider the minimal resolutions of singularities $V'$, $(\tilde{V}_{\rm st}, D_{\rm st})$ of $V$ and $(\bar{V}_{\rm st}, D_{\rm st})$, respectively.

Since $C_0^2 = C_1^2 = 0$ the linear systems $|C_0|$ and $|C_1|$ define a morphism $\Phi = \Phi_0 \times \Phi_1 : \tilde{V}_{\rm st} \to \mathbb{P}^1 \times \mathbb{P}^1$ with $\Phi_i = \Phi_{|C_i|}$, $i = 0, 1$. We notice that $C_1$ is a section of $\Phi_0$ and so the restriction $\Phi_0|V' : V' \to \mathbb{P}^1$ is an $\mathbb{A}^1$-fibration. We can choose the coordinates in such a way that

$$C_0 = \Phi_0^{-1}(\infty)\,, \qquad \Phi(C_1) = \mathbb{P}^1 \times \{\infty\} \quad \text{and} \quad C_2 \cup \ldots \cup C_n \subseteq \Phi_0^{-1}(0)\,.$$

The divisor $D_{\mathrm{ext}} := C_0 \cup C_1 \cup \Phi_0^{-1}(0)$ is called the *extended divisor*. It will be studied systematically in Section 5.

**Remark 2.18.** If $V$ carries a $\mathbb{C}^*$-action, then we can find equivariant standard completions $(\bar V_{\mathrm{st}}, D_{\mathrm{st}})$ and $(\tilde V_{\mathrm{st}}, D_{\mathrm{st}})$, see Theorem 2.9. Thus $\Phi$ is also equivariant with respect to a suitable $\mathbb{C}^*$-action on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Lemma 2.19.** *With the notation as in 2.17, $\Phi$ is birational and induces an isomorphism $\tilde V_{\mathrm{st}} \backslash \Phi_0^{-1}(0) \cong (\mathbb{P}^1 \backslash \{0\}) \times \mathbb{P}^1$. In particular, $\Phi_0^{-1}(0)$ is the only possible degenerate fiber of the $\mathbb{P}^1$-fibration $\Phi_0 : \tilde V_{\mathrm{st}} \to \mathbb{P}^1$.*

*Proof.* Since by construction $\Phi^{-1}(\infty, \infty) = C_0 \cap C_1$ consists of one point, the map is birational and so $\tilde V_{\mathrm{st}}$ is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$. Because of $C_0^2 = C_1^2 = 0$ no blowup can occur along $C_0 \cup C_1$, whence $\Phi$ is an isomorphism in a neighborhood of $C_0 \cup C_1$.

Now assume that for some point $x \in (\mathbb{P}^1 \backslash \{0, \infty\}) \times (\mathbb{P}^1 \backslash \{\infty\})$ the fibre $\Phi^{-1}(x)$ is a curve. Then this curve meets neither $C_0 \cup C_1$ nor the divisor $D_{\mathrm{st}} \ominus C_0 \ominus C_1$ since by construction, the latter one is contained in $\Phi_0^{-1}(0)$. Thus $\Phi^{-1}(x)$ is contained in $V'$. Since $V$ being affine does not contain complete curves this is only possible if $\Phi^{-1}(x)$ is contained in the exceptional divisor of $V' \to V$. Because $\Phi^{-1}(x)$ contracts to a smooth point in $\mathbb{P}^1 \times \mathbb{P}^1$ it must contain a $(-1)$-curve, which gives a contradiction since $V'$ is the minimal resolution of $V$. □

**Lemma 2.20.** *In the notation of 2.17, if for some standard completion $(\bar V_{\mathrm{st}}, D_{\mathrm{st}})$ of a Gizatullin surface $V$ the extended divisor $D_{\mathrm{ext}}$ is linear then $V$ is toric.*

*On the other hand, for any* equivariant *standard completion $(\bar V_{\mathrm{st}}, D_{\mathrm{st}})$ of a toric Gizatullin surface $V$ the extended divisor $D_{\mathrm{ext}}$ is linear.*

*Proof.* Since $C_1^2 = 0$ both on $\tilde V_{\mathrm{st}}$ and on $Q = \mathbb{P}^1 \times \mathbb{P}^1$, no blowup is done under $\Phi = (\Phi_0, \Phi_1) : \tilde V_{\mathrm{st}} \to Q$ with center on $C_1$. Thus we may assume that the center of the first blowup in $\Phi$ is the fixed point $(0, 0) \in C_2 \ominus C_1$ of the standard $\mathbb{T}$-action on $Q$. We claim that this action lifts to $\tilde V_{\mathrm{st}}$ stabilizing $V'$ and then descends to $V$.

Indeed, by Lemma 2.19, $\Phi_0^{-1}(0)$ is the only possible degenerate fiber of $\Phi_0$. Thus, with $E$ the exceptional set of the minimal resolution $V' \to V$, both $D_{\mathrm{st}}, E$ are disjoint subchains of the linear chain $D_{\mathrm{ext}}$. Since $V$ is affine, contracting $E$ every component of $D_{\mathrm{ext}} \ominus (D_{\mathrm{st}} + E)$ meets the image of $D_{\mathrm{st}}$. Since $D_{\mathrm{st}}$ is connected it follows that there is exactly one such component, say, $E_0$ which separates $D_{\mathrm{st}}$ and $E$. Moreover since $D_{\mathrm{st}}$ and $E$ are both minimal, $E_0$ is the only $(-1)$-curve in $D_{\mathrm{ext}}$.

Therefore all blowups in $\Phi | D_{\mathrm{ext}} : D_{\mathrm{ext}} \to C_0 + C_1 + C_2$ are inner except for the first one. Hence the standard torus action on $Q$ lifts through $\Phi$ to $\tilde V_{\mathrm{st}}$ leaving $D_{\mathrm{ext}}$ stable. It stabilizes as well $D_{\mathrm{st}}, E$ and $V' = \tilde V_{\mathrm{st}} \setminus D_{\mathrm{st}}$ and so by Lemma 2.2 descends to $V$ with an open orbit. Thus indeed $V$ is toric.

As for the converse, note that by Lemma 2.15 $D_{\mathrm{ext}}$ is part of a cycle of rational curves. Hence being connected and simply connected, it is a linear chain. □

**Lemma 2.21.** *Assume that $(\bar V_{\mathrm{st}}, D_{\mathrm{st}})$ is an equivariant completion of a normal affine $\mathbb{C}^*$-surface $V$. With the notation as in 2.17, if $V$ is non-toric then one of the curves $C_2, \ldots, C_n$ is parabolic.*

*Proof.* We note that the fiber $\Phi_0^{-1}(0)$ is invariant under the $\mathbb{C}^*$-action. Since $V$ is non-toric, by Proposition 2.20 the dual graph of $D_{\mathrm{ext}} = C_0 \cup C_1 \cup \Phi_0^{-1}(0)$ contains a branching point $C_k$, $k \geq 2$. Thus the $\mathbb{C}^*$-action has at least 3 fixed points on this component $C_k$ which is then parabolic, as needed. □

**2.22.** *Proof of Theorem 2.13(a).* Let $(\bar{V}_{\rm st}, D_{\rm st})$ be an equivariant standard completion of $V$. With the notations as in 2.17, by Lemma 2.21 there is a parabolic component, say, $C_{s+1}$ in $D_{\rm st}$. After moving the two zero weights in the zigzag via a sequence of inner elementary transformations to the components $C_s$ and $C_{s+1}$ we get a new equivariant completion $(\tilde{V}, D)$ of $V$. Note that moving these zeros the curve $C_{s+1}$ is not blown down, and that the inverse transformation $D \dashrightarrow D_{\rm st}$ is as well inner, cf. Lemma 2.12 in [FKZ]. The linear system $|C_s|$ gives a morphism $\psi : \tilde{V} \to \mathbb{P}^1$ equivariant with respect to a suitable $\mathbb{C}^*$-action on $\mathbb{P}^1$, where $\psi(C_s) = \infty$. The curves $C_{s\pm1}$ being disjoint sections of $\psi$ and $C_{s+1}$ being parabolic, $\psi$ is the orbit map. We let $\bar{V}$ be the surface obtained from $\tilde{V}$ by contracting all curves in $D$ besides $C_{s\pm1}$ and $C_s$. Obviously $\tilde{V}$ is then the minimal resolution of the singularities of $\bar{V}$ sitting on the boundary.

Given a second equivariant standard completion $(V'_{\rm st}, D'_{\rm st})$, with the same procedure we get surfaces $\tilde{V}'$ and $\bar{V}'$ fibered equivariantly over $\mathbb{P}^1$. As before $\bar{V}'$ is a completion of $V$ by three curves $C'_{t-1}$, $C'_t$ and $C'_{t+1}$ so that $C'_{t-1}$ and $C'_{t+1}$ are sections of the $\mathbb{P}^1$-fibration and $C'_t$ is the fiber over $\infty$. The identity map on $V$ extends to an equivariant birational map $h : \bar{V} \dashrightarrow \bar{V}'$ compatible with the orbit maps to $\mathbb{P}^1$. In particular, $h$ respects sections of the $\mathbb{P}^1$-fibrations and so $C_{s+1}$ is the proper transforms of one of the sections $C'_{t+1}$ or $C'_{t-1}$ in $\bar{V}'$, and similarly for $C_{s-1}$. Performing, if necessary, elementary transformations at the fiber $C_s$ we may also assume that $C_s$ is the proper transform of $C'_t$.

Now $h$ defines a biregular map on the complements of discrete sets, so by Zariski's main theorem, it is everywhere regular and an isomorphism. This isomorphism lifts to the minimal resolutions of singularities giving an equivariant isomorphism $\tilde{h} : (\tilde{V}, D) \to (\tilde{V}', D')$. Since $(\tilde{V}, D) \dashrightarrow (\bar{V}_{\rm st}, D_{\rm st})$ and $(\bar{V}'_{\rm st}, D'_{\rm st}) \dashrightarrow (\tilde{V}', D') \cong (\tilde{V}, D)$ are both composed of inner elementary transformations it follows that $D'_{\rm st} \dashrightarrow D_{\rm st}$ is as well inner. Thus using Proposition 3.4 in [FKZ], either $D_{\rm st} = D'_{\rm st}$, or $D_{\rm st}$ is the reversion of $D'_{\rm st}$, proving (b). □

**Remark 2.23.** For an arbitrary normal affine $\mathbb{C}^*$-surface $V$ the dual graph of a standard equivariant completion can be easily deduced from the description in Corollary 3.18(b). It is easy to see that, if the surface is not a Gizatullin one, it admits in general many different equivariant standard completions.

## 3. Equivariant completions of $\mathbb{C}^*$-surfaces

### 3.1. **Generalities.**

**3.1.** For an arbitrary normal compact complex surface $X$, there is a $\mathbb{Q}$-valued intersection theory for divisors on $X$ (see [Mu, §II.4], [Sa]). This is a pairing

$$\mathrm{Div}_{\mathbb{Q}}(X) \times \mathrm{Div}_{\mathbb{Q}}(X) \to \mathbb{Q}, \quad (D_1, D_2) \mapsto D_1.D_2 \in \mathbb{Q}\,,$$

sharing the usual properties of intersections on smooth surfaces:

1. The pairing is bilinear.
2. The projection formula with respect to proper mappings $f : X \to Y$ of normal surfaces holds:

$$f^*(D).E = D.f_*(E) \quad \text{for } D \in \mathrm{Div}_{\mathbb{Q}}(Y) \text{ and } E \in \mathrm{Div}_{\mathbb{Q}}(X)\,.$$

3. The adjunction formula holds, i.e. if $C \subseteq X$ is an integral curve and $D$ is a Cartier divisor on $X$ then $C.D = \deg_C(\mathcal{O}_X(D)|C)$.

For a sequence of real numbers $k_0, \ldots, k_n$ with $k_0, \ldots, k_{n-1} \geq 2$ and $k_n \geq 1$ we let $[k_0, \ldots, k_n]$ be the (minus) continued fraction defined inductively via

$$[k_0] = k_0 \quad \text{and} \quad [k_0, \ldots, k_n] = k_0 - \frac{1}{[k_1, \ldots, k_n]}\,.$$

**Proposition 3.2.** *Let $X$ be a normal surface and let $C_0, C_1, \ldots, C_n$ be a chain of irreducible curves with $C_{i-1}.C_i = 1$ for $i = 1, \ldots, n$ and $C_i.C_j = 0$ for $i \neq j$ otherwise, and with dual graph*

$$\begin{array}{ccccc} C_0 & C_1 & & C_{n-1} & C_n \\ \circ\!\!-\!\!\!-\!\!\!-\!\!\!- & \!\!\!\circ\cdots & & \cdots\circ\!\!-\!\!\!-\!\!\!-\!\!\!- & \!\!\!\circ \\ -k_0 & -k_1 & & -k_{n-1} & -k_n \end{array}\,,$$

*where $k_i = -C_i^2 \geq 2$ $\forall i = 1, \ldots, n$ (however, we allow $X$ and the $C_i$ to be singular so that $k_i \in \mathbb{Q}$). Assume that $C_1 \cup \ldots \cup C_n$ can be contracted via a map $\pi : X \to X'$, and let $C_0' = \pi(C_0)$ be the image of $C_0$ in $X'$. Then*

$$-C_0'^2 = [k_0, k_1, \ldots, k_n] = k_0 - \frac{1}{[k_1, \ldots, k_n]}\,. \tag{10}$$

*In particular, in the case where $k_i \in \mathbb{N}$ $\forall i$ we have $C_0^2 = \lfloor C_0'^2 \rfloor$.*

*Proof.* We write $\pi^*(C_0') = C_0 + r_1C_1 + \ldots + r_nC_n$. By the projection formula

$$\pi^*(C_0').C_0 = C_0'^2 \quad \text{and} \quad \pi^*(C_0').C_i = 0 \text{ for } 1 \leq i \leq n.$$

This leads to the equalities

$$C_0'^2 = C_0^2 + r_1 \quad \text{and} \quad \frac{r_{i-1}}{r_i} = k_i - \frac{r_{i+1}}{r_i} \text{ for } 1 \leq i \leq n\,,$$

with the convention that $r_0 = 1$ and $r_{n+1} = 0$. Hence by induction

$$\frac{r_{i-1}}{r_i} = [k_i, \ldots, k_n]\,.$$

In particular, $\frac{r_0}{r_1} = [k_1, \ldots, k_n]$. As $-C_0'^2 = -C_0^2 - r_1$, (10) follows. The last assertion also follows as $C_0^2 = C_0'^2 - r_1 \in \mathbb{Z}$, where $0 < r_1 = [k_1, \ldots, k_n]^{-1} < 1$ by our assumption. $\square$

**Example 3.3.** Suppose that $X$ as in 3.2 is smooth and that $C_1, \ldots, C_n$ is a chain of smooth $(-2)$-curves. In this case $[2, \ldots, 2] = (n+1)/n$ and so, $C_0'^2 = C_0^2 + n/(n+1)$. For instance, if $C_0$ is a $(-1)$-curve in $X$ then the self-intersection number of $C_0'$ is $-1/(n+1)$.

**Remark 3.4.** If the curves $C_1, \ldots, C_n$ as in 3.2 above are smooth and sitting in the smooth locus of $X$ then by a result of Grauert [Gr] these curves can be contracted in the category of normal analytic spaces, provided that $k_i \geq 2$ $\forall i = 1, \ldots, n$. However, in general $X'$ is not necessarily a scheme, see for instance [Sch].

**Lemma 3.5.** *Let $D \in \operatorname{Div}_{\mathbb{Q}}(C)$ be a $\mathbb{Q}$-divisor on a smooth complete curve $C$ and let $\mathcal{O}_C[D]$ be the sheaf of $\mathcal{O}_C$-algebras*

$$\mathcal{O}_C[D] := \bigoplus_{k \geq 0} \mathcal{O}_C(\lfloor kD \rfloor) \cdot u^k\,,$$

*where $u$ is an indeterminate. The (relative) spectrum $X = \operatorname{Spec}\mathcal{O}_C[D]$ is then a normal surface, and the zero section $S \subseteq X$ corresponding to the projection $\mathcal{O}_C[D] \to \mathcal{O}_C$ has selfintersection $-\deg(D)$.*

*Proof.* Consider $d \in \mathbb{N}$ such that the divisor $D' = dD$ is Cartier on $C \simeq S$. If $\zeta$ is a local generator of $\mathcal{O}_C(dD))$ in a neighbourhood of a point $s \in S$ as an $\mathcal{O}_C$-module then $dS$ is given by the zeros of the local section $\zeta u^d$ in $\mathcal{O}_C[D]$. Thus $dS$ is Cartier on $X$. By adjunction $dS.S = \deg(dS|S)$. Under the canonical identification $S \cong C$ we have $dS|S = -dD$, so $dS^2 = -d\deg(D)$, proving the lemma. □

3.2. **Equivariant completions of hyperbolic $\mathbb{C}^*$-surfaces.**

**3.6.** In this subsection $V$ denotes a hyperbolic $\mathbb{C}^*$-surface. According to [FlZa$_1$, I], such a surface is isomorphic to $\operatorname{Spec} A_0[D_+, D_-]$, where $D_\pm$ is a pair of $\mathbb{Q}$-divisors on the normal affine curve $C = \operatorname{Spec} A_0$ with $D_+ + D_- \le 0$. This means that

$$A = A_{\le 0} \oplus_{A_0} A_{\ge 0} \subseteq K[u, u^{-1}]$$

with $K = \operatorname{Frac}(A_0)$,

$$A_{\ge 0} = \bigoplus_{i \ge 0} H^0(C, \mathcal{O}_C(\lfloor iD_+ \rfloor)) \cdot u^i \quad \text{and} \quad A_{\le 0} = \bigoplus_{i \le 0} H^0(C, \mathcal{O}_C(\lfloor -iD_- \rfloor)) \cdot u^i .$$

Our goal is to describe a canonical completion of such a $\mathbb{C}^*$-surface $V$ in terms of the divisors $D_\pm$.

**3.7.** Let us consider the same pair of $\mathbb{Q}$-divisors $D_\pm$ on the smooth completion $\bar{C}$ of $C$. Identifying the function field $K = \operatorname{Frac}(\bar{C})$ with the constant sheaf $K$ on $\bar{C}$, we form the sheaf of $\mathcal{O}_{\bar{C}}$-algebras

$$\mathcal{O}_{\bar{C}}[D_+, D_-] \subseteq K[u, u^{-1}]$$

by defining it on affine open subsets as in 3.6. The resulting normal $\mathbb{C}^*$-surface $V_0 = \operatorname{Spec}\mathcal{O}_{\bar{C}}[D_+, D_-]$ contains $V$ as an open subset and can be completed as follows.

**Proposition 3.8.** *There is a natural $\mathbb{C}^*$-equivariant completion of $V_0$ given by*

$$\bar{V} = \bar{V}_- \cup \bar{V}_+ \cup V_0 ,$$

*where*

$$\bar{V}_+ = \operatorname{Spec}\mathcal{O}_{\bar{C}}[-D_+] \quad \textit{and} \quad \bar{V}_- = \operatorname{Spec}\mathcal{O}_{\bar{C}}[-D_-] .$$

*Moreover, the canonical projection $\pi : V_0 \to \bar{C}$ extends to a $\mathbb{P}^1$-fibration also denoted by $\pi : \bar{V} \to \bar{C}$. The boundary divisor $\bar{D} = \bar{V} \backslash V_0$ consists of two disjoint components $\bar{C}_\pm$ which correspond to the zero sections in $\bar{V}_\pm$, respectively.*

*Proof.* We let $\{p_i\}$ be the points of $\bar{C}$ with $D_+(p_i) + D_-(p_i) < 0$. The fiber over $p_i$ of the orbit map $\pi : V_0 \to \bar{C}$ induced by the inclusion $\mathcal{O}_{\bar{C}} \hookrightarrow \mathcal{O}_{\bar{C}}[D_+, D_-]$, is reducible and consists of two orbit closures $O_i^\pm$, see [FlZa$_1$, I.4]. Let us consider the $\mathbb{C}^*$-surfaces

$$V_- = \operatorname{Spec}\mathcal{O}_{\bar{C}}[-D_-, D_-] \quad \text{and} \quad V_+ = \operatorname{Spec}\mathcal{O}_{\bar{C}}[D_+, -D_+] .$$

There are natural identifications

$$V_\pm = V_0 \backslash \bigcup_i O_i^\mp \quad \text{and} \quad V_+ \cup V_- = V_0 \backslash F$$

and open embeddings $V_\pm \hookrightarrow \bar{V}_\pm$, where $F$ denotes the fixed point set of the original $\mathbb{C}^*$-action on $V$. The complements $\bar{C}_\mp = \bar{V}_\pm \backslash V_\pm$ are the zero sections in $\bar{V}_\mp$ and so are both isomorphic to $\bar{C}$. Pasting first $V_0$ and $\bar{V}_+$ along their common open subset $V_+$ and gluing then $\bar{V}_-$ and the resulting surface $V'$ along their common open subset $V_-$ gives the desired equivariant completion $\bar{V}$ of $V_0$. □

**Remarks 3.9.** 1. The completion of Proposition 3.8 can be constructed with any pair of divisors $(D_+, D_-)$ on $\bar{C}$. It is not necessary to assume that they are zero in the points at infinity. For instance, if $p \in \bar{C}\backslash C$ is a point at infinity and if we replace a pair of divisors $(D_+, D_-)$ by $(D_+ - p, D_- + p)$ then the corresponding completions $\bar{V}$ and $\bar{V}'$ are easily seen to differ by an elementary transformation at the fiber $\pi^{-1}(p) \cong \mathbb{P}^1$.

2. We say that two pairs of $\mathbb{Q}$-divisors $(D_+, D_-)$ and $(D'_+, D'_-)$ on $\bar{C}$ are equivalent if $D'_\pm = D_\pm \pm \operatorname{div}(f)$ for some nonzero meromorphic function $f$ on $\bar{C}$. By the same arguments as in [FlZa$_1$, Theorem 4.3(b)] the hyperbolic $\mathbb{C}^*$-surfaces $V_0 = \operatorname{Spec} \mathcal{O}_{\bar{C}}[D_+, D_-]$ and $V'_0 = \operatorname{Spec} O_{\bar{C}}[D'_+, D'_-]$ over $\bar{C}$ are equivariantly isomorphic if and only if the pairs $(D_+, D_-)$ and $(D'_+, D'_-)$ are equivalent on $\bar{C}$. It is easily seen that equivalent pairs of divisors on $\bar{C}$ lead to equivariantly isomorphic completions of $V_0$ in a canonical way.

3. However, starting from equivalent pairs of divisors $(D_+, D_-)$ and $(D'_+, D'_-)$ on $C$ and extending them by zero in the points at infinity (as we do in 3.7), does not lead in general to equivalent pairs of divisors on $\bar{C}$. Thus the completion constructed here depends on the equivalence class of the pair $(D_+, D_-)$ on $C$. Using (1) and (2) it follows that the completions $\bar{V}$ and $\bar{V}'$ associated to two equivalent pairs $(D_+, D_-)$ and $(D'_+, D'_-)$ of $\mathbb{Q}$-divisors on $C$ differ by elementary transformations at the fibers at infinity.

4. In the completion $\bar{V}$ the curves $C_\pm$ are parabolic, and $C_+$ is easily seen to be repulsive whereas $C_-$ is attractive.

Next we describe the singularities of the above completion $\bar{V}$ and the intersection pairing on its $\mathbb{C}^*$-invariant divisors. We use the following notation.

**3.10.** Following [FlZa$_1$, I.4.21] we let $\{p_i\}$ be the set of points of $\bar{C}$ with $D_+(p_i) + D_-(p_i) < 0$, and $\{q_j\}$ be the points of $\bar{C}$ with $D_+(q_j) = -D_-(q_j) \notin \mathbb{Z}$. We write

$$D_+(p_i) = -\frac{e_i^+}{m_i^+} \quad \text{and} \quad D_-(p_i) = \frac{e_i^-}{m_i^-} \quad \text{with} \quad \gcd(e_i^\pm, m_i^\pm) = 1 \quad \text{and} \quad \pm\, m_i^\pm > 0\,.$$

Since $D_+(p_i) + D_-(p_i) < 0$ and $m_i^+ m_i^- < 0$, the determinant

$$\Delta_i = -\begin{vmatrix} e_i^+ & e_i^- \\ m_i^+ & m_i^- \end{vmatrix} = m_i^+ m_i^- (D_+(p_i) + D_-(p_i)) \tag{11}$$

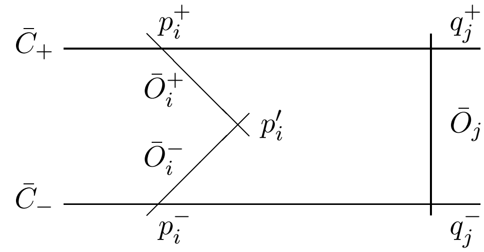
is positive.

The fiber over $p_i$ in $\bar{V}$ consists of two orbit closures $\bar{O}_i^\pm$ which meet the curves $\bar{C}_\pm$ in points, say, $p_i^\pm$. Moreover, $\bar{O}_i^+$ and $\bar{O}_i^-$ meet in a unique point $p'_i$; $p_i^\pm$ and $p'_i$ are the only fixed points over $p_i$ of the $\mathbb{C}^*$-action on $\bar{V}$.

Similarly, we let $\bar{O}_j$ be the orbit closure in $\bar{V}$ of the orbit over $q_j$, and we write

$$D_+(q_j) = -\frac{e_j}{m_j} \quad \text{with} \quad \gcd(e_j, m_j) = 1 \quad \text{and} \quad m_j > 0\,.$$

The fiber over $q_j$ is irreducible and meets $\bar{C}_\pm$ in points $q_j^\pm$.




According to [FlZa$_1$] besides these points $\{p_i, q_j\}$, the fibers of $\pi$ over all other points are smooth and reduced.

**3.11.** Letting further $\mathbb{Z}_d = \langle\zeta\rangle$ be a cyclic group generated by a primitive $d$-th root of unity $\zeta$, we consider the $\mathbb{Z}_d$-action on $\mathbb{A}^2_{\mathbb{C}}$ via

$$\zeta.(x,y) := (\zeta x, \zeta^e y)\,, \tag{12}$$

where $\gcd(d,e) = 1$. If $(V,p) \cong (\mathbb{C}^2/\mathbb{Z}_d, \bar 0)$ analytically then we say that $V$ has a *cyclic quotient singularity of type* $(d,e)$ at $p$. Thus a cyclic quotient singularity of type $(d,e)$ is also a cyclic quotient singularity of type $(d,\tilde e)$, where $\tilde e$ is the unique integer with $e \equiv \tilde e \mod d$ and $0 \le \tilde e < d$. The case $d = 1$ corresponds to a smooth point. A cyclic quotient singularity of type $(d, d-1)$ is an $A_{d-1}$-singularity.

**Lemma 3.12.** *For an equivariant completion $\bar V$ of $V_0$ as in Proposition 3.8, the following hold.*

*(a) $\bar V$ has a cyclic quotient singularity of type $(m_i^+, -e_i^+)$ at $p_i^+ \in \bar V$ and a cyclic quotient singularity of type $(-m_i^-, -e_i^-)$ at $p_i^- \in \bar V$. In particular, $p_i^\pm$ is a smooth point of $\bar V$ if and only if $D_\pm(p_i)$ is integral; that is $m_i^\pm = \pm 1$.*

*(b) $\bar V$ has a cyclic quotient singularity of type $(m_j, \mp e_j)$ at $q_j^\pm \in \bar V$. In particular, $q_j^\pm \in \bar V$ is a smooth point if and only if $D_\pm(q_j)$ is integral that is, $m_j = 1$.*

*(c) For a given value of $i$, let $a, b \in \mathbb{Z}$ be integers with $\begin{vmatrix} a & e_i^+ \\ b & m_i^+ \end{vmatrix} = 1$, and let $e^{(i)} = \begin{vmatrix} a & e_i^- \\ b & m_i^- \end{vmatrix}$. Then $V_0$ has a cyclic quotient singularity of type $(\Delta_i, e^{(i)})$ at $p_i'$, see (11). Thus $p_i' \in V_0$ is a smooth point if and only if $\Delta_i = 1$.*

*Proof.* The point $p_i^+$ lies on $\operatorname{Spec} \mathcal{O}_{\bar C}[-D_+]$. As $-D_+(p_i) = e_i^+/m_i^+$, by [FlZa$_1$, Proposition I.3.8] $\bar V$ has a cyclic quotient singularity of type $(m_i^+, -e_i^+)$ at $p_i^+$. The other assertions in (a) and (b) follow with the same argument. For (c) see Theorem 4.15 in [FlZa$_1$, I]. $\square$

**Proposition 3.13.** *The intersection numbers on $\bar V$ are as follows.*

*(a) $\bar C_\pm^2 = \deg D_\pm$ and $\bar C_+.\bar C_- = 0$.*

*(b) $\bar O_i^+.\bar C_+ = \frac{1}{m_i^+}$, $\quad \bar O_i^-.\bar C_- = -\frac{1}{m_i^-}$ and $\bar O_i^+.\bar C_- = \bar O_i^-.\bar C_+ = 0$.*

*(c) $\bar O_j.\bar C_\pm = \frac{1}{m_j}$ and $\bar O_j^2 = 0$.*

*(d) $\bar O_i^+.\bar O_i^- = \frac{1}{\Delta_i}$, $\quad (\bar O_i^+)^2 = \frac{m_i^-}{\Delta_i m_i^+}$, $\quad (\bar O_i^-)^2 = \frac{m_i^+}{\Delta_i m_i^-}$.*

*Proof.* The first part of (a) follows from Lemma 3.5, and the second part is an immediate consequence of the construction, since $\bar C_+$ and $\bar C_-$ do not meet.

Again by construction the curves $\bar O_i^\pm$ and $\bar C_\mp$ do not meet and so, the intersection numbers $\bar O_i^\pm.\bar C_\mp$ are equal 0. By Proposition 4.18 in [FlZa$_1$, I] the full fiber over $p_i$ is given by

$$\pi^*(p_i) = m_i^+\bar{O}_i^+ - m_i^-\bar{O}_i^- \,.$$

Since its intersection with $\bar{C}_\pm$ is equal to 1, we have

$$\bar{O}_i^\pm.\bar{C}_\pm = \pm 1/m_i^\pm,$$

proving (b). (c) follows along the same kind of arguments.

To compute the intersection numbers in (d) we note that the rational function $u$ on $\bar{V}$ as in 3.6 has a simple pole along $\bar{C}_+$ and a simple zero along $\bar{C}_-$. According to Theorem 4.18 in [FlZa$_1$, I], the restriction of $\operatorname{div}(u)$ on $V_0$ is given by $-\sum_j e_j\bar{O}_j - \sum_i(e_i^+\bar{O}_i^+ - e_i^-\bar{O}_i^-)$ and so we obtain on $\bar{V}$

$$\operatorname{div}(u) = -\bar{C}_+ + \bar{C}_- - \sum_j e_j\bar{O}_j - \sum_i (e_i^+\bar{O}_i^+ - e_i^-\bar{O}_i^-). \tag{13}$$

Multiplying with $\bar{O}_i^+$ we get by (b)

$$e_i^+\bar{O}_i^+\bar{O}_i^+ - e_i^-\bar{O}_i^+\bar{O}_i^- = -\bar{O}_i^+\bar{C}_+ = -1/m_i^+ \,. \tag{14}$$

As $m_i^+\bar{O}_i^+ - m_i^-\bar{O}_i^-$ is numerically equivalent to any fiber of $\pi$, the product of this divisor with $m_i^\pm\bar{O}_i^\pm$ is zero. This leads to the equalities

$$(m_i^+\bar{O}_i^+)^2 = (m_i^-\bar{O}_i^-)^2 = (m_i^+\bar{O}_i^+).(m_i^-\bar{O}_i^-). \tag{15}$$

Hence we can rewrite (14) as

$$\Delta_i(\bar{O}_i^+)^2 = m_i^-/m_i^+.$$

Using this and (15), (d) follows. $\square$

3.3. **Equivariant resolution of singularities.** In this subsection we consider the minimal resolution of singularities of $\bar{V}$, which is equivariant by 2.1. To describe the boundary divisor we introduce the following notation.

**Notation 3.14.** We abbreviate by a box $\square$ with rational weight $e/m$ the weighted linear graph

$$\overset{C_1}{\underset{-k_1}{\circ}} \!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!-\!\! \overset{C_n}{\underset{-k_n}{\circ}} \quad = \quad \overset{e/m}{\square} \tag{16}$$

with $k_1, \ldots, k_n \geq 2$, where $m/e = [k_1, \ldots, k_n]$, $0 < e < m$ and $\gcd(m, e) = 1$. A chain of rational curves $(C_i)$ on a smooth surface with dual graph (16) contracts to a cyclic quotient singularity of type $(m, e)$ [Hi$_1$]. It is convenient to introduce the weighted box $\overset{0}{\square}$ for the empty chain. Given extra curves $E$, $F$ we also abbreviate

$$\overset{E}{\circ}\!\!-\!\!-\!\!-\!\!\overset{C_1}{\circ}\!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!-\!\!\overset{C_n}{\circ} \quad = \quad \overset{E}{\circ}\!\!-\!\!-\!\!-\!\!\overset{e/m}{\square} \quad = \quad \overset{(e/m)^*}{\square}\!\!-\!\!-\!\!-\!\!\overset{E}{\circ} \tag{17}$$

and

$$\overset{C_1}{\circ}\!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!-\!\!\overset{C_n}{\circ}\!\!-\!\!-\!\!-\!\!\overset{F}{\circ} \quad = \quad \overset{e/m}{\square}\!\!-\!\!-\!\!-\!\!\overset{F}{\circ} \quad = \quad \overset{F}{\circ}\!\!-\!\!-\!\!-\!\!\overset{(e/m)^*}{\square} \,. \tag{18}$$

The orientation of the chain of curves $(C_i)_i$ in (16) plays an important role. Indeed, $[k_n, \ldots, k_1] = m/e'$, where $0 < e' < m$ and $ee' \equiv 1 \pmod{m}$ [Fu, Ru], and the box

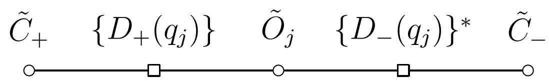
$\square$ marked with $(e/m)^* := e'/m$ corresponds to the reversed chain in (16). Note that contracting the curves $C_1, \ldots, C_n$ leads in both cases to a quotient singularity of type $(m, e)$ on the ambient surface sitting on $E$ and $F$, respectively, however with a different orientation; see e.g. [Mi, Lemma 5.3.3(1)].

**3.15.** Next we consider the minimal resolution of singularities $\varphi : \tilde{V} \to \bar{V}$ of the surface $\bar{V}$. By [OrWa] this resolution is equivariant and all fibers of $\tilde{\pi} := \pi \circ \varphi : \tilde{V} \to \mathbb{P}^1$ are chains of rational curves (cf. also 2.1-2.2). The proper transforms $\tilde{C}_\pm$ on $\tilde{V}$ of the curves $\bar{C}_\pm$ are sections of $\tilde{\pi}$. The boundary divisor $\tilde{D} = \varphi^{-1}(\bar{D})$ can be read off from the following proposition. We recall that $\{r\} = r - \lfloor r \rfloor$, respectively, $\{D\} = D - \lfloor D \rfloor$ stands for the fractional part of a real $r$, respectively, of a $\mathbb{Q}$-divisor $D$.

**Proposition 3.16.** *(a) The fibers $F_p = \tilde{\pi}^{-1}(p)$ in $\tilde{V}$ over the points $p \in \bar{C} \backslash C$ are reduced, isomorphic to $\mathbb{P}^1$ and satisfy $F_p.\tilde{C}_\pm = 1$. Moreover, $F_p.E = 0$ for all curves $E$ in $\tilde{D}$ different from $\tilde{C}_\pm$.*

*(b)* $\tilde{C}_\pm^2 = \deg \lfloor D_\pm \rfloor$.

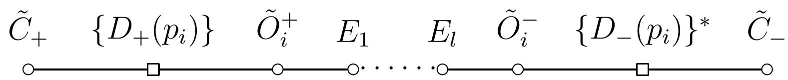
*(c) The fibers over the points $q_j$ together with the curves $\tilde{C}_\pm$ are as follows:*



,

*where the proper transform $\tilde{O}_j$ of $\bar{O}_j$ is a $(-1)$-curve. All these curves except $\tilde{O}_j$ are components of the boundary divisor $\tilde{D}$ of $\tilde{V}$.*

*(d) The fibers over the points $p_i$ together with the curves $\tilde{C}_\pm$ are as follows:*



.

*Here the chain of rational curves $E_1, \ldots, E_l$ corresponds to the cyclic quotient singularity at the fixed point $p_i'$ of the type described in Lemma 3.12(c). Moreover, the curves $\tilde{O}_i^\pm$ are the proper transforms of $\bar{O}_i^\pm$, and at least one of them is a $(-1)$-curve.*

*Proof.* (a) is obvious from our construction, see 3.10. By symmetry it is sufficient to prove (b) for the curve $\tilde{C}_+$. According to Lemma 3.13(a), $(\bar{C}_+)^2 = \deg D_+$. By Proposition 3.2

$$(\tilde{C}_+)^2 = (\bar{C}_+)^2 - \sum_{p \in C} \{D_+(p)\} = \deg D_+ - \deg \{D_+\} = \deg \lfloor D_+(p) \rfloor ,$$

proving (b).

To show (c) we may assume that the set $\{p_i, q_j\}$ consists of a single point $q$ so that $D_\pm = \mp e/m[q]$. By Lemma 3.12, in this case we deal with the minimal resolutions of the cyclic quotient singularities of $\bar{V}$ of type $(m, \mp e)$ at the points $q^\pm \in \bar{C}_\pm$, respectively, resulting in chains of smooth rational curves with weights defined via the continuous fraction expansions of $\mp m/e$, see 3.14. As the fiber over $q$ is a chain of smooth rational curves, it remains to check that the orientations of the boxes labeled by $\{D_\pm(q)\}$ are as indicated. By (b), $\tilde{C}_\pm^2 = \lfloor D_\pm(q) \rfloor$, and by Proposition 3.13(a), $\bar{C}_\pm^2 = D_\pm(q)$. So comparing with Proposition 3.2 the orientation of the chain is indeed as indicated. Since the fiber $F_q$ can be blown down to a smooth one, one of its components is a

$(-1)$-curve. This can be only the component $\tilde{O}_j$ because the resolution of singularities is minimal.

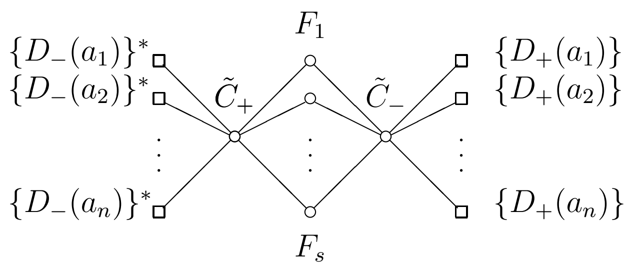
The proof of (d) is similar and is left to the reader. □

**Remark 3.17.** It is easily seen that any irreducible curve on $\tilde{V}$ stable under the $\mathbb{C}^*$-action on $\tilde{V}$ is one of the curves appearing in the proposition.

**Corollary 3.18.** *If $(\tilde{V}, \tilde{D})$ is the equivariant completion of the resolution of singularities of $V$ constructed in 3.16 and $\tilde{\pi} : \tilde{V} \to \bar{C}$ is the extension of the orbit morphism $\pi : V \to C$, then the following hold.*

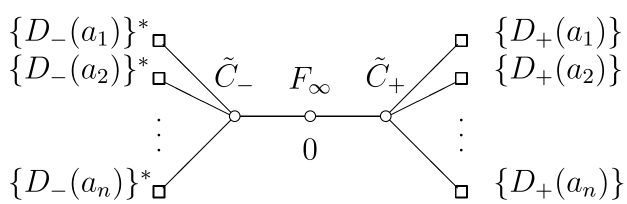
*(a) Every degenerate fiber of the map $\tilde{\pi} : \tilde{V} \to C$ is a linear chain of rational curves meeting the sections $\tilde{C}_\pm$ in the end components.*

*(b) Let $\bar{C}$ be a completion of $C$ with $\operatorname{card}(\bar{C} \setminus C) = s$, and let $\{a_1, \ldots, a_n\} = \{p_i, q_j\}$ be the set of points of $C$ with $D_+(a_i) \neq 0$ or $D_-(a_i) \neq 0$. Then the boundary divisor $\tilde{D} = \tilde{V} \setminus V$ has dual graph*




*Besides possibly $\tilde{C}_\pm \cong \bar{C}$ all the curves are rational, and $F_1, \ldots, F_s$ are the fibers over the points at infinity.*

*(c) In particular, if $C = \mathbb{A}^1$ then the boundary divisor $\tilde{D}$ consists of smooth rational curves, and the dual graph $\Gamma(\tilde{D})$ is*

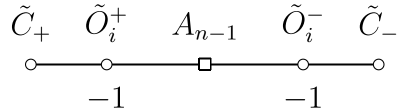



*(d) $V$ is a Gizatullin surface, i.e. the dual graph $\Gamma(\tilde{D})$ is a linear chain of rational curves, if and only if $C \cong \mathbb{A}^1_{\mathbb{C}}$ and each of the fractional parts $\{D_\pm\}$ is either zero or supported at one point: $\operatorname{supp}(\{D_\pm\}) \subseteq \{p_\pm\}$.*

*(e) The dual graph of $\tilde{D}$ is circular if and only if $s = 2$ and the divisors $D_\pm$ are integral. Moreover, in this case the dual graph is $((0, \deg D_+, 0, \deg D_-))$, which has standard form $((0, 0, 0, \deg(D_+ + D_-)))$.*

**Remark 3.19.** We note that every surface as in (e) can be obtained from a Hirzebruch surface by blowing up at some distinct points of two disjoint sections (not at the same fiber) and deleting two other fibers and the proper transforms of these sections. The $\mathbb{C}^*$-action is vertical and the sections are parabolic curves.

**Examples 3.20.** 1. It can happen that both $\tilde{O}_i^{\pm}$ are $(-1)$-curves. Indeed, assume that for some $i$ the coefficients $D_{\pm}(p_i)$ at $p_i$ are both integral and $-n := D_+(p_i)+D_-(p_i) < 0$. In this case by Lemma 3.12 the points $p_i^{\pm} \in \bar{C}_{\pm}$ are smooth and $p_i' \in \bar{V}$ is a cyclic quotient singularity of type $(n, n-1)$ (with $\Delta_i = n$). Since $n/(n-1) = [2, \dots, 2]$ $(n-1$ times) the fiber of $\tilde{V} \to \bar{C}$ over $p_i$ together with the curves $\tilde{C}_{\pm}$ is




with a chain $A_{n-1} = [[(-2)_{n-1}]]$ of $(-2)$-curves of length $n-1$ in the middle. Indeed, by Proposition 3.13(d), $(\bar{O}_i^{\pm})^2 = -1/n$. Applying Proposition 3.2 we obtain $(\tilde{O}_i^{\pm})^2 = \lfloor -1/n \rfloor = -1$.

2. Let $C$ be a nodal cubic in $\mathbb{P}^2$. We claim that the smooth affine surface $V = \mathbb{P}^2 \setminus C$ does not admit a $\mathbb{C}^*$-action. Indeed, $C$ has dual graph $((9))$ with standard form $((0, 0, (-2)_6, -3))$, so this graph is not birationally equivalent to a one in (e) above. Hence $V$ does not admit a hyperbolic $\mathbb{C}^*$-action. We will see below that the dual graphs of equivariant completions of parabolic and elliptic $\mathbb{C}^*$-surfaces are trees, which excludes the existence of a parabolic or elliptic $\mathbb{C}^*$-action on $V$.

3.4. **Parabolic and elliptic $\mathbb{C}^*$-surfaces.** In this subsection we give a short description of the boundary divisors of parabolic and elliptic $\mathbb{C}^*$-surfaces.

**3.21.** *Parabolic case.* We let $V = \operatorname{Spec} A_0[D]$ be a parabolic $\mathbb{C}^*$-surface, where $D$ is a $\mathbb{Q}$-divisor on a smooth affine curve $C = \operatorname{Spec} A_0$. The projection $A_0[D] \to A_0$ provides a section $\iota : C \to V$ with image $C_0 = \iota(C)$.

We recall that $V$ has a cyclic quotient singularity of type $(m, e)$ at $\iota(p) \in C_0$ if $D(p) = -e/m$, see [FlZa$_1$, Prop. I.3.8].

Letting as before $\bar{C}$ be a smooth completion of $C$ with $s$ points at infinity, we consider $D$ as a $\mathbb{Q}$-divisor on $\bar{C}$ and we identify the function field $K = \operatorname{Frac}(C)$ with the constant sheaf $K$ on $\bar{C}$. We form a sheaf of $\mathcal{O}_{\bar{C}}$-algebras

$$\mathcal{O}_{\bar{C}}[D] \subseteq K[u, u^{-1}]$$

as in 3.5. The corresponding normal $\mathbb{C}^*$-surface $V_0 = \operatorname{Spec} \mathcal{O}_{\bar{C}}[D]$ can be completed as follows.

**Proposition 3.22.** *There is a natural $\mathbb{C}^*$-equivariant completion of $V$ given by*

$$\bar{V} = V_0 \cup V_\infty,$$

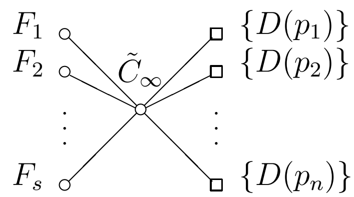
*where $V_0$ and $V_\infty = \operatorname{Spec} \mathcal{O}_{\bar{C}}[-D]$ are pasted along $V^* := V_0 \cap V_\infty = \operatorname{Spec} \mathcal{O}_{\bar{C}}[D, -D]$ via $u \mapsto u^{-1}$. Moreover, the canonical projections $\pi : V_0 \to \bar{C}$ and $\pi : V_\infty \to \bar{C}$ coincide on the intersection and so provide a $\mathbb{P}^1$-fibration also denoted $\pi : \bar{V} \to \bar{C}$. The boundary divisor $\bar{D} = \bar{V} \backslash V$ has a decomposition*

$$\bar{D} = \bar{C}_\infty \cup F_1 \cup \ldots \cup F_s,$$

*where $F_1, \dots, F_s$ are the fibers of $\pi$ over $\bar{C} \backslash C$ and $\bar{C}_\infty$ corresponds to the section in $\bar{V}_\infty$ induced by the projection $\mathcal{O}_{\bar{C}}[-D] \to \mathcal{O}_{\bar{C}}$.*

*Proof.* By Proposition 4.1 and Remark 4.20 in [FlZa$_1$, I], $V^* = \operatorname{Spec} \mathcal{O}_{\bar{C}}[D, -D]$ can be identified with the open subset $V_0 \backslash \bar{C}_0$ of $V_0$, and similarly, with the open subset $V_\infty \backslash \bar{C}_\infty$ of $V_\infty$. Thus pasting $V_0$ and $V_\infty$ along $V^*$ gives an equivariant completion of $V$, cf. Proposition 3.8. The above description of the boundary divisor is now straightforward. $\square$

We let further $\tilde{V}$ be the minimal resolution of singularities of $\bar{V}$, and $\tilde{C}_0$, $\tilde{C}_\infty$ be the proper transforms of the sections $\bar{C}_0$ and $\bar{C}_\infty$, respectively. For every point $p \in C$ with $D(p) = -e/m$ the surface $\bar{V}$ has a cyclic quotient singularity of type $(m, m-e)$ at the point $p' \in \tilde{C}_\infty$ over $p$, cf. Lemma 3.12(a). Thus using 3.18(b) the dual graph of the boundary divisor is as follows:




where $\{p_i\}$ are the points of $C$ with $\{D(p_i)\} \neq 0$. Thus the dual graph of the boundary divisor $D = \tilde{V} \setminus V$ is a linear chain of rational curves if and only if $C \cong \mathbb{A}^1_{\mathbb{C}}$ and $\operatorname{supp}(\{D\})$ is either empty or consists of one point.

**3.23.** *Elliptic case.* We let now $V = \operatorname{Spec} A$, where $A = \bigoplus_{i \geq 0} A_i$ with $A_0 = \mathbb{C}$ is a positively graded normal 2-dimensional $\mathbb{C}$-algebra of finite type. So $V$ is an elliptic $\mathbb{C}^*$-surface. By the results of Dolgachev and Pinkham, see [FlZa$_1$, I], the projective curve $C = \operatorname{Proj} A$ is smooth, and there is a $\mathbb{Q}$-divisor $D$ on $C$ with $\deg D > 0$ such that

$$A_n = H^0(C, \mathcal{O}_C\lfloor nD \rfloor) \cdot u^n \subseteq \operatorname{Frac}(\mathcal{O}_C)[u], \quad \forall n \geq 0\,.$$

The elliptic $\mathbb{C}^*$-surface $V$ can be obtained in the following way. Consider the surface

$$S_0 = \operatorname{Spec}(\mathcal{O}_C[D])$$

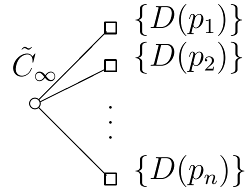
with a parabolic $\mathbb{C}^*$-action provided by the grading of $\mathcal{O}_C[D]$. The inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C[D]$ gives the orbit map $S_0 \to C$, and the projection $\mathcal{O}_C[D] \to \mathcal{O}_C$ gives a section $\iota : C \to S_0$. The natural map $A \to \mathcal{O}_C[D]$ yields a morphism $\pi : S_0 \to V$, which is the contraction of the curve $C_0 = \iota(C) \hookrightarrow S_0$. As in the parabolic case, $S_0$ has a cyclic quotient singularity of type $(m, e)$ at $\iota(p) \in C_0$, where $D(p) = -e/m$. We obtain now a completion $\bar{S}_0$ of $S_0$ as follows.

**Proposition 3.24.** *There is a natural $\mathbb{C}^*$-equivariant completion $\bar{S}$ of $S_0$ given by*

$$\bar{S} = S_0 \cup S_\infty\,,$$

*where $S_0$ and $S_\infty = \operatorname{Spec} \mathcal{O}_C[-D]$ are pasted along $S_0 \cap S_\infty = \operatorname{Spec} \mathcal{O}_C[D, -D]$ via $u \mapsto u^{-1}$. The canonical projections $S_0 \to C$ and $S_\infty \to C$ provide a projection $\pi : \bar{S} \to C$, and the section $C_\infty = \bar{S} \setminus S_0 \subseteq S_\infty$ of $\pi$ is induced by the projection $\mathcal{O}_C[-D] \to \mathcal{O}_C$.*

The proof is the same as in the parabolic case. Consider further the minimal resolution of singularities $\sigma : \tilde{S} \to \bar{S}$, and let $\tilde{C}_\infty$ be the proper transform of $C_\infty$. For every point $p \in C$ with $D(p) = -e/m$ the surface $\bar{S}$ has a cyclic quotient singularity of type $(m, m-e)$ at the point $p' \in C_\infty$ over $p$. Thus similarly as before the boundary divisor $\tilde{S} \setminus S_0$ has dual graph



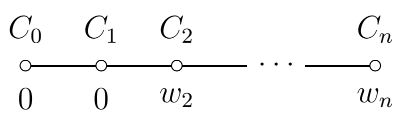

where $(p_i)$ are the points of $C$ with $\{D(p_i)\} \neq 0$.

Since $V$ is obtained from $S_0$ by contracting $C_0$, contracting $C_0$ on $\bar{S}$ yields a completion $\bar{V}$ of $V$. The minimal resolution of singularities $\tilde{V} \to \bar{V}$ of $\bar{V}$ is also equivariant, and the boundary divisor $\tilde{V} \setminus V$ is as shown in the above diagram. This divisor is a linear chain of rational curves provided that $C$ is rational and $\{D\}$ is concentrated in at most 2 points.

## 4. Boundary zigzags of Gizatullin $\mathbb{C}^*$-surfaces

In this section we address Gizatullin surfaces. By definition (see the Introduction) these are normal affine surfaces admitting completion by a zigzag, i.e. by an SNC divisor whose components are rational curves and the dual graph is linear.

4.1. **Smooth Gizatullin surfaces.** By Theorem 2.15 in [FKZ] any Gizatullin surface admits a completion with a standard zigzag $[[0, 0, w_2, \ldots, w_n]]$, $n \geq 1$, as boundary:
(19)




, where $\begin{cases} w_i \leq -2 \quad \forall i & \text{if } n \geq 3 \\ w_2 \leq 0,\ w_2 \neq -1 & \text{if } n = 2\,. \end{cases}$

By Corollary 3.5 in [FKZ] this zigzag is unique up to reversing the sequence of weights $(w_2, \ldots, w_n)$. The following lemma shows that actually every such zigzag can be the boundary of a smooth Gizatullin surface.

**Lemma 4.1.** *(*[Gi, I] *or also* [Du$_2$, I]*) Every standard zigzag (19) occurs as boundary divisor of a smooth Gizatullin surface $X$.*

*Proof.* We start with the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and the curve $C_0 + C_1 + C_2$ on $Q$, where

(20) $$C_0 = \{\infty\} \times \mathbb{P}^1, \quad C_1 = \mathbb{P}^1 \times \{\infty\} \quad \text{and} \quad C_2 = \{0\} \times \mathbb{P}^1\,.$$

In case $n = 1$ we let $X = Q$ and $D = C_0 + C_1$ with $C_0, C_1$ as above. If $n \geq 2$ then performing a sequence of outer blowups over a point $x_0 \in C_2 \backslash C_1$ we obtain a linear chain of rational curves $D = C_0 + C_1 + \ldots + C_n$ with dual graph $Z = [[0, 0, 0]]$ if $n = 2$ (here no blowup is necessary) and $Z = [[0, 0, -1, (-2)_{n-3}, -1]]$ if $n \geq 3$, respectively. Performing further blowups with centers at distinct points of the curves $C_i$ different from the double points of $D$, we can achieve the prescribed weights $C_i^2 = w_i \leq -2$, $i = 2, \ldots, n$.

Letting $\bar{X}$ be the resulting smooth projective surface dominating $Q$, we denote by $\bar{D} = \bar{C}_0 + \bar{C}_1 + \ldots + \bar{C}_n$ the proper transform of $D$ in $\bar{X}$. It remains to check that the smooth open surface $X = \bar{X} \backslash \bar{D}$ is affine. For this it is enough to show that, for a sequence of positive multiplicities $m_0, \ldots, m_n$, the divisor $D' = \sum_{i=0}^n m_i \bar{C}_i$ on $\bar{X}$ is ample. It is easily seen that $\bar{D}$ meets every irreducible curve $C$ on $\bar{X}$ different from all the $\bar{C}_i$, hence $C \cdot \bar{D}' > 0$. Also $\bar{C}_i \cdot D' > 0$ for every $i = 0, \ldots, n$ provided that

$m_{i+1} + m_{i-1} > -m_i w_i$ for all $i$. The latter can be achieved recursively starting with $m_n = 1$. Now such a divisor $D'$ is ample by the Nakai-Moishezon criterion. $\square$

4.2. **Toric Gizatullin surfaces.** In this part we answer the question what further restrictions on the boundary zigzag of a Gizatullin surface are imposed by the presence of a $\mathbb{C}^*$-action. The answer provided by Proposition 4.3 and Theorem 4.4 below depends on whether the surface is smooth or not. Let us first examine the toric case.

**Lemma 4.2.** *(a) Every smooth toric affine surface is isomorphic either to $\mathbb{C}^* \times \mathbb{C}^*$, to $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$ or to $\mathbb{A}^2_{\mathbb{C}}$. Every normal singular toric affine surface is isomorphic to $V_{d,e} := \mathbb{A}^2/\langle\zeta\rangle$, where the primitive $d$-th root of unity $\zeta$ acts on $\mathbb{A}^2$ via $\zeta.(x,y) = (\zeta x, \zeta^e y)$ for some $d > 1$, $e \in \mathbb{Z}$ with $\gcd(e,d) = 1$.*

*(b) A $\mathbb{C}^*$-surface $V = \operatorname{Spec} A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ is toric if and only if $(D_+, D_-) \sim \left(-\frac{e^+}{m^+}[p_0],\, \frac{e^-}{m^-}[p_0]\right)$ for some point $p_0 \in \mathbb{A}^1$.*

*Proof.* (a) is well known and can be found in e.g. [FlZa$_1$, I, Example 2.3 and II, Example 2.8]. To deduce (b), if $V = \operatorname{Spec} A_0[D_+, D_-]$ and $(D_+, D_-) \sim \left(-\frac{e^+}{m^+}[p_0],\, \frac{e^-}{m^-}[p_0]\right)$, then $V$ is toric as was shown in the proof of Theorem 4.15(c) in [FlZa$_1$, I]. Conversely assume that for some pair of $\mathbb{Q}$-divisors $(D_+, D_-)$ the surface $V = \operatorname{Spec} A_0[D_+, D_-]$ is toric. According to [FlZa$_2$], Theorem 4.5 and its proof the pair $(D_+, D_-)$ has then the claimed form, so the lemma follows. $\square$

**Proposition 4.3.** *(a) Any standard zigzag* (19) *occurs as the boundary divisor of a normal toric affine surface*[6].

*(b) A standard zigzag* (19) *occurs as the boundary divisor of a smooth toric affine surface if and only if it is* $[[0,0]]$ *or* $[[0,0,0]]$.

*Proof.* To show (a), given a standard zigzag $[[0, 0, w_2, \ldots, w_n]]$ as in (19) we write

$$\frac{m}{e} = [-w_2 + 1, -w_3, \ldots, -w_n] \quad \text{with} \quad \gcd(e, m) = 1\,.$$

We also consider the pair of $\mathbb{Q}$-divisors $(D_+, D_-) = (-\frac{e}{m}[0], 0)$ on the affine line $C = \mathbb{A}^1$. By Lemma 4.2(b), $V = \operatorname{Spec} A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ is a toric surface. According to Corollaries 2.9, 3.18(c) and Proposition 3.16(d), $V$ has a $\mathbb{T}$-equivariant completion $\tilde{V}$ with boundary divisor

$$\tilde{D}' \;=\; \underset{0}{\overset{\tilde{C}_-}{\circ}} \!-\!\!-\! \underset{0}{\overset{F_\infty}{\circ}} \!-\!\!-\! \underset{-1}{\overset{\tilde{C}_+}{\circ}} \!-\!\!-\!\!-\!\!-\! \overset{\frac{e}{m}}{\square} \;=\; \underset{0}{\overset{\tilde{C}_-}{\circ}} \!-\!\!-\! \underset{0}{\overset{F_\infty}{\circ}} \!-\!\!-\! \underset{-1}{\overset{\tilde{C}_+}{\circ}} \!-\!\!-\! \underset{w_2-1}{\circ} \!-\!\!-\! \underset{w_3}{\circ} \!-\!\!-\! \cdots \!-\!\!-\! \underset{w_n}{\circ}\;.$$

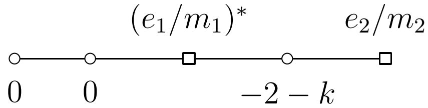
Contracting $\tilde{C}_+$ we perform a $\mathbb{T}$-equivariant outer elementary transformation which consists in blowing up at the only fixed point on $\tilde{C}_- \ominus F_\infty$ of the torus action on $\tilde{V}$ and then blowing down the proper transform of $\tilde{C}_-$. This results in a new equivariant completion of $V$ with the given standard zigzag $[[0, 0, w_2, \ldots, w_n]]$ as boundary, proving (a).

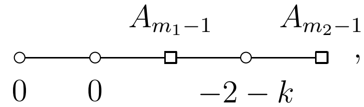
Now (b) follows from Lemma 4.2(a) by virtue of the uniqueness (up to reversion) of a standard zigzag in its birational equivalence class, see Corollary 3.5 in [FKZ]. $\square$

[6]Hence also of a normal surface with a hyperbolic (elliptic, parabolic) $\mathbb{C}^*$-action.

### 4.3. **Smooth Gizatullin $\mathbb{C}^*$-surfaces.**

**Theorem 4.4.** *A standard zigzag occurs as the boundary divisor of a smooth affine hyperbolic $\mathbb{C}^*$-surface if and only if it can be written in one of the forms* $[[0,0]]$, $[[0,0,0]]$,

$(i)$ 

*or* $(ii)$ 

,

*where as before* $A_k = [[(-2)_k]]$, $k \geq 0$, $m_i \geq 1$, $\gcd(e_i, m_i) = 1$ *for* $i = 1, 2$, *and either*

$$\frac{e_1}{m_1} + \frac{e_2}{m_2} = 1 \quad \text{or} \quad \frac{e_1}{m_1} + \frac{e_2}{m_2} = 1 - \frac{1}{m_1 m_2}\,. \tag{21}$$

*Proof.* We suppose first that $V = \operatorname{Spec} A_0[D_+, D_-]$ is a smooth affine surface with a hyperbolic $\mathbb{C}^*$-action, completed by a standard zigzag. By Corollary 3.18(d) $A_0 \cong \mathbb{C}[t]$ and the support of each of the fractional parts $\{D_\pm\}$ is empty or consists of just one point $p_\pm$. Actually we establish below that (i) holds if $p_+ = p_-$ and (ii) holds if $p_+ \neq p_-$.

If $V$ is a smooth toric surface then the assertion follows from Proposition 4.3(b). So we may assume for the rest of the proof that $V$ is not toric.

• Suppose first that the fractional parts $\{D_\pm\}$ are supported at the same point $p_+ = p_-$ or that one or both of them are zero. By a coordinate change of the base and passing to an equivalent pair $(D_+, D_-)$ we may assume that $p_\pm = 0 \in \mathbb{A}^1_{\mathbb{C}}$ and

$$(D_+, D_-) = \left(\left(\frac{e_1}{m_1} - 1\right)[0], \frac{e_2}{m_2}[0] - D'\right) \quad \text{with } 0 \leq e_2 < m_2,\ \gcd(e_i, m_i) = 1,\ i = 1, 2\,,$$

where $D'$ is an effective integral divisor of degree, say, $k + 1 \geq 0$ with $0 \notin \operatorname{supp}(D')$. Actually $k \geq 0$ since otherwise, $D_\pm$ being concentrated at one point, by Lemma 4.2(b) $V$ would be a smooth toric surface, which has been excluded.

The fibers of $\pi : V \to \mathbb{A}^1_{\mathbb{C}}$ over the points $p_i \in \operatorname{supp} D'$ are reducible and singular at the points $p'_i$, see 3.10. According to Lemma 3.12(c) $p'_i \in V$ is a smooth point if and only if $\Delta_i = D'(p_i) = 1$. Since $V$ is supposed to be smooth, $D'$ is supported at $k + 1$ distinct points.

⋄ The fiber in $V$ over $0 \in \mathbb{A}^1_{\mathbb{C}}$ is irreducible (and so $V$ is automatically smooth along this fiber) if and only if

$$D_+(0) + D_-(0) = 0 \quad \Longleftrightarrow \quad \frac{e_1}{m_1} + \frac{e_2}{m_2} = 1\,.$$

The latter agrees with the first equality in (21). Note that this is also true if $m_1 = 1$ or $m_2 = 1$ since in this case the corresponding boxes in (i) are empty.

⋄ The fiber over 0 is reducible if and only if $D_+(0) + D_-(0) < 0$ i.e., $e_1/m_1 + e_2/m_2 < 1$. Moreover

$$D_+(0) = \frac{e_1 - m_1}{m_1} = -\frac{e_0^+}{m_0^+} \quad \text{and} \quad D_-(0) = \frac{e_2}{m_2} = \frac{e_0^-}{m_0^-}\,,$$

so again by Lemma 3.12(c), $V$ is smooth along this fiber if and only if

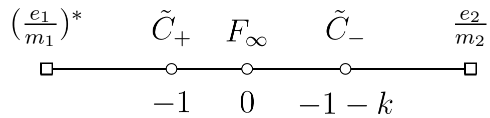
$$\Delta_0 = -\begin{vmatrix} e_0^+ & e_0^- \\ m_0^+ & m_0^- \end{vmatrix} = -\begin{vmatrix} m_1 - e_1 & -e_2 \\ m_1 & -m_2 \end{vmatrix} = 1 \quad \Longleftrightarrow \quad \frac{e_1}{m_1} + \frac{e_2}{m_2} = 1 - \frac{1}{m_1 m_2}\,.$$

The latter agrees with the second equality in (21).

By Proposition 3.16(b) we have

$$\tilde{C}_+^2 = \deg \lfloor D_+ \rfloor = -1 \quad \text{and} \quad \tilde{C}_-^2 = \deg \lfloor D_- \rfloor = -1 - k\,. \tag{22}$$

Thus by virtue of Corollary 3.18 the boundary divisor of the completion $\tilde{V}$ constructed in 3.16 has the form



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Performing an elementary transformation at $F_\infty \cap \tilde{C}_-$ by blowing up this point and blowing down the proper transform of $F_\infty$, we arrive at a linear chain with two zero weights in the middle. By virtue of Lemma 2.12(a) in [FKZ] applying further a sequence of elementary transformations we can move this pair of zero weights to the left to obtain a standard zigzag of type (i).

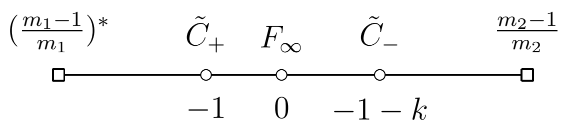
• If now $\{D_\pm\} \neq 0$ and $p_+ \neq p_-$ then we can write

$$D_+ = \frac{e_1}{m_1}[p_+] \quad \text{and} \quad D_- = \frac{e_2}{m_2}[p_-] - D', \qquad \text{where} \quad m_1,\, m_2 \geq 2$$

and $D'$ is an effective integral divisor of degree $k \geq 0$, whose support does not contain the points $p_\pm$. As before, the condition that $V$ is smooth forces by Lemma 3.12(c) that $D'$ is supported at $k$ distinct points and $e_1 = e_2 = -1$, so that

$$D_+ = \frac{-1}{m_1}[p_+] \quad \text{and} \quad D_- = \frac{-1}{m_2}[p_-] - \sum_{i=1}^{k} p_i \quad \text{with} \quad p_i \neq p_\pm \;\forall i\,.$$

Again (22) hold and so, the boundary divisor $\tilde{D}$ of the smooth equivariant completion $\tilde{V}$ of $V$ is in this case



,

see Examples 3.3 and 3.20. Performing a sequence of inner elementary transformations we can transform this into the standard zigzag (ii), as required.

• Vice versa, given a linear chain $\Gamma$ as in (i) or (ii), we choose the divisors $D_\pm$ as in the proof above. This yields a smooth affine surface $V = \operatorname{Spec} A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ equipped with a hyperbolic $\mathbb{C}^*$-action, which admits an equivariant completion by a standard zigzag $\tilde{D}_{\mathrm{st}}$ with dual graph $\Gamma$. □

**Remark 4.5.** Reversing the grading on $A_0[D_+, D_-]$ or, equivalently, switching $\lambda \longmapsto \lambda^{-1}$ in the $\mathbb{C}^*$-action amounts to interchanging $D_+$ and $D_-$. This also amounts to reversing the standard zigzags in (i) or in (ii).

The following corollary is similar to Russell's description of the Ramanujam-Morrow graphs [Ru, 3.3].

**Corollary 4.6.** *A standard zigzag $[[0, 0, w_2, \ldots, w_n]]$ occurs as the boundary divisor of a smooth $\mathbb{C}^*$-surface if and only if one of the following conditions is satisfied.*

*(i′) For some $i$ with $2 \leq i \leq n$, the zigzag $[[w_2, \ldots, w_{i-1}, -1, w_{i+1} \ldots, w_n]]$ is contractible to $[[0]]$ or to $[[-1]]$.*

*(ii′) $[[0, 0, w_2, \ldots, w_n]] = [[0, 0, (-2)_\alpha,\, -2-k,\, (-2)_\beta]]$ for some $\alpha, \beta, k \geq 0$ with $\alpha + \beta = n - 2$.*

*Proof.* We must show that (i′) is equivalent to condition (i) of Theorem 4.4. Consider first the case that $e_1/m_1 + e_2/m_2 = 1$ in 4.4(i). This means that $m := m_1 = m_2$ and $e_2 = m - e$, where $e := e_1$. Replacing in the zigzag from 4.4(i) the weight $-2-k$ by $-1$ and choosing $e'$, $0 \le e' < m$, with $ee' \equiv 1 \mod m$, by virtue of 3.14 we obtain

$$\begin{array}{ccccc} (\frac{e}{m})^* & & -1 & & \frac{m-e}{m} \\ \square & \text{———} & \circ & \text{———} & \square \end{array} \quad = \quad \begin{array}{ccccc} \frac{e'}{m} & & -1 & & (\frac{m-e'}{m})^* \\ \square & \text{———} & \circ & \text{———} & \square \end{array} \;.$$

Letting now in Proposition 3.16(c) $D_+ = e'/m[0]$ and $D_- = -e'/m[0]$, the latter chain occurs as the dual graph of the fiber over $0 \in \mathbb{A}^1$ of a $\mathbb{P}^1$-fibration $\tilde{\pi} : \tilde{V} \to \mathbb{P}^1$ on a smooth surface $\tilde{V}$. Therefore it contracts to $[[0]]$.

Similarly, by Proposition 4.9(b) in [FKZ], the condition

$$\frac{e_1}{m_1} + \frac{e_2}{m_2} = 1 - \frac{1}{m_1 m_2} \tag{23}$$

is equivalent to the contractibility to $[[-1]]$ of the graph

$$\begin{array}{ccccc} (\frac{e_1}{m_1})^* & & -1 & & \frac{e_2}{m_2} \\ \square & \text{———} & \circ & \text{———} & \square \end{array} \;.$$

Now the proof is completed. □

**Examples 4.7.** 1. Every zigzag $[[0, 0, w_2]]$ with $w_2 \le -2$ is of type (ii) in Theorem 4.4 with $m_1 = m_2 = 1$, so that the boxes labelled by $A_{m_i-1}$, $i = 1, 2$, are empty.

2. The zigzag $[[0, 0, w_2, w_3]]$ with $w_2, w_3 \le -2$ satisfies one of the conditions in Corollary 4.6 (and so, corresponds to a smooth Gizatullin surface with a hyperbolic $\mathbb{C}^*$-action) if and only if at least one of the weights $w_2, w_3$ is equal to $-2$.

Indeed, the linear chains $[[-1, w_3]]$ and $[[w_2, -1]]$ cannot be contracted to $[[0]]$ whatever are the weights $w_i \le -2$, $i = 1, 2$. Moreover, under the above condition, and only then, one of these chains contracts to $[[-1]]$ and so, the zigzag $[[0, 0, w_2, w_3]]$ satisfies (i′) and, simultaneously, (ii′).

3. A graph $[[0, 0, w_2, w_3, w_4]]$ ($w_i \le -2$) corresponds to a smooth Gizatullin surface with a hyperbolic $\mathbb{C}^*$-action if and only if either two of the weights $w_2, w_3, w_4$ are equal to $-2$ or $(w_2, w_4)$ is one of the pairs $(-2, -3)$ or $(-3, -2)$.

Indeed, in the first case (ii′) in Corollary 4.6 is fulfilled, and in the second one (i′) holds. Actually the chain $[[w_2, -1, w_4]]$ contracts to $[[-1]]$ (to $[[0]]$, respectively) if and only if $(w_2, w_4)$ is one of the pairs $(-2, -3)$ or $(-3, -2)$ ($(-2, -2)$, respectively). Moreover, the chains $[[-1, w_3, w_4]]$ and $[[w_2, w_3, -1]]$ cannot be contracted to $[[0]]$, and they are contracted to $[[-1]]$ if and only if $w_3 = w_4 = -2$, respectively, $w_2 = w_3 = -2$.

An elliptic or parabolic Gizatullin $\mathbb{C}^*$-surface is necessarily toric, see Corollary 4.4 in [FlZa$_1$, II]. In particular, if such a surface is smooth then it is equivariantly isomorphic to $\mathbb{A}^2$ or $\mathbb{A}^1 \times \mathbb{C}^*$ with a linear $\mathbb{C}^*$-action. Therefore the above examples and Lemma 4.1 lead to the following corollary.

**Corollary 4.8.** *There exist smooth Gizatullin surfaces that do not admit any $\mathbb{C}^*$-action.*

**Remark 4.9.** Every Gizatullin surface admits two non-conjugated $\mathbb{C}_+$-actions. However by Theorem 3.3 in [FlZa$_2$], if a normal affine surface $V \not\cong \mathbb{C}^* \times \mathbb{C}^*$ admits two distinct, up to switching $\lambda \longmapsto \lambda^{-1}$ in one of them, $\mathbb{C}^*$-actions then it also admits a

$\mathbb{C}_+$-action. Moreover by *loc. cit.* $V$ is a Gizatullin surface provided that these $\mathbb{C}^*$-actions are non-conjugate and remain non-conjugate after switching $\lambda \longmapsto \lambda^{-1}$ in one of them.

## 5. Extended graphs of Gizatullin $\mathbb{C}^*$-surfaces

These graphs were used by Gizatullin [Gi], and systematically studied by Dubouloz [Du$_2$]. Here we express the extended graph of a hyperbolic Gizatullin surface $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ in terms of the divisors $D_\pm$ on $\mathbb{A}^1$. In 5.13 and 5.14 we apply these descriptions to study Danilov-Gizatullin $\mathbb{C}^*$-surfaces.

### 5.1. **Extended graphs.**

**Definition 5.1.** Let $V$ be a Gizatullin surface and $(\bar V, D)$ a completion of $V$ by a zigzag. By Proposition 2.9(b) we can transform $(\bar V, D)$ into a standard completion $(\bar V_{\mathrm{st}}, D_{\mathrm{st}})$ so that

$$D_{\mathrm{st}} = C_0 + \ldots + C_n$$

as in (19). We also consider the minimal resolutions of singularities $V'$, $(\tilde V, D)$ and $(\tilde V_{\mathrm{st}}, D_{\mathrm{st}})$ of $V$, $(\bar V, D)$ and $(\bar V_{\mathrm{st}}, D_{\mathrm{st}})$, respectively.

As in 2.17 the linear systems $|C_0|$ and $|C_1|$ define a morphism $\Phi = \Phi_0 \times \Phi_1 : \tilde V_{\mathrm{st}} \to \mathbb{P}^1 \times \mathbb{P}^1$ with $\Phi_i = \Phi_{|C_i|}$, $i = 0, 1$. As before we choose the coordinates in such a way that

$$C_0 = \Phi_0^{-1}(\infty)\,, \qquad \Phi(C_1) = \mathbb{P}^1 \times \{\infty\} \quad \text{and} \quad C_2 \cup \ldots \cup C_n \subseteq \Phi_0^{-1}(0)\,.$$

We recall that the divisor

$$D_{\mathrm{ext}} = C_0 \cup C_1 \cup \Phi_0^{-1}(0)$$

is the *extended divisor* and its dual graph, also denoted by $D_{\mathrm{ext}}$, the *extended graph* of $(\tilde V_{\mathrm{st}}, V)$ or of $V$, for short.

**Remarks 5.2.** 1. By Corollary 3.5 in [FKZ] the standard zigzag $D_{\mathrm{st}} \subseteq D_{\mathrm{ext}}$ as above is uniquely determined up to reversing the chain $C_2, \ldots, C_n$ in (19). However, the extended divisor $D_{\mathrm{ext}}$ usually depends on the completion.

2. As follows from Definition 5.1, the extended graph $D_{\mathrm{ext}}$ can be blown down to

$$\begin{array}{ccccc} C_0 & & C_1 & & C_2 \\ \circ & \text{———} & \circ & \text{———} & \circ \\ 0 & & 0 & & 0 \end{array} \,. \tag{24}$$

In particular, $D_{\mathrm{ext}}$ is a tree, and the intersection form $I(D_{\mathrm{ext}})$ has exactly one positive and one zero eigenvalues (see [FKZ, 4.1]).

We let $\kappa(C)$ denote the number of irreducible components of a curve $C$ and $\rho(V) = \mathrm{rk}(\mathrm{Pic}(V))$ denote the Picard number of $V$.

**Corollary 5.3.** *With $E$ being the exceptional locus of the minimal resolution of singularities $V' \to V$ we have*

$$\rho(V) = \kappa(D_{\mathrm{ext}}) - \kappa(D_{\mathrm{st}}) - \kappa(E) - 1\,. \tag{25}$$

*Proof.* Indeed, $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}[C_1] \oplus \mathbb{Z}[C_2]$, hence $\operatorname{Pic}(\tilde{V}_{\text{st}}) \cong \mathbb{Z}^{\kappa(D_{\text{ext}})-1}$ is freely generated by the components of $D_{\text{ext}} \ominus C_0$. Now $D_{\text{st}} = \tilde{V}_{\text{st}} \setminus V'$, so

$$\rho(V) = \rho(V') - \rho(E) = \rho(\tilde{V}_{\text{st}}) - \rho(D_{\text{st}}) - \rho(E) = \kappa(D_{\text{ext}}) - 1 - \kappa(D_{\text{st}}) - \kappa(E)\,,$$

and the result follows. $\square$

**Examples 5.4.** 1. For an affine toric surface $V$ the Picard number $\rho(V)$ vanishes, hence by (25), $D_{\text{ext}} \ominus D_{\text{st}} \ominus E$ consists of just one component.

2. For $V = \mathbb{P}^2 \setminus C$, where $C$ is a smooth conic, $\operatorname{Pic}(V) = \mathbb{Z}/4\mathbb{Z}$ and so $\rho(V) = 0$. Hence $D_{\text{ext}} \ominus D_{\text{st}}$ consists of one component, where $D_{\text{st}} = [[0, 0, -2, -2, -2]]$.

3. Since any Danilov-Gizatullin surface $V_{k+1}$ (see 5.13 below) is smooth and $\rho(V_{k+1}) = 1$ , $D_{\text{ext}} \ominus D_{\text{st}}$ consists of 2 components.

4. As was shown in Lemma 2.20, $V$ is toric if and only if its extended divisor $D_{\text{ext}}$ has a linear dual graph.

5.2. **Extended graphs on Gizatullin $\mathbb{C}^*$-surfaces.**

**Definition 5.5.** A *feather* $\mathfrak{F}$ is a linear chain of smooth rational curves with dual graph

$$\overset{B}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{e/m}{\square} \tag{26}$$

where $B$ has self-intersection $\leq -1$ and $e, m$ are coprime integers with $m > 0$, cf. (17). Note that the box can also be empty. The curve $B$ will be called the *bridge curve.* A collection of feathers $\{\mathfrak{F}_\rho\}$ consists of feathers $\mathfrak{F}_\rho$, $\rho = 1, \ldots, r$, that are pairwise disjoint. Such a collection will be denoted by a plus box $\boxplus$.

We say that this collection of feathers $\{\mathfrak{F}_\rho\}$ is attached to the curve $C_i$ in a chain (19) if the bridge curves $B_\rho$ meet $C_i$ in pairwise distinct points, and all the feathers are disjoint with the curves $C_j$ for $j \neq i$. In a diagram we will write in brief

$$\overset{C_i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{\{\mathfrak{F}_\rho\}}{\boxplus}\ .$$

**Examples 5.6.** 1. A curve $B$ with self-intersection $\leq -1$ is a feather; the box in (26) is in this case empty and there is only a bridge curve.

2. The $A_k$*-feather* is

$$\underset{-1}{\overset{E}{\circ}}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{A_k}{\square}\ .$$

Thus the $A_k$-feather represents the contractible linear chain $[[-1, (-2)_k]]$ and the $A_0$-feather represents a single $(-1)$-curve $E$.

**Definition 5.7.** A feather collection will be called *admissible* if it contains at most one feather which is not an $A_k$-feather.

In the next proposition we describe the extended graphs of Gizatullin surfaces with a hyperbolic $\mathbb{C}^*$-action. We recall that for such a surface $V = \operatorname{Spec} A_0[D_+, D_-]$, necessarily $A_0 = \mathbb{C}[t]$ and both $\operatorname{supp}(\{D_\pm\})$ consist of at most one point.

**Proposition 5.8.** *(a) The minimal resolution $V' \to V$ of a non-toric Gizatullin surface $V$ with a hyperbolic $\mathbb{C}^*$-action admits a standard equivariant completion* $(\tilde{V}_{\text{st}}, D_{\text{st}})$

*with* $D_{\rm st} = C_0 + \ldots + C_n$ *as in Definition 5.2 and with the following extended graph* $D_{\rm ext}$ *:*

$$\begin{array}{cccccccccc} & & & & \overset{\{\mathfrak{F}_\rho\}_{\rho\ge 1}}{\boxplus} & & & & & \\ & & & & \vert & & & & & \\ \underset{0}{\overset{C_0}{\circ}} \text{---} & \underset{0}{\overset{C_1}{\circ}} \text{---} & \underset{w_2}{\overset{C_2}{\circ}} \text{---} & \cdots \text{---} & \underset{w_s}{\circ} \text{---} & \underset{w_{s+1}}{\overset{C_{s+1}}{\circ}} \text{---} & \cdots \text{---} & \underset{w_n}{\overset{C_n}{\circ}} \text{---} & \overset{\mathfrak{F}_0}{\boxplus} \end{array}, \tag{27}$$

*where* $w_i \le -2$ $\forall i \ge 2$, $\mathfrak{F}_0$ *is a single feather (maybe empty) and* $\{\mathfrak{F}_\rho\}_{\rho\ge 1}$ *is a nonempty admissible feather collection.*

(b) *If, moreover,* $\mathrm{supp}(\{D_+\}) \cup \mathrm{supp}(\{D_-\})$ *consists of at most one point then, after possibly reversing the chain* $(C_2, \ldots, C_n)$ *in the standard zigzag, we can achieve additionally that*

   (i) *the chain* $\overset{C_{s+1}}{\circ} \text{---} \cdots \text{---} \overset{C_n}{\circ} \text{---} \overset{\mathfrak{F}_0}{\boxplus}$ *is either empty or is not contractible to a smooth point, and*

   (ii) *all the* $\mathfrak{F}_\rho$, $\rho \ge 1$, *are* $A_{s_\rho}$*-feathers for some* $s_\rho \ge 0$.

(c) *If* $\mathrm{supp}(\{D_+\}) \cup \mathrm{supp}(\{D_-\})$ *consists of two points then the chain in (i) contracts to a smooth point.*

*Proof.* Let as before $V = \mathrm{Spec}\, A_0[D_+, D_-]$ be a DPD-presentation of $V$ with $A_0 = \mathbb{C}[t]$. Since $V$ is a Gizatullin surface, we have $\mathrm{supp}(\{D_\pm\}) \subseteq \{p_\pm\}$ for some points $p_+, p_- \in \mathbb{P}^1$. So $p_+, p_-$ are among the points $\{p_i, q_j\}$ considered in 3.10, and $\mathrm{supp}(\{D_\pm\})$ can also be empty or equal. We will construct a standard equivariant completion $(\tilde{V}_{\rm st}, D_{\rm st})$ of $V'$ starting from the natural completion $(\tilde{V}, \tilde{D})$ as obtained in 3.16.

• Let us first consider the case where $p_+ = p_-$. In this case, after passing to an equivalent pair $(D_+, D_-)$ if necessary, none of the $q_j$ is present besides possibly $p_+$, and for all the $p_i$ different from $p_+$ the numbers $D_\pm(p_i)$ are integral. According to Example 3.20, the fiber of $\tilde{\pi} : \tilde{V} \to \mathbb{P}^1$ over $p_i \ne p_+$ together with the sections $\tilde{C}_\pm$ of $\pi$ is

$$\overset{\tilde{C}_+}{\circ} \text{---} \overset{\tilde{O}_i^+}{\circ} \text{---} \overset{A_{s_i}}{\square} \text{---} \overset{\tilde{O}_i^-}{\circ} \text{---} \overset{\tilde{C}_-}{\circ}$$

with $s_i = -1 - (D_+(p_i) + D_-(p_i))$. The fiber $\tilde{\pi}^{-1}(p_+)$ together with the sections $\tilde{C}_\pm$ is in case $D_+(p_+) + D_-(p_+) = 0$

$$\overset{\tilde{C}_+}{\circ} \text{---} \overset{\{D_+(p_+)\}}{\square} \text{---} \overset{\tilde{O}_{p_+}}{\circ} \text{---} \overset{\{D_-(p_+)\}^*}{\square} \text{---} \overset{\tilde{C}_-}{\circ},$$

and in case $D_+(p_+) + D_-(p_+) < 0$

$$\overset{\tilde{C}_+}{\circ} \text{---} \overset{\{D_+(p_+)\}}{\square} \text{---} \overset{\tilde{O}_{p_+}^+}{\circ} \text{---} \overset{R_{p_+}}{\square} \text{---} \overset{\tilde{O}_{p_+}^-}{\circ} \text{---} \overset{\{D_-(p_+)\}^*}{\square} \text{---} \overset{\tilde{C}_-}{\circ}, \tag{28}$$

where $R_{p_+}$ stands for the minimal resolution of the cyclic quotient singularity in the fiber $\pi^{-1}(p_+)$, see Proposition 3.16(c). By Corollary 3.18, in both cases the boundary zigzag is

$$\overset{\{D_+(p_+)\}^*}{\square} \text{---} \overset{\tilde{C}_+}{\circ} \text{---} \underset{0}{\overset{F_\infty}{\circ}} \text{---} \overset{\tilde{C}_-}{\circ} \text{---} \overset{\{D_-(p_-)\}}{\square}, \tag{29}$$

where $p_+ = p_-$ and, according to Proposition 3.16(d),

$$\tilde{C}_+^2 + \tilde{C}_-^2 = \deg\left(\lfloor D_+ \rfloor + \lfloor D_- \rfloor\right) \le \deg\left(D_+ + D_-\right) \le 0\,. \tag{30}$$

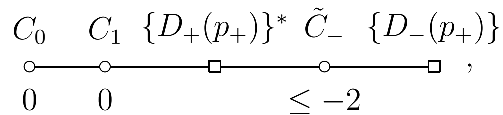
*Claim.* *($\alpha$) If $\tilde{C}_+^2 + \tilde{C}_-^2 \ge -1$ then $V$ is a toric surface. ($\beta$) If $\tilde{C}_+^2 + \tilde{C}_-^2 \le -2$ then moving the zero weight of $F_\infty$ in (29) to the left yields a standard zigzag.*

*Proof of the claim.* ($\alpha$) If $\tilde{C}_+^2 + \tilde{C}_-^2 = 0$ then by virtue of (30), $D_+ = -D_-$ and both divisors are integral. Hence the pair $(D_+, D_-)$ is equivalent to $(0,0)$ and $V \cong \mathbb{A}^2$ is toric. If $\tilde{C}_+^2 + \tilde{C}_-^2 = -1$ then either $D_+ = -D_-$ and $D_\pm(a)$ are integers except for one point $a = p$, or there is a point $p \in \mathbb{A}^1$ where $D_+(p) + D_-(p) < 0$, and for all the other points $q \in \mathbb{A}^1$, $q \ne p$, we have $D_+(q) + D_-(q) = 0$ and $\{D_\pm(q)\} = 0$. Anyhow, passing to an equivalent pair of divisors $(D_+, D_-)$ we obtain

$$D_+ = -\frac{e_+}{m_+}[p] \qquad \text{and} \qquad D_- = \frac{e_-}{m_-}[p]\,.$$

By Lemma 4.2(b), in this case $V$ is toric. This shows ($\alpha$).

To show ($\beta$) we perform inner (hence equivariant) elementary transformations in (29) which replace the curves $\tilde{C}_+$ and $F_\infty$ by two others with self-intersection 0 making the new weight of $\tilde{C}_-$ equal to $\tilde{C}_+^2 + \tilde{C}_-^2 \le -2$. Further we perform inner elementary transformations moving the two zeros to the left to obtain the boundary zigzag on $\tilde{V}_{\rm st}$ in the standard form



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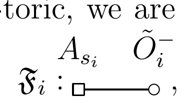
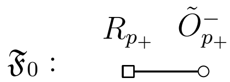
see Lemma 2.12 in [FKZ]. These elementary transformations do not contract the components to the right of $\tilde{C}_-$ preserving their weights. $\square$

Since by our assumption the surface $V$ is non-toric, we are in case ($\beta$) above. We attach to the curve $\tilde{C}_-$ the collection of feathers



, and in case $D_+(p_+) + D_-(p_+) \ne 0$ to the last curve of the weighted $\{D_-(p_+)\}$-box also the feather

(31)



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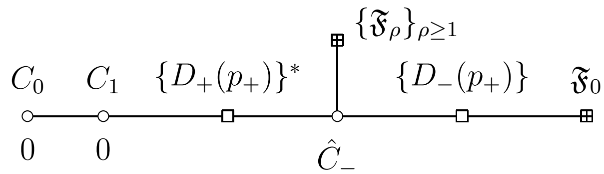
This leads to the graph

(32)



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where $\hat{C}_-$ is the proper transform of $\tilde{C}_-$ with $\hat{C}_-^2 \le -2$. We claim that (32) is already the full extended graph $D_{\rm ext}$ or, equivalently, that the curves in (32) besides $C_0, C_1$ constitute the full fiber $\Phi_0^{-1}(0)$.

In fact, all the components of $D_{\rm ext}$ are $\mathbb{C}^*$-stable, since so are the curves $C_0, C_1$ and the linear systems $|C_0|$ and $|C_1|$ on $\tilde{V}_{\rm st}$. Moreover, since the extended graph is a tree, a curve which occurs in $\Phi_0^{-1}(0) \ominus D_{\rm st}$ meets the boundary zigzag $D_{\rm st}$ in at most one point. Thus the proper transforms on $\tilde{V}_{\rm st}$ of the curves $\tilde{O}_{p_i}^+$ and $\tilde{O}_{p_+}^+$, respectively, $\tilde{O}_{p_+}$ or of an irreducible fiber of $\tilde{\pi} : \tilde{V} \to \mathbb{P}^1$ cannot appear in $\Phi_0^{-1}(0)$. All the other $\mathbb{C}^*$-invariant curves belong already to the boundary zigzag $D_{\mathrm{st}}$ or are in one of the feathers (indeed, in $\tilde{V}$ the only $\mathbb{C}^*$-invariant curves are those in the fibers and the curves $\tilde{C}_\pm$), proving the claim. $\square$

Now (ii) is clear from the construction. To deduce (i), assume in contrary that the chain in (i) is contractible to a smooth point. This is only possible in the case where $D_+(p_+) + D_-(p_+) < 0$, since otherwise the feather $\mathfrak{F}_0$ is empty by construction. Moreover, $\tilde{O}^-_{p_+}$ in the feather $\mathfrak{F}_0$ in (31) must be a $(-1)$-curve, since otherwise the chain in (i) would be minimal, contrary to our assumption. Thus as well the part

$$P_{p_+}: \quad \overset{R_{p_+}}{\square}\!\!-\!\!-\!\!\overset{\tilde{O}^-_{p_+}}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{\{D_-(p_+)\}^*}{\square}$$

of the fiber $\tilde{\pi}^{-1}(p_+)$ in (28) is contractible to a smooth point. After contracting this, the resulting dual graph of the blown down fiber must be $\overset{\{D_+(p_+)\}^*}{\square}\!\!-\!\!-\!\!-\!\!-\!\!\overset{-1}{\circ}$, or it is just irreducible. The former case is not possible since obviously a linear graph with one $(-1)$-end vertex and the rest of vertices of weights at most $-2$, cannot be contracted to a single vertex of weight 0. In the latter case $D_+(p_+)$ is integral i.e., $m_+ = 1$. If we now interchange $D_+$ and $D_-$ then the chain in (i) is either not contractible, or, by the same argument, in addition $D_-(p_+) = 1$. In the latter case the chain in (i) is empty, as desired, see (32). In case $p_+ = p_-$ this proves (a) and (b).

• Suppose further that $p_+$ and $p_-$ are distinct, so that $\{D_-(p_+)\} = \{D_+(p_-)\} = 0$. With the same reasoning as before, the completion $\tilde{V}$ from Proposition 3.15 has the boundary divisor as in (29), and $\tilde{V}_{\mathrm{st}}$ has the extended graph as in (32) with the only difference that now one of the $\{\mathfrak{F}_\rho\}_{\rho \geq 1}$ might be not an $A_s$-feather. We leave the details to the reader. This completes the proof of (a).

To show (c) we note that by our construction, up to reversing, the chain in (i) is the part

$$P_{p_-}: \quad \overset{R_{p_-}}{\square}\!\!-\!\!-\!\!\overset{\tilde{O}^-_{p_-}}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{\{D_-(p_-)\}^*}{\square}$$

of the linear chain $\tilde{\pi}^{-1}(p_-) + \tilde{C}_+ + \tilde{C}_-$ :

$$\overset{\tilde{C}_+}{\circ}\!\!-\!\!-\!\!-\!\!\overset{\tilde{O}^+_{p_-}}{\circ}\!\!-\!\!-\!\!\overset{R_{p_-}}{\square}\!\!-\!\!-\!\!\overset{\tilde{O}^-_{p_-}}{\circ}\!\!-\!\!-\!\!-\!\!\overset{\{D_-(p_-)\}^*}{\square}\!\!-\!\!-\!\!-\!\!\overset{\tilde{C}_-}{\circ}\,,$$

where $\{D_-(p_-)\}^*$ is associated to the cyclic quotient singularity of type $(-m^-, -e^-)$. We claim that $\tilde{O}^-_{p_-}$ is a $(-1)$-curve. Indeed, if $\tilde{O}^-_{p_-}$ were not a $(-1)$-curve then $\tilde{O}^+_{p_-}$ would be since at least one of these is a $(-1)$-curve by Proposition 3.13. But then the fiber $\tilde{\pi}^{-1}(p_-)$ would represent a linear chain with just one $(-1)$-curve at the end and so it cannot be contracted to a 0-curve by the same argument as before. Now tracing the process of blowing down to a non-degenerate fiber $[[0]]$, it is easily seen that $P_{p_-}$ is indeed contractible to a smooth point. $\square$

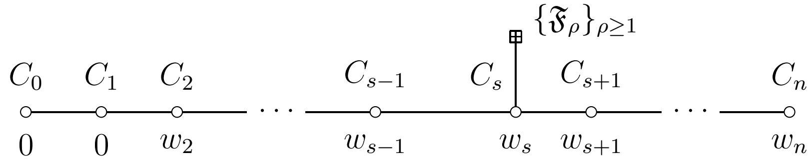
**Remark 5.9.** The curve on the left of $\hat{C}_-$ in (32) has self-intersection $-2$ if and only if $m^+/e_+ \leq 2$, where $e_+$ is the integer with $0 \leq e_+ < m^+$ and $e_+ \equiv -e^+ \mod m^+$. Similarly, the curve on the right of $\hat{C}_-$ in (32) has self-intersection $-2$ if and only if

$-m^-/e_- \le 2$, where $e_-$ is the unique integer with $0 \le e_- < -m^-$ and $e_- \equiv -e^-$ mod $m^-$.

The following result is a useful supplement to Corollary 4.6.

**Corollary 5.10.** *If the surface $V$ in Proposition 5.8 is smooth then the following hold.*

*(a) In case where* $\operatorname{supp}(\{D_+\}) \cup \operatorname{supp}(\{D_-\}) = \{p_0\}$*, up to reversing the zigzag, the extended graph $D_{\mathrm{ext}}$ of $V$ is*



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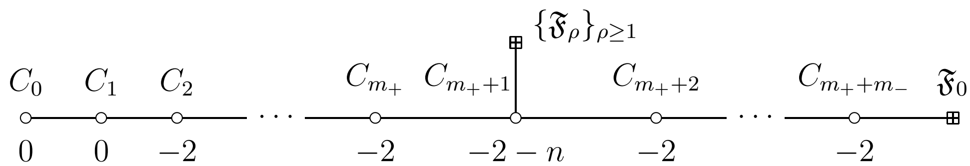
*where $w_i \le -2$ $\forall i \ge 2$, every feather $\mathfrak{F}_\rho$ consists of a single $(-1)$-vertex and the number of these feathers is equal to $|w_s| - 1$. Moreover, the length $\kappa(D_{\mathrm{st}})$ of the boundary zigzag $D_{\mathrm{st}}$ can be $n+1$ or $n$ depending on whether $C_n$ is in $D_{\mathrm{st}}$ or not, and*

$$\rho(V) = \operatorname{rk}(\operatorname{Pic}(V)) = \begin{cases} |w_s| - 2 & \text{if } \kappa(D_{\mathrm{st}}) = n+1 \\ |w_s| - 1 & \text{if } \kappa(D_{\mathrm{st}}) = n\,. \end{cases}$$

*(b) In case where* $\operatorname{supp}(\{D_\pm\}) = \{p_\pm\}$ *with $p_+ \ne p_-$, up to equivalence of the pair $(D_+, D_-)$ we have $D_+(p_+) = -1/m_+$, $D_-(p_+) = 0$ and $D_+(p_-) = 0$, $D_-(p_-) = -1/m_-$ with $m_+, m_- \ge 2$. The extended graph $D_{\mathrm{ext}}$ of $V$ is*



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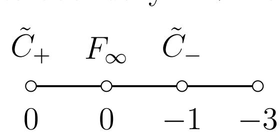
*where $\mathfrak{F}_1$ is a feather consisting of a single $(-m_+)$-curve $\tilde{O}^-_{p_+}$, $\mathfrak{F}_\rho$, $\rho > 1$, are $n$ feathers consisting of $(-1)$-curves $\tilde{O}^-_{p_\rho}$ and $\mathfrak{F}_0$ is a feather consisting of a single $(-1)$-curve $\tilde{O}^-_{p_-}$.*

**Examples 5.11.** 1. It is clear that the curves $\tilde{C}_\pm$ in the completion constructed in Proposition 3.16 are pointwise fixed by the $\mathbb{C}^*$-action. Thus the component $C_s$ as in Proposition 5.8 joined by bridges with a feather collection is parabolic.

2. In the case when $D_+ = 0$ and $D_- = -2/3[a]$ for some $a \in \mathbb{A}^1$, we let $\tilde{V}$ be the resolution of the completion of $V = \operatorname{Spec} A_0[D_+, D_-]$ constructed in Proposition 3.8. Then $V$ is a toric surface and its boundary in $\tilde{V}$ has dual graph



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Blowing up the intersection point $\tilde{C}_+ \cap F_\infty$ and contracting the proper transforms of $\tilde{C}_\pm$ leads to an equivariant completion of $V$ by a standard zigzag $[[0, 0, -2]]$ and without any $\mathbb{C}^*$-parabolic component.

The latter cannot happen for a non-toric $\mathbb{C}^*$-surface, see Lemma 2.21.

In the following result we analyze as to when the extended graph determines uniquely a non-toric Gizatullin $\mathbb{C}^*$-surface.

**Proposition 5.12.** *Suppose that two non-toric Gizatullin $\mathbb{C}^*$-surfaces have the same extended graphs and the same positions of the feathers on the parabolic component. Then these surfaces are equivariantly isomorphic, after possibly switching $\lambda \longmapsto \lambda^{-1}$ in one of them.*

*Proof.* Let $V$ be a Gizatullin $\mathbb{C}^*$-surface with standard completion $(\bar{V}_{\mathrm{st}}, \bar{D}_{\mathrm{st}})$, and let $\tilde{V}_{\mathrm{st}}$ be the minimal resolution of singularities of $\bar{V}_{\mathrm{st}}$. As in 5.1 we consider the extended divisor

$$D_{\mathrm{ext}} = C_0 \cup C_1 \cup \Phi_0^{-1}(0)$$

with dual graph as in (27) (see Proposition 5.8). By 2.13 the standard completion is uniquely determined up to reversing the boundary zigzag. Moreover $(\bar{V}_{\mathrm{st}}, \bar{D}_{\mathrm{st}})$ can be obtained via the construction in the proof of Proposition 5.8, possibly after replacing the given $\mathbb{C}^*$-action by its inverse action and consequently the pair $(D_+, D_-)$ in *loc. cit.* by $(D_-, D_+)$. It is enough to show that the pair $(D_+, D_-)$ can be recovered by the standard boundary divisor with dual graph (27) in Proposition 5.8. Indeed, according to Theorem 4.3.b in [FlZa$_1$, I], a hyperbolic $\mathbb{C}^*$-surface $V$ with orbit map $\pi : V \to C = V//\mathbb{C}^*$ is uniquely determined by the equivalence class of the pair $(D_+, D_-)$, up to an equivariant isomorphism which respects the orbit map. We recall that the equivalence relation is $(D_+, D_-) \rightsquigarrow (D_+ + D, D_- - D)$ for some principal divisor $D$ on $C$.

Since $V$ is not toric the extended divisor $D_{\mathrm{ext}}$ is not linear (see 5.4(4)). With the notation as in the proof of Proposition 5.8, the extended divisor together with the boundary zigzag determine the graph (29) uniquely up to the weights $\tilde{C}_\pm^2$, whereas the sum $\tilde{C}_+^2 + \tilde{C}_-^2$ is also determined. So they determine uniquely the position of $F_\infty$ and the values $\{D_+(p_+)\}$ and $\{D_-(p_-)\}$. The orbit map $\tilde{\pi} : \tilde{V} \to \mathbb{P}^1$ yields isomorphisms $\tilde{C}_\pm \xrightarrow{\cong} \mathbb{P}^1$ and sends $F_\infty$ to $\infty \in \mathbb{P}^1$. The positions of the points $p_\pm \in \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ are determined uniquely by the extended divisor. Indeed under the orbit map, $p_+$ corresponds to the intersection point of $\tilde{C}_+$ with the next component to the left in (29), and similarly $p_-$ to the intersection of $\tilde{C}_-$ with the next component to the right. For a point $p \in \mathbb{A}^1$ with $p \neq p_\pm$ we have $D_+(p) + D_-(p) = k \neq 0$ if and only if there is an $A_k$-feather attached to $\bar{C}_- \cong \mathbb{P}^1$ at the point $p$. By (30) we know that $\deg(\lfloor D_+ \rfloor + \lfloor D_- \rfloor) = \tilde{C}_+^2 + \tilde{C}_-^2$. Hence if $p_+ = p_-$ the pair $(D_+, D_-)$ is determined uniquely, up to equivalence, by the extended divisor.

Assume now that $p_+ \neq p_-$. In this case we may assume that $\{D_-(p_+)\} = \{D_+(p_-)\} = 0$. Further, the feather $\mathfrak{F}_0$ consists of

$$\mathfrak{F}_0 : \quad \underset{R_{p_-}}{\square} \!\!-\!\!-\!\!-\!\!-\!\! \underset{\tilde{O}_{p_-}^-}{\circ} \quad ,$$

where $\tilde{O}_{p_-}^-$ is the bridge curve. The fiber over $p_-$ together with the curves $\tilde{C}_\pm$ has dual graph

$$\underset{\tilde{C}_+}{\circ} \!\!-\!\!-\!\!-\!\!-\!\! \underset{\tilde{O}_{p_-}^+}{\circ} \!\!-\!\!-\!\!-\!\!-\!\! \underset{R_{p_-}}{\square} \!\!-\!\!-\!\!-\!\!-\!\! \underset{\tilde{O}_{p_-}^-}{\circ} \!\!-\!\!-\!\!-\!\!-\!\! \underset{\{D_-(p_-)\}^*}{\square} \!\!-\!\!-\!\!-\!\!-\!\! \underset{\tilde{C}_-}{\circ} \quad ,$$

hence this fiber is known from the extended graph up to the weight of $\tilde O^+_{p_-}$. However, the fibre is contractible to a 0-curve, and this determines the weight of $\tilde O^+_{p_-}$ uniquely. Contracting the boxes leads to the fiber $\bar\pi^{-1}(p_-) = \bar O^-_{p_-} + \bar O^+_{p_-}$ on $\bar V$ and yields the values of $\bar O^-_{p_-}.\bar O^+_{p_-}$, $(\bar O^-_{p_-})^2$ and $(\bar O^+_{p_-})^2$. By Proposition 3.13(d) we can now recover the values of $\Delta(p_-)$, $m^+(p_-) = 1$ and $m^-(p_-)$. Hence also $D_+(p_-) = 0$ and $D_-(p_-) = \frac{\Delta(p_-)}{m^-(p_-)}$ are known, and similarly for the point $p_+$. $\square$

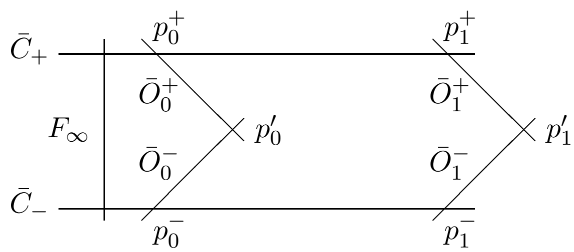
5.3. **Danilov-Gizatullin $\mathbb{C}^*$-surfaces.** The following class of examples was elaborated by Danilov and Gizatullin [DaGi] (see also the Introduction). Answering our question on the uniqueness of $\mathbb{C}^*$-actions [FlZa$_2$], P. Russell showed that there are several non-conjugated $\mathbb{C}^*$-actions on a Danilov-Gizatullin surface. We expose here these $\mathbb{C}^*$-actions in a somewhat different manner.

**Example 5.13.** Given a pair of natural numbers $k, r$ with $1 \le r \le k$ and a pair of distinct points $p_0$, $p_1 \in \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$, we consider the smooth affine hyperbolic $\mathbb{C}^*$-surface $V = V_{k,r} = \operatorname{Spec} A_0[D_+, D_-]$, where $A_0 = \mathbb{C}[t]$,

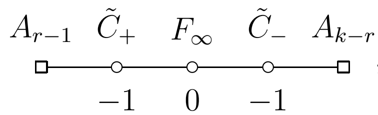
$$D_+ = -\frac{1}{r}[p_0] \quad\text{and}\quad D_- = -\frac{1}{k+1-r}[p_1]\,. \tag{33}$$

We call these *Danilov-Gizatullin $\mathbb{C}^*$-surfaces.*

By Lemma 3.12, the equivariant completion $\bar V$ of $V$ as constructed in Proposition 3.8 has an $A_{r-1}$-singularity at the point $p_0^+$ and an $A_{k-r}$-singularity at $p_1^-$, whereas the other points shown at the following diagram are smooth:




here $\bar O_0^\pm := \bar O^\pm_{p_0}$ and $\bar O_1^\pm := \bar O^\pm_{p_1}$. By Corollary 3.18, the boundary zigzag $\bar D \subseteq \bar V$ is



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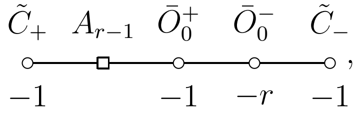
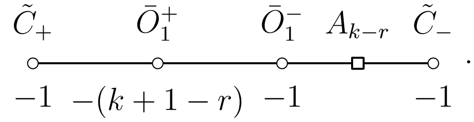
where $F_\infty$ denotes the fiber of $\pi$ over $\infty \in \mathbb{P}^1$. Contracting successively all $(-1)$-curves provides an equivariant completion $\bar V_{k,r}$ of $V_{k,r} := V$ by a single smooth rational curve, say, $S$ of self-intersection $k+1$. For a fixed $k$, by a theorem of Danilov-Gizatullin [DaGi] the $k$ affine surfaces $V_{k,r}$, $1 \le r \le k$, are all isomorphic. However, by Theorem 4.3(b) in [FlZa$_1$, I] they are not equivariantly isomorphic since the fractional parts of the pairs $(D_+, D_-)$ are all distinct for distinct values of $r$. Thus the Danilov-Gizatullin surface $V_{k+1} \cong V_{k,r}$ possesses at least $k$ different $\mathbb{C}^*$-actions that are not conjugated in the automorphism group $\operatorname{Aut}(V_{k+1})$. Furthermore the action of the automorphism $\lambda \longmapsto \lambda^{-1}$ of the group $\mathbb{C}^*$ amounts to interchanging $D_+$ and $D_-$, which reduces the number of essentially different $\mathbb{C}^*$-structures on $V_{k+1}$ to $\lfloor \frac{k+1}{2} \rfloor$.

Let us study the extended divisors $D_{\text{ext}}$ of the $\mathbb{C}^*$-surfaces $V_{k,r} \cong V_{k+1}$. The fibers over $p_0$ and $p_1$ together with the curves $\tilde{C}_\pm$ have dual graphs




, respectively,

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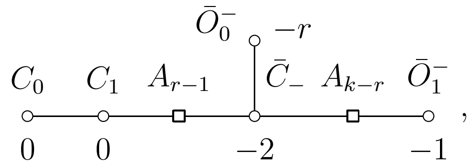
Thus moving the zero on the boundary to the left by means of elementary transformations leads to the extended graph

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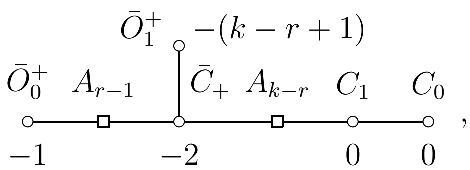
where the feathers are formed by $\bar{O}_0^-$ and $\bar{O}_1^-$. Similarly, moving the zero to the right leads to the extended graph

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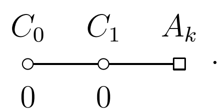
where now the feathers are $\bar{O}_0^+$ and $\bar{O}_1^+$. In both cases, the standard boundary zigzag $D_{\text{st}}$ is $[[0, 0, (-2)_k]]$ with dual graph

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**Proposition 5.14.** *The Danilov-Gizatullin surface $V_{k+1}$ ($k \geq 0$) carries exactly $k$ different, up to conjugation in the automorphism group, $\mathbb{C}^*$-actions, and all of them are hyperbolic.*

Let us give two alternative proofs.

*1-st proof.* A smooth elliptic or parabolic Gizatullin $\mathbb{C}^*$-surface is necessarily isomorphic to $\mathbb{A}^2$, see Corollary 4.4 in [FlZa$_1$, II]. Hence the Gizatullin surface $V_{k+1}$ with the Picard group $\mathrm{Pic}(V_{k+1}) \cong \mathbb{Z}$ cannot carry any elliptic or parabolic $\mathbb{C}^*$-action.

We have shown in 5.13 above that there are at least $k$ mutually non-conjugated hyperbolic $\mathbb{C}^*$-actions on $V_{k+1}$. To show that any such action on $V_{k+1}$ is conjugated to one of these is the same as to show that, given an isomorphism

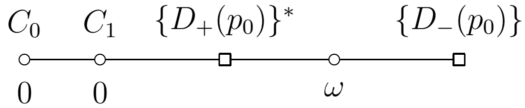
$$V_{k+1} \cong \operatorname{Spec} A_0[D_+, D_-]$$

with $A_0 = \mathbb{C}[t]$ and some pair of $\mathbb{Q}$-divisors $D_\pm$ on $\mathbb{A}^1 = \operatorname{Spec} A_0$ with $D_+ + D_- \leq 0$, up to equivalence $(D_+, D_-)$ must be one of the pairs (33). Since $V_{k+1}$ is a Gizatullin surface, the supports of $\{D_+\}$ and $\{D_-\}$ consist of at most one point. Let as before $p_0, \ldots, p_l$ be the points with $D_+(p_i) + D_-(p_i) < 0$, and $q_1, \ldots, q_s$ the points with $D_+(q_j) + D_-(q_j) = 0$. Replacing $D_+$, $D_-$ by an equivalent pair we may suppose that $\{D_\pm(q_j)\} \neq 0$. Thus necessarily $s \leq 1$. If $s = 1$ then by Corollary 4.24 in [FlZa$_1$, I], $\mathrm{Pic}(V_{k+1})$ would have torsion. Since $\mathrm{Pic}(V_{k+1}) \cong \mathbb{Z}$, this case is impossible and so $s = 0$. On the other hand, again by Corollary 4.24 in [FlZa$_1$, I], we have $l = 1$.

First we assume that both $\{D_+(p_0)\}$ and $\{D_-(p_0)\}$ are nonzero. Then necessarily $\{D_+(p_1)\} = \{D_-(p_1)\} = 0$. As $p_0' \in V_{k+1}$ is a smooth point, by Lemma 3.12(c) we have

$$D_+(p_0) + D_-(p_0) = \frac{\Delta(p_0)}{m_0^+ m_0^-} = \frac{1}{m_0^+ m_0^-}\,.$$

This implies $\lfloor D_+(p_0)\rfloor + \lfloor D_-(p_0)\rfloor = -1$. The standard boundary zigzag of $V_{k+1}$ is

(34) 

$= \quad [[0, 0, (-2)_k]]\,,$

where $\omega = \sum_{i=0,1} (\lfloor D_+(p_i)\rfloor + \lfloor D_-(p_i)\rfloor) = D_+(p_1) + D_-(p_1) - 1 = -2$. Therefore $D_+(p_1) + D_-(p_1) = -1$. Moreover, the boxes in (34) are $A_{r-1}$ and $A_{k-r}$-boxes for some $r$ with $0 < r < k+1$, so that

$$\{D_+(p_0)\} = \frac{r-1}{r} \qquad \text{and} \qquad \{D_-(p_1)\} = \frac{k-r}{k+1-r}\,. \tag{35}$$

Passing to an equivalent pair of divisors we may assume that $D_+(p_0) = \frac{-1}{r}$, hence $\lfloor D_+(p_0)\rfloor = -1$, $\lfloor D_-(p_0)\rfloor = 0$ and $D_-(p_0) = \frac{k-r}{k+1-r} = \frac{e_0^-}{m_0^-}$, where $e_0^- = -(k-r)$ and $m_0^- = -(k+1-r)$. Again by smoothness of the point $p_0' \in V_{k+1}$, the determinant (11) is equal to 1:

$$\Delta(p_0) = -\begin{vmatrix} 1 & -(k-r) \\ r & -(k+1-r) \end{vmatrix} = 1\,.$$

Hence $(k+1-r) - (k-r)r = 1$ and so, $(k-r)(r-1) = 0$. This forces $k = r$ or $r = 1$ i.e., $D_+(p_0)$ or $D_-(p_0)$ is integral, contrary to our assumption.

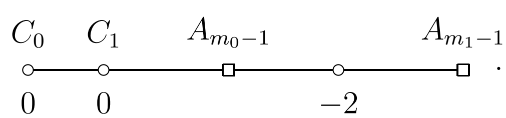
Therefore, up to interchanging $p_0$ and $p_1$, the only possibility is

$$\{D_+(p_0)\} \neq 0 \qquad \text{and} \qquad \{D_-(p_1)\} \neq 0\,,$$

whereas $D_-(p_0)$ and $D_+(p_1)$ are integral. After passing again to an equivalent pair $(D_+, D_-)$ we may suppose that $D_-(p_0) = D_+(p_1) = 0$. We write now

$$D_+(p_0) = -\frac{e_0}{m_0} \qquad \text{and} \qquad D_-(p_1) = -\frac{e_1}{m_1}\,,$$

where $m_0, m_1 > 0$. By smoothness of the points $p_i' \in V_{k+1}$ we have $\Delta(p_i) = 1$ for $i = 0, 1$, hence $e_0 = e_1 = 1$. Thus the zigzag (34) of the equivariant standard completion $(\tilde{V}_{k+1})_{\mathrm{st}}$ of $V_{k+1}$ is

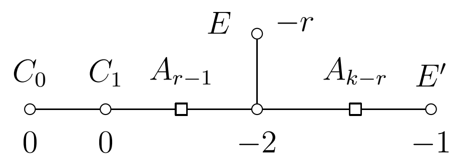


.

This yields $m_0 + m_1 = k - 1$, so $(D_+, D_-)$ is one of the pairs in (33), as required. $\square$

*2-nd proof.* We must show that any hyperbolic $\mathbb{C}^*$-action $\Lambda$ on $V_{k+1}$ is conjugate to one of those constructed in Example 5.13. Since these $k$ $\mathbb{C}^*$-actions on $V$ are mutually non-conjugate, this would complete the proof.

The surface $V_{k+1}$ admits an equivariant completion $(\bar{V}_{k+1})_{\mathrm{st}}$ by a standard zigzag $D = C_0 + C_1 + \ldots + C_{k+1}$ such that $C_j^2 = -2$ for $j \geq 2$. As before, the complete linear systems $|C_0|$ and $|C_1|$ yield a morphism $\Phi = (\Phi_0, \Phi_1) : (\bar{V}_{k+1})_{\mathrm{st}} \to Q = \mathbb{P}^1 \times \mathbb{P}^1$. Since the $\mathbb{C}^*$-action on $(\bar{V}_{k+1})_{\rm st}$ stabilizes $C_0$ and $C_1$ it preserves the corresponding linear systems and hence induces a linear $\mathbb{C}^*$-action $(x,y) \to (\lambda^n x, \lambda^m y)$ on $Q$ such that $\Phi$ is equivariant. We note that the numbers $n$ and $m$ uniquely determine the part of the extended graph $D_{\rm ext}$ between $C_2$ and the parabolic component $C_r$. Indeed $C_r$ appears as the $(-1)$-curve in the resolution graph $\Gamma_0$ of the curve singularity $x^m = y^n$. Unless $n = r-2$ and $m = r-1$ for some $r$ the part of $\Gamma_0$ between $C_2$ and $C_r$ contains vertices of weight $\le -3$ which contradicts our assumption. Thus $n = r-2$ and $m = r-1$ and so, besides $C_3, \ldots, C_r$, $\Gamma_0$ must contain an extra vertex $E$ of weight $E^2 = -r$ which is the proper transform of a feather (unless, maybe, in the case where $r = 3$). The only way to get $C_j^2 = -2$ for $j \ge r$ is to construct a linear chain $C_{r+1}, \ldots, C_{k+1}, E'$ with $C_j^2 = -2$, where $E'$ with $(E')^2 = -1$ is the second feather and a neighbor of $C_{k+1}$. This produces exactly the same extended graph




as in Example 5.13 i.e., the same extended graph as one of the standard actions. Now the pair of divisors $(D_+, D_-)$ can be read up from this graph and so it coincides with the corresponding pair (33). Hence the corresponding $\mathbb{C}^*$-actions on $V_{k+1}$ are conjugate. $\square$

**Remarks 5.15.** (1) Every Gizatullin surface $V$ admits at least two different affine rulings (that is, $\mathbb{A}^1$-fibrations) $v_\pm : V \to \mathbb{A}^1$. They are provided by the projections $\Phi_0^\pm : \bar{V}_{\rm st}^\pm \to \mathbb{P}^1$ as in Definition 5.1, where $(\bar{V}_{\rm st}^\pm, D_{\rm st}^\pm)$ are two equivariant completions of $V$ by standard zigzags $D_{\rm st}^+$, $D_{\rm st}^-$ which differ by reversion moving a pair of zeros from the left to the right. By Lemma 2.19 $v_\pm : V \to \mathbb{A}^1$ is a smooth $\mathbb{A}^1$-fibration over $\mathbb{A}^1 \setminus \{0\}$.

If moreover $V = \operatorname{Spec} A_0[D_+, D_-]$ is not toric and carries a $\mathbb{C}^*$-action then taking the two unique equivariant completions there are unique affine rulings $v_\pm$ that are equivariant with respect to a suitable $\mathbb{C}^*$-action on $\mathbb{A}^1$. Moreover we can describe their fibers over 0 in terms of $D_\pm$: they are disjoint unions of the $\mathbb{C}^*$-orbit closures $\bar{O}_i^\mp \cong \mathbb{A}^1$, one for each point $p_i \in \mathbb{A}^1_{\mathbb{C}}$ with $(D_+ + D_-)(p_i) < 0$, see Proposition 3.25 in [FlZa$_1$], where also the multiplicity of $\bar{O}_i^\mp$ in $\operatorname{div}(v_\pm)$ is computed.

(2) In particular, to any given hyperbolic $\mathbb{C}^*$-action on a Danilov-Gizatullin surface $V_{k+1}$ corresponds such a unique pair $v_\pm$ of equivariant affine rulings $V_{k+1} \to \mathbb{A}^1_{\mathbb{C}}$. Given $r$ with $1 \le r \le k$ as in Example 5.13 above, $v_\pm^{-1}(0)$ consists of the corresponding feather components $\bar{O}_0^\mp, \bar{O}_1^\mp$, with multiplicities

$$\operatorname{div}(v_+) = [\bar{O}_0^-] + r[\bar{O}_1^-] \quad \text{and} \quad \operatorname{div}(v_-) = (k+r-1)[\bar{O}_0^+] + [\bar{O}_1^+]\,,$$

see Proposition 3.25 in [FlZa$_1$]. Alternatively, these equalities can be seen following the construction of the feather components in the second proof above. Since conjugate affine rulings must have equal sequences of multiplicities of degenerate fibers, we obtain the following corollary[7].

[7]This was first proved by Peter Russell, see [CNR], and in some particular cases by Adrien Dubouloz.

**Corollary 5.16.** *(a) The Danilov-Gizatullin surface $V_{k+1}$ $(k \geq 0)$ carries at least $\lfloor \frac{k+1}{2} \rfloor$ different, up to conjugation in the automorphism group, affine rulings $V_{k+1} \to \mathbb{A}^1_{\mathbb{C}}$ with a unique degenerate fiber.*

*(b) Given integer $r \neq \frac{k+1}{2}$ with $1 \leq r \leq k$, the equivariant affine rulings $v_\pm : V_{k,r} = V_{k+1} \to \mathbb{A}^1_{\mathbb{C}}$ canonically attached to the corresponding $\mathbb{C}^*$-action on $V_{k+1}$ are not conjugate.*

See also [Du$_3$] for another approach to (a).

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