EMBEDDINGS OF $\mathbb{C}^*$-SURFACES INTO WEIGHTED PROJECTIVE SPACES

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Abstract. Let $V$ be a normal affine surface which admits a $\mathbb{C}^*$- and a $\mathbb{C}_+$-action. Such surfaces were classified e.g., in [FlZa1, FlZa2], see also the references therein. In this note we show that in many cases $V$ can be embedded as a principal Zariski open subset into a hypersurface of a weighted projective space. In particular, we recover a result of D. Daigle and P. Russell, see Theorem A in [DR].

1. Introduction

If $V = \text{Spec } A$ is a normal affine surface equipped with an effective $\mathbb{C}^*$-action, then its coordinate ring $A$ carries a natural structure of a $\mathbb{Z}$-graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$. As was shown in [FlZa1], such a $\mathbb{C}^*$-action on $V$ has a hyperbolic fixed point if and only if $C = \text{Spec } A_0$ is a smooth affine curve and $A_{\pm 1} \neq 0$. The structure of the graded ring $A$ can be elegantly described in this case in terms of a pair $(D_+, D_-)$ of $\mathbb{Q}$-divisors on $C$ with $D_+ + D_- \leq 0$. More precisely, $A$ is the graded subring $A = A_0[D_+, D_-] \subseteq K_0[u, u^{-1}], \quad K_0 := \text{Frac } A_0,$

where for $i \geq 0$

$A_i = \{ f \in K_0 \mid \text{div } f + iD_+ \geq 0 \} u^i \quad \text{and} \quad A_{-i} = \{ f \in K_0 \mid \text{div } f + iD_- \geq 0 \} u^{-i}.$

This presentation of $A$ (or $V$) is called in [FlZa1] a DPD-presentation. Furthermore, two pairs $(D_+, D_-)$ and $(D_+', D'_-)$ define equivariantly isomorphic surfaces over $C$ if and only if they are equivalent that is,

$D_+ = D_+' + \text{div } f \quad \text{and} \quad D_- = D_-'+ - \text{div } f \quad \text{for some } f \in K_0^\times.$

Our main result (Theorem 2.4) states that if such a surface $V$ admits also a $\mathbb{C}_+$-action then it can be $\mathbb{C}^*$-equivariantly embedded (up to normalization) into a weighted projective space minus a hyperplane; see also Remark 2.5 and Corollary 2.6 below. In particular we recover the following difficult result of Daigle and Russell (see [DR, Theorem A]; cf. also Remark 3.4 below).

Theorem 1.1. Let $V$ be a normal Gizatullin surface$^1$ with a finite divisor class group. Then $V$ can be embedded into a weighted projective plane $\mathbb{P}(a, b, c)$ minus a hypersurface. More precisely:

(a) If $V = V_{d,e}$ is toric$^2$ then $V$ is equivariantly isomorphic to the open part$^3$ $\mathbb{D}_+(z)$ of the weighted projective plane $\mathbb{P}(1, e, d)$ equipped with homogeneous coordinates $(x : y : z)$ and with the 2-torus action $(\lambda_1, \lambda_2).(x : y : z) = (\lambda_1 x : \lambda_2 y : z)$.

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$^1$That is, $V$ admits a completion by a linear chain of smooth rational curves; see Section 3 below.

$^2$See 3.1(a) below.

$^3$We use the standard notation $V_+(f) = \{ f = 0 \}$ and $\mathbb{D}_+(f) = \{ f \neq 0 \}$. 

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(b) If $V$ is non-toric then $V \cong \mathbb{D}_+(xy - z^m) \subseteq \mathbb{P}(a,b,c)$ for some positive integers $a, b, c$ satisfying $a + b = cm$ and $\gcd(a,b) = 1$.

2. Embeddings of $\mathbb{C}^*$-surfaces into weighted projective spaces

According to Proposition 4.8 in [FlZa1] every normal affine $\mathbb{C}^*$-surface $V$ is equivariantly isomorphic to the normalization of a weighted homogeneous surface $V'$ in $\mathbb{A}^4$. In some cases (described in loc.cit.) $V'$ can be chosen to be a hypersurface in $\mathbb{A}^3$. Cf. also [Du] for affine embeddings of some other classes of surfaces.

In Theorem 2.4 below (see also Remark 2.5) we show that any normal hyperbolic $\mathbb{C}^*$-surface $V$ with a $\mathbb{C}_+$-action is the normalization of a principal Zariski open subset of some weighted projective hypersurface.

For our purposes it is convenient to consider also weighted projective spaces with any weights in $\mathbb{Z}$ as introduced in [BS]. More precisely, if $A$ is a finitely generated $\mathbb{Z}$-graded algebra over $\mathbb{C}$ then we can form Proj$A$ to be the scheme covered by the affine pieces $D_+(f) = \text{Spec} A(f)$, where $f \in A$ is homogeneous of non-zero degree and $A(f) = (A)$0. In particular for any $d_0, \ldots, d_n \in \mathbb{Z}$ we can form a weighted projective space $\mathbb{P}(d_0, \ldots, d_n) = \text{Proj} \mathbb{C}[T_0, \ldots, T_d]$, where $\deg T_i = d_i$ for $i = 0, \ldots, d$. We note that this space is in general not complete.

In the proofs we use the following observation from [Fl]; this Proposition was formulated in loc.cit. only for positively graded algebras. We note that this result – with exactly the same proof – is also valid for $\mathbb{Z}$-graded rings as stated here.

Proposition 2.1. Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a graded $R_0$-algebra of finite type containing the field of rational numbers $\mathbb{Q}$ and the group $E_d \cong \mathbb{Z}/d\mathbb{Z}$ of $d$th roots of unity, where $d > 0$. If $z \in R_d$ then $E_d$ acts on $R$ and then also on $R/(z - 1)$ via

$$\zeta \cdot a = \zeta^i \cdot a \quad \text{for} \quad a \in R_i, \quad \zeta \in E_d,$$

with ring of invariants $(R/(z - 1))^{E_d} \cong (R[1/z])_0$. Consequently

$$(\text{Spec } R/(z - 1))/E_d \cong \mathbb{D}_+(z)$$

is isomorphic to the complement of the hypersurface $\{z = 0\}$ in Proj$(R)$.

We also recall the following result.

Proposition 2.2. Let $V = \text{Spec } A$ be a normal hyperbolic $\mathbb{C}^*$-surface with DPD-presentation

$$A = A_0[D_+, D_-] \subseteq \text{Frac}(A_0)[u, u^{-1}],$$

where $(D_+, D_-)$ is a pair of $\mathbb{Q}$-divisors on the curve $C = \text{Spec } A_0$ with $D_+ + D_- \leq 0$. Then the following are equivalent.

(a) $V$ carries a $\mathbb{C}_+$-action;

(b) $A_0 \cong \mathbb{C}[t]$, and after interchanging $(D_+, D_-)$, if necessary, the fractional part $\{D_+\}$ of $D_+$ is supported at one point.

For a proof we refer the reader to [FlZa2], Corollary 3.23.

2.3. We let now $V = \text{Spec } A_0[D_+, D_-]$ be a normal hyperbolic $\mathbb{C}^*$-surface carrying also a $\mathbb{C}_+$-action. Using Proposition 2.2 we can assume that $A_0 = \mathbb{C}[t]$ and that, after
interchanging \((D_+, D_-)\) and passing to an equivalent pair, if necessary,
\[
D_+ = \frac{-e_+}{d} [0] \quad \text{with} \quad 0 < e_+ \leq d, \\
D_- = \frac{-e_-}{d} [0] - \frac{1}{k} D_0 \quad \text{with} \quad k > 0, \ e_+ + e_- \geq 0
\]
and an integral effective divisor \(D_0\), where \(D_0(0) = 0\). We choose a polynomial \(Q \in \mathbb{C}[t]\) with \(D_0 = \text{div}(Q)\); so \(Q(0) \neq 0\).

**Theorem 2.4.** Let \(F\) be the polynomial
\[
F = x^k y - s^{k(e_++e_-)} Q(s^d/z) z^{\deg Q} \in \mathbb{C}[x, y, z, s],
\]
which is weighted homogeneous of degree \(k(e_++e_-)+d \deg Q\) with respect to the weights
\[
deg x = e_+, \quad deg y = ke_- + d \deg Q, \quad deg z = d, \quad deg s = 1.
\]
Then the surface \(V\) as in 2.3 above is equivariantly isomorphic to the normalization of the principal Zariski open subset \(\mathcal{D}_+(z)\) of the hypersurface \(\mathbb{V}_+(F)\) in the weighted projective 3-space
\[
\mathbb{P} = \mathbb{P}(e_+, ke_- + d \deg Q, d, 1).
\]

**Proof.** With \(s = \sqrt[3]{d}\) the field \(L = \text{Frac}(A)[s]\) is a cyclic extension of \(K = \text{Frac}(A)\). Its Galois group is the group of \(d\)th roots of unity \(E_d\) acting on \(L\) via the identity on \(K\) and by \(\zeta.s = \zeta \cdot s\) if \(\zeta \in E_d\). The normalization \(A'\) of \(A\) in \(L\) is stabilized by the action of \(E_d\) with invariant ring \(A = A'^{E_d}\). According to Proposition 4.12 in [FlZa]
\[
A' = \mathbb{C}[s][D'_+, D'_-] \subseteq \mathbb{C}(s)[u, u^{-1}]
\]
with \(D'_+ = \pi_d^*(D_+)\), where \(\pi_d : \mathbb{A}^1 \to \mathbb{A}^1\) is the covering \(s \mapsto s^d\). Thus

\[
(D'_+, D'_-) = \left( -e_+[0], -e_-[0] - \frac{1}{k} \pi_d^*(D_0) \right) = \left( -e_+[0], -e_-[0] - \frac{1}{k} \text{div}(Q(s^d)) \right).
\]

The element \(x = s^{e_+} u \in A'_1\) is a generator of \(A'_1\) as a \(\mathbb{C}[s]\)-module. According to Example 4.10 in [FlZa] the graded algebra \(A'\) is isomorphic to the normalization of

\[
B = \mathbb{C}[x, y, s]/(x^k y - s^{k(e_++e_-)} Q(s^d)).
\]

More precisely, \(B\) can be considered as the subalgebra of \(L\) generated over \(\mathbb{C}\) by the elements
\[
s, \quad x = s^{e_+} u, \quad \text{and} \quad y = x^{-k} s^{k(e_++e_-)} Q(s^d).
\]

Here the action of \(E_d\) is given by

\[
\zeta.s = \zeta s, \quad \zeta.x = \zeta^{e_+} x, \quad \zeta.y = \zeta^{e_-} y.
\]

In particular this action stabilizes \(B\). Assigning to \(x, y, z, s\) the degrees as in (4), \(F\) as in (3) is indeed weighted homogeneous. Since \(F(x, y, 1, s) = x^k y - s^{k(e_++e_-)} Q(s^d)\), the graded algebra

\[
R = \mathbb{C}[x, y, z, s]/(F)
\]
satisfies \(R/(z-1) \cong B\). Applying Proposition 2.1 \(\text{Spec } B^{E_d}\) is isomorphic to \(\mathcal{D}_+(z) \cap \mathbb{V}_+(F)\) in the weighted projective space \(\mathbb{P}\). Thus the normalizations of \(\text{Spec } B^{E_d}\) and \(\mathcal{D}_+(z) \cap \mathbb{V}_+(F)\) are isomorphic as well. As normalization commutes with taking invariants the normalization of \(B^{E_d}\) is just \(A'^{E_d} = A\), proving our result. \(\Box\)
Remark 2.5. In general not all weights of the weighted projective space \( \mathbb{P} \) in (5) are positive. Indeed it can happen that \( ke_- + d \deg Q \leq 0 \). In this case we can choose \( \alpha \in \mathbb{N} \) with \( ke_- + d(\deg Q + \alpha) > 0 \) and consider instead of \( F \) the polynomial
\[
\tilde{F} = x^ky - s^{k(e_+ + e_-)}Q(s^d/z)z^{\deg Q + \alpha} \in \mathbb{C}[x, y, z, s],
\]
which is now weighted homogeneous of degree \( k(e_+ + e_-) + d(\deg Q + \alpha) \) with respect to the positive weights
\[
\text{deg } x = e_+ , \quad \text{deg } y = ke_- + d(\deg Q + \alpha) , \quad \text{deg } z = d , \quad \text{deg } s = 1.
\]
As before \( V = \text{Spec } A \) is isomorphic to the normalization of the principal open subset \( \mathbb{D}_+(z) \) of the hypersurface \( \mathbb{V}_+(F) \) in the weighted projective space
\[
\mathbb{P} = \mathbb{P}(e_+, ke_- + d(\deg Q + \alpha), d, 1).
\]
In certain cases it is unnecessary in Theorem 2.4 to pass to normalization.

Corollary 2.6. Assume that in (2) one of the following conditions is satisfied.

(i) \( k = 1 \);  
(ii) \( e_+ + e_- = 0 \), and \( D_0 \) is a reduced divisor.

Then \( V = \text{Spec } A \) is equivariantly isomorphic to the principal open subset \( \mathbb{D}_+(z) \) of the weighted projective hypersurface \( \mathbb{V}_+(F) \) as in (3) in the weighted projective space \( \mathbb{P} \) from (5).

Proof. In case (i) the hypersurface in \( \mathbb{A}^3 \) with equation
\[
F(x, y, 1, s) = xy - s^{e_+ + e_-}Q(s^d) = 0
\]
is normal. In other words, the quotient \( R/(z-1) \) of the graded ring \( R = \mathbb{C}[x, y, z, s]/(F) \) is normal and so is its ring of invariants \( (R/(z-1))^{Eq} \). Comparing with Theorem 2.4 the result follows.

Similarly, in case (ii)
\[
F(x, y, 1, s) = x^ky - Q(s^d).
\]
Since the divisor \( D_0 \) is supposed to be reduced and \( D_0(0) = 0 \), the polynomials \( Q(t) \) and then also \( Q(s^d) \) both have simple roots. Hence the hypersurface \( F(x, y, 1, s) = 0 \) in \( \mathbb{A}^3 \) is again normal, and the result follows as before.

Remark 2.7. The surface \( V \) as in 2.3 is smooth if and only if \( D_0 \) is reduced and \( -m_+(D_+(0) + D_-(0)) = 1 \), where \( m_\pm > 0 \) are the denominators in the irreducible representation of \( D_\pm(0) \), see Proposition 4.15 in [FIZa1]. It can happen, however, that \( V \) is smooth but the surface \( \mathbb{V}_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P} \) has non-isolated singularities. For instance, if in 2.3 \( D_0 = 0 \) (and so \( Q = 1 \)), then \( V \) is an affine toric surface\(^4\). In fact, every affine toric surface different from \( (\mathbb{A}^1)^2 \) or \( \mathbb{A}^1 \times \mathbb{A}^1 \) appears in this way, see Lemma 4.2(b) in [FKZ4].

In this case the integer \( k > 0 \) can be chosen arbitrarily. For any \( k > 1 \), the affine hypersurface \( \mathbb{V}_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P} \) with equation \( x^ky - s^{k(e_+ + e_-)} = 0 \) has non-isolated singularities and hence is non-normal. Its normalization \( V = \text{Spec } A \) can be given as the Zariski open part \( \mathbb{D}_+(z) \) of the hypersurface \( \mathbb{V}_+(x^{y'} - s^{e_+ + e_-}) \) in \( \mathbb{P}' = \mathbb{P}(e_+, e_-, d, 1) \) (which corresponds to the choice \( k = 1 \)). Indeed, the element \( y' = s^{e_+ + e_-}/x \in K \) with \( y'^k = y \) is integral over \( A \). However cf. Theorem 1.1(a).

\(^4\)See 3.1(a) below.
Example 2.8. (Danilov-Gizatullin surfaces) We recall that a Danilov-Gizatullin surface $V(n)$ of index $n$ is the complement to a section $S$ in a Hirzebruch surface $\Sigma_d$, where $S^2 = n > d$. By a remarkable result of Danilov and Gizatullin [DaGi, Theorem 5.8.1] up to an isomorphism such a surface only depends on $n$ and neither on $d$ nor on the choice of the section $S$; see also [CNR], [FKZ$_3$] for alternative proofs.

According to [FKZ$_1$, §5], up to conjugation $V(n)$ carries exactly $(n - 1)$ different $\mathbb{C}^*$-actions. They admit DPD-presentations with $A_0 = \mathbb{C}[t]$ and

$$(D_+, D_-) = \left( -\frac{1}{d}[0], -\frac{1}{n - d}[1] \right), \quad \text{where} \quad d = 1, \ldots, n - 1.$$ Applying Theorem 2.4 with $e_+ = 1$, $e_- = 0$, and $k = n - d$, the $\mathbb{C}^*$-surface $V(n)$ is the normalization of the principal open subset $\mathbb{D}_+(z)$ of the hypersurface $V_+(F_{n,d}) \subseteq \mathbb{P}(1,d,d,1)$ of degree $n$, where

$$F_{n,d}(x,y,z,s) = x^{n-d}y - s^{n-d}(s^d - z).$$ Taking here $d = 1$ it follows that $V(n)$ is isomorphic to the normalization of the hypersurface $x^{n-1}y - (s - 1)s^{n-1} = 0$ in $\mathbb{A}^3$.

As our next example, let us consider yet another remarkable class of surfaces. These were studied from different viewpoints e.g., in [MM, Theorem 1.1], [FKZ$_3$, Theorem 1.1(iii)], [GMMR, 3.8-3.9], [KK, Theorem 1.1. and Example 1], [Za, Theorem 1(b) and Lemma 7]. Collecting results from loc.cit. and from this section, we obtain the following equivalent characterizations.

Theorem 2.9. For a smooth affine surface $V$, the following conditions are equivalent.

(i) $V$ is not Gizatullin and admits an effective $\mathbb{C}^*$-action and an $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ with exactly one degenerate fiber, which is irreducible$^5$.
(ii) $V$ is $\mathbb{Q}$-acyclic, $\bar{k}(V) = -\infty$ $^6$ and $V$ carries a curve $\Gamma \cong \mathbb{A}^1$ with $\bar{k}(V \setminus \Gamma) \geq 0$.
(iii) $V$ is $\mathbb{Q}$-acyclic and admits an effective $\mathbb{C}^*$- and $\mathbb{C}_+$-actions. Furthermore, the $\mathbb{C}^*$-action possesses an orbit closure $\Gamma \cong \mathbb{A}^1$ with $\bar{k}(V \setminus \Gamma) \geq 0$.
(iv) The universal cover $\tilde{V} \to V$ is isomorphic to a surface $x^k y - (s^d - 1) = 0$ in $\mathbb{A}^3$, with the Galois group $\pi_1(V) \cong E_d$ acting via $\zeta_1(x,y,s) = (\zeta x, \zeta^{-k} y, \zeta^e s)$, where $k > 1$ and $\gcd(e,d) = 1$.
(v) $V$ is isomorphic to the $\mathbb{C}^*$-surface with DPD presentation $\text{Spec} \mathbb{C}[t][D_+, D_-]$, where

$$(D_+, D_-) = \left( -\frac{e}{d}[0], \frac{e}{d}[0] - \frac{1}{k}[1] \right) \quad \text{with} \quad 0 < e \leq d, \quad \gcd(e,d) = 1, \quad \text{and} \quad k > 1.$$ (vi) $V$ is isomorphic to the Zariski open subset$^7$

$$\mathbb{D}_+(x^k y - s^d) \subseteq \mathbb{P}(e, d - ke, 1), \quad \text{where} \quad 0 < e \leq d, \quad \gcd(e,d) = 1, \quad \text{and} \quad k > 1.$$ In view of the references cited above it remains to show that the surfaces in (v) and (vi) are isomorphic. By Corollary 2.6(ii) with $e_+ = -e_- = e$, the surface $V$ as in (v) is

$^5$Since $V$ is not Gizatullin there is actually a unique $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$. A surface $V$ as in (i) is necessarily a $\mathbb{Q}$-homology plane (or $\mathbb{Q}$-acyclic) that is, all higher Betti numbers of $V$ vanish.

$^6$As usual, $\bar{k}$ stands for the logarithmic Kodaira dimension.

$^7$In the case where $d - ke < 0$, see Remark 2.5.
isomorphic to the principal open subset $\mathbb{D}_+(z)$ in the weighted projective hypersurface

$$V_+(x^k y - (s^d - z)) \subseteq \mathbb{P}(e, d - ke, d, 1).$$

Eliminating $z$ from the equation $x^k y - (s^d - z) = 0$ yields $(vi)$.

These surfaces admit as well a constructive description in terms of a blowup process starting from a Hirzebruch surface, see [GMMR, 3.8] and [KK, Example 1].

An affine line $\Gamma \cong \mathbb{A}^1$ on $V$ as in $(ii)$ is distinguished because it cannot be a fiber of any $\mathbb{A}^1$-fibration of $V$. There is always a family of such affine lines on $V$, see [Za].

Some of the surfaces as in Theorem 2.9 can be properly embedded in $\mathbb{A}^3$ as Bertin surfaces $x^e y - x - s^d = 0$, see [FlZa2, Example 5.5] or [Za, Example 1].

### 3. Gizatullin surfaces with a finite divisor class group

A Gizatullin surface is a normal affine surface completed by a zigzag i.e., a linear chain of smooth rational curves. By a theorem of Gizatullin [Gi] such a surface can be characterized by the property that it admits two $\mathbb{C}^*$-actions with different general orbits, unless it is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$.

In this section we give an alternative proof of the Daigle-Russell Theorem 1.1 cited in the Introduction. It will be deduced from the following result proven in [FKZ2, Corollary 5.16].

**Proposition 3.1.** Every normal Gizatullin surface with a finite divisor class group is isomorphic to one of the following surfaces.

(a) The toric surfaces $V_{d,e} = \mathbb{A}^2 / E_d$, where the group $E_d \cong \mathbb{Z}/d\mathbb{Z}$ of $d$-th roots of unity acts on $\mathbb{A}^2$ via

$$\zeta.(x, y) = (\zeta x, \zeta^e y).$$

(b) The non-toric $\mathbb{C}^*$-surfaces $V = \text{Spec} \mathbb{C}[t][D_+, D_-]$, where

$$(10) \quad (D_+, D_-) = \left(\frac{e}{m}[p], \frac{e}{m}[p] - c[q]\right) \quad \text{with} \quad c \geq 1, \quad p, q \in \mathbb{A}^1, \quad p \neq q,$$

and with coprime integers $e, m$ such that $1 \leq e < m$.

Conversely, any normal affine $\mathbb{C}^*$-surface $V$ as in (a) or (b) is a Gizatullin surface with a finite divisor class group.

Let us now deduce Theorem 1.1.

#### 3.2. Proof of Theorem 1.1

To prove (a), we note that according to 2.1 the cyclic group $E_d$ acts on the ring $\mathbb{C}[x,y,z]/(z-1) \cong \mathbb{C}[x,y]$ via $\zeta.x = \zeta x$, $\zeta.y = \zeta^e y$, and $\zeta.z = z$, where

$$\deg x = 1, \quad \deg y = e, \quad \text{and} \quad \deg z = d.$$

Hence $\mathbb{D}_+(z) = \text{Spec} \mathbb{C}[x,y]^{E_d} = V_{d,e}$, as required in (a).

To show (b) we consider $V = \text{Spec} A$ as in 3.1(b), where

$$A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}(t)[u, u^{-1}].$$

By definition (see (1)) the homogeneous pieces $A_{\pm 1}$ of $A$ are generated as $\mathbb{C}[t]$-modules by the elements

$$u_+ = tu \quad \text{and} \quad u_- = (t-1)^e u^{-1},$$

and similarly $A_{\pm m}$ by

$$v_+ = t^e u^m \quad \text{and} \quad v_- = t^{-e}(t-1)^m u^{-m}.$$
Thus
\[ u^m_+ = t^{m-e}v_+, \quad u^m_- = t^ev_-, \quad \text{and} \quad u_+u_- = t(t-1)^e. \]
The algebra \( A \) is the integral closure of the subalgebra generated by \( u_+, v_+ \) and \( t \).

Consider now the normalization \( A' \) of \( A \) in the field \( L = \text{Frac}(A)[u'_+] \), where
\begin{equation}
(11) \quad u'_+ = \sqrt[ d ]{ v_+ } \quad \text{with} \quad d = cm. \end{equation}
Clearly the elements \( \sqrt[ d ]{ v_+ } = t^{ \frac{ e }{ m } } u_+ \) and also \( t^{ \frac{ e }{ m } } \) both belong to \( L \). Since \( e \) and \( m \) are coprime we can choose \( \alpha, \beta \in \mathbb{Z} \) with \( \alpha(e - m) + \beta m = 1 \). It follows that the element \( \tau := t^{ \frac{ m }{ d } } = t^{ \alpha \frac{ e }{ m } - \beta e } \) is as well in \( L \) whence being integral over \( A \) we have \( \tau \in A' \).

The element \( u'_+ \) as in \( (11) \) and then also \( u'_- := \sqrt[ d ]{ v_- } = (t-1)(\sqrt[ d ]{ v_+ })^{-1} \) belongs to \( A' \). Now \( v_+v_- = (t-1)^cm \), so taking \( d \)th roots we get for a suitable choice of the root \( u'_- \),
\begin{equation}
(12) \quad u'_+u'_- = \tau^m - 1. \end{equation}
We note that \( u_+, v_+ \) and \( t \) are contained in the subalgebra \( B = \mathbb{C}[u'_+, u'_-, \tau] \subseteq A' \). The equation \( (12) \) defines a smooth surface in \( A^3 \). Hence \( B \) is normal and so
\[ A' = B \cong \mathbb{C}[u'_+, u'_-, \tau] / (u'_+u'_- - (\tau^m - 1)). \]

By Lemma 3.3 below, for a suitable \( \gamma \in \mathbb{Z} \) the integers \( a = e - \gamma m \) and \( d \) are coprime. We may assume as well that \( 1 \leq a < d \). We let \( E_d \) act on \( A' \) via \( \zeta.u'_+ = \zeta^a u'_+ \) and \( \zeta.A = \text{id}_A \). Since \( \gcd(a, d) = 1 \), \( A \) is the invariant ring of this action. We claim that the action of \( E_d \) on \((u'_+, u'_-, \tau)\) is given by
\begin{equation}
(13) \quad \zeta.u'_+ = \zeta^a u'_+, \quad \zeta.u'_- = \zeta^{-a} u'_-, \quad \zeta.\tau = \zeta^c \tau, \end{equation}
where \( b = d - a \). Indeed, the equality \( u'_+ = t^{ \frac{ e }{ m } } u_+ = \tau^{e-m} u_+ \) implies that \( \zeta.\tau^{e-m} = \zeta^{ac} \tau^{e-m} \). Since \( \tau = \tau^{a(e - m)} t^\beta \) the element \( \zeta \in E_d \) acts on \( \tau \) via \( \zeta.\tau = \zeta^{a\alpha} \tau \). In view of the congruence \( \alpha \equiv 1 \mod m \) the last expression equals \( \zeta^c \tau \). Now the last equality in \( (13) \) follows. In the equation \( u'_+u'_- = \tau^m - 1 \) the term on the right is invariant under \( E_d \). Hence also the term on the left is. This provides the second equality in \( (13) \).

The algebra \( B = \mathbb{C}[u'_+, u'_-, \tau] \) is naturally graded via
\[ \deg u'_+ = a, \quad \deg u'_- = b, \quad \text{and} \quad \deg \tau = c. \]
According to Proposition 2.1 \( \text{Spec } A = \text{Spec } A^{E_d} \) is the complement of the hypersurface \( \mathbb{V}_+(f) \) of degree \( d = a + b \) in the weighted projective plane
\[ \mathbb{P}(a, b, c) , \quad \text{where} \quad f = u'_+u'_- - \tau^m , \]
proving (b).

To complete the proof we still have to show the following elementary lemma.

**Lemma 3.3.** Assume that \( e, m \in \mathbb{Z} \) are coprime. Then for every \( c \geq 2 \) there exists \( \gamma \in \mathbb{Z} \) such that \( \gamma m - e \) and \( c \) are coprime.

**Proof.** Write \( c = c'd \) such that \( c' \) and \( m \) are coprime and every prime factor of \( d \) divides \( m \). Then for any \( \gamma \in \mathbb{Z} \) the integers \( \gamma m - e \) and \( d \) are coprime. Hence it is enough to establish the existence of \( \gamma \in \mathbb{Z} \) such that \( \gamma m - e \) and \( c' \) are coprime. However, the latter is evident since the residue classes of \( \gamma m, \gamma \in \mathbb{Z} \), in \( \mathbb{Z}/c'\mathbb{Z} \) cover this group. \( \square \)
Remark 3.4. We can also recover the criterion given in Theorem A(3) in [DR] for when two surfaces as in Theorem 1.1 are isomorphic. More precisely we can argue in the cases (a) and (b) of this theorem as follows.

(a) It is a classical fact that two toric surfaces \(V_{d,e}^d\) and \(V_{d',e'}^{d'}\) are isomorphic if and only if \((d,e) = (d',e')\) or \(d = d'\) and \(ee' \equiv 1 \mod d\), see e.g. [FlZa1, Remark 2.5]. Hence two triples \((1,e,d)\) and \((1,e',d')\) as in Theorem 1.1(a) define isomorphic surfaces if and only if \((d,e) = (d',e')\) or \(d = d'\) and \(ee' \equiv 1 \mod d\). We note that here the abstract isomorphism type and equivariant isomorphism type amount to the same.

(b) As follows from Theorem 0.2 in [FKZ2], the integers \(c, m\) in Theorem 1.1(b) are invariants of the (abstract) isomorphism type of \(V\). Indeed, the fractional parts of both divisors \(D^\pm\) as in (10) being nonzero and concentrated at the same point, there is a unique DPD presentation for \(V\) up to interchanging \(D^+\) and \(D^-\), passing to an equivalent pair and applying an automorphism of the affine line \(A^1 = \text{Spec} \mathbb{C}[t]\).

Furthermore, from the proof of Theorem 1.1 one can easily derive that

\[
a \equiv e \mod m \quad \text{and} \quad b = mc - a \equiv -e \mod m .
\]

Therefore also the pair \((a,b)\) is uniquely determined by the (abstract) isomorphism type of \(V\) up to a transposition and up to replacing \((a,b)\) by \((a',b') = (a-sm, b+sm)\), while keeping \(\gcd(a',b') = 1\).

References


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