UNIQUENESS OF $\mathbb{C}^*$- AND $\mathbb{C}_+$-ACTIONS ON GIZATULLIN SURFACES

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Abstract. A *Gizatullin surface* is a normal affine surface V over $\mathbb{C}$, which can be completed by a zigzag; that is, by a linear chain of smooth rational curves. In this paper we deal with the question of uniqueness of $\mathbb{C}^*$-actions and $\mathbb{A}^1$-fibrations on such a surface V up to automorphisms. The latter fibrations are in one to one correspondence with $\mathbb{C}_+$-actions on V considered up to a "speed change".

Non-Gizatullin surfaces are known to admit at most one $\mathbb{A}^1$-fibration $V \to S$ up to an isomorphism of the base S. Moreover an effective $\mathbb{C}^*$-action on them, if it does exist, is unique up to conjugation and inversion $t \mapsto t^{-1}$ of $\mathbb{C}^*$. Obviously uniqueness of $\mathbb{C}^*$-actions fails for affine toric surfaces; however we show in this case that there are at most two conjugacy classes of $\mathbb{A}^1$-fibrations. There is a further interesting family of non-toric Gizatullin surfaces, called the Danilov-Gizatullin surfaces, where there are in general several conjugacy classes of $\mathbb{C}^*$-actions and $\mathbb{A}^1$-fibrations, see e.g., [FKZ$_1$].

In the present paper we obtain a criterion as to when $\mathbb{A}^1$-fibrations of Gizatullin surfaces are conjugate up to an automorphism of V and the base S. We exhibit as well a large subclasses of Gizatullin $\mathbb{C}^*$-surfaces for which a $\mathbb{C}^*$-action is essentially unique and for which there are at most two conjugacy classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$.

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Introduction

Let V be a normal affine surface admitting an effective action of the group $\mathbb{C}^*$. It is a natural question as to when any two such actions on V are conjugate in the automorphism group $\mathrm{Aut}(V)$. Similarly, given an $\mathbb{C}_+$-action on V one may ask whether its associated $\mathbb{A}^1$-fibration $V \to S$ is unique up to conjugation; that is, up to an automorphism of V and an isomorphism of the the base S.

Recall [FKZ$_2$] that a *Gizatullin surface* is a normal affine surface completable by a *zigzag* that is, by a linear chain of smooth rational curves. The uniqueness of $\mathbb{C}^*$-actions on normal affine surfaces, up to conjugation and inversion, is known to hold for all non-Gizatullin surfaces (see [Be] for the smooth case, [FlZa$_3$, Theorem 3.3] for the general one). Similarly in these cases there is at most one $\mathbb{A}^1$-fibration $V \to S$ over an affine base up to an isomorphism of S, so any two $\mathbb{C}_+$-actions define the same $\mathbb{A}^1$-fibration. However uniqueness fails for every affine toric surface, which admits a sequence of pairwise non-conjugate $\mathbb{C}^*$-actions.

Another important class of counterexamples is provided by the *Danilov-Gizatullin surfaces*. By definition such a surface V is the complement of an ample section, say S, in a Hirzebruch surface Σ_n. A surprising theorem established in [DaGi][1] says that the isomorphism type of $V = V_{k+1} = \Sigma_n \setminus S$ depends only on $k = S^2 - 1$ and neither on n nor on S. Answering our question, Peter Russell observed that the Danilov-Gizatullin theorem actually provides k pairwise non-conjugate $\mathbb{C}^*$-actions on V_{k+1}. We reproved in [FKZ$_2$, 5.3] this result showing moreover that these k $\mathbb{C}^*$-actions exhaust all $\mathbb{C}^*$-actions on V_{k+1} up to conjugation. At least half of them stay non-conjugate up to inversion in $\mathbb{C}^*$. Moreover by [FKZ$_2$, 5.16] in this case there are at least $\lfloor \frac{k+1}{2} \rfloor$ different conjugacy classes of $\mathbb{A}^1$-fibrations.

Let us recall that every Gizatullin surface $V \not\cong \mathbb{A}^1 \times \mathbb{C}^*$ can be completed by a *standard* zigzag

$$\begin{array}{ccccccc} C_0 & & C_1 & & C_2 & & & & C_n \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ \\ 0 & & 0 & & w_2 & & & & w_n \end{array}, \tag{1}$$

with $w_i = C_i^2 \le -2 \ \forall i \ge 2$. Although this completion is not unique the sequence of weights $(w_2, \ldots, w_n)$ is up to reversion an invariant of V [Gi], cf. also [Du, FKZ$_2$].

[1] See [CNR, Corollary 4.8] for an alternative approach.

The linear system $|C_0|$ provides a $\mathbb{P}^1$-fibration $\Phi_0 : \bar{V} \to \mathbb{P}^1$, which restricts to an $\mathbb{A}^1$-fibration $\Phi_0 : V \to \mathbb{A}^1$ (similarly, reversing the zigzag gives a second $\mathbb{A}^1$-fibration $\Phi_0^\vee : V \to \mathbb{A}^1$). This $\mathbb{P}^1$-fibration lifts to the minimal resolution of singularities $\tilde{V}$ of $\bar{V}$. Our results are formulated in terms of the so called *extended boundary divisor*

$$D_{\text{ext}} := C_0 + C_1 + \tilde{\Phi}_0^{-1}(0) \subseteq \tilde{V}$$

considered in [Gi, Du, FKZ$_2$], where $\tilde{\Phi}_0$ is the induced fibration. Its structure is well known, see Proposition 1.11. We introduce *rigid* and *distinguished* extended divisors that are characterized by their weighted dual graph, see 1.20 and 2.13 for details. The main result of the paper (see Theorem 5.2) can be stated as follows.

Theorem 0.1. *Let V be a Gizatullin surface whose extended divisor D_{ext} is distinguished and rigid. Then Φ_0 and $\Phi_0^\vee$ are up to conjugation the only $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$.*

In the special case of surfaces $xy = p(z)$ in $\mathbb{A}^3$, this result was obtained in terms of locally nilpotent derivations by Daigle [Dai] and Makar-Limanov [ML$_2$]. A similar uniqueness result was obtained by Daigle and Russell [DR] for normal affine Gizatullin surfaces under the assumption that the divisor class group[2] is finite.

Our approach has important applications to the classification of $\mathbb{C}^*$-actions on V. In [FKZ$_2$] we conjectured that among smooth affine $\mathbb{C}^*$-surfaces, the toric surfaces and the Danilov-Gizatullin surfaces are the only exceptions to uniqueness of a $\mathbb{C}^*$-action. In Theorem 0.2 below we confirm this conjecture in the particular case of Gizatullin surfaces with a rigid extended divisor. Recall [FlZa$_1$] that every normal affine surface V with a hyperbolic $\mathbb{C}^*$-action admits a *DPD presentation* $V = \operatorname{Spec} A_0[D_+, D_-]$, where D_+, D_- are two $\mathbb{Q}$-divisors on the smooth affine curve $C = \operatorname{Spec} A_0$ with $D_+ + D_- \le 0$, and A_0 is the ring of invariants; see Section 3.1 for details. For a Gizatullin $\mathbb{C}^*$-surface V one has [FlZa$_2$]: $A_0 = \mathbb{C}[t]$, and each of the fractional parts $\{D_\pm\} = D_\pm - \lfloor D_\pm \rfloor$ is concentrated on at most one point $\{p_\pm\}$. To formulate our second main result we consider the following 3 conditions on the pair (D_+, D_-).

(α_+) $\operatorname{supp}\{D_+\} \cup \operatorname{supp}\{D_-\}$ is empty or consists of one point, say, p satisfying either $D_+(p) + D_-(p) = 0$ or

$$D_+(p) + D_-(p) \le -\max\left(\frac{1}{m^{+2}}, \frac{1}{m^{-2}}\right),$$

where $\pm m^\pm$ is the minimal positive integer such that $m^\pm D_\pm(p) \in \mathbb{Z}$.

(α_*) $\operatorname{supp}\{D_+\} \cup \operatorname{supp}\{D_-\}$ is empty or consists of one point p, where $D_+(p) + D_-(p) \le -1$ or both fractional parts $\{D_+(p)\}$, $\{D_-(p)\}$ are nonzero.

(β) $\operatorname{supp}\{D_+\} = \{p_+\}$ and $\operatorname{supp}\{D_-\} = \{p_-\}$ for two different points p_+, p_-, where $D_+(p_+) + D_-(p_+) \le -1$ and $D_+(p_-) + D_-(p_-) \le -1$.

Theorem 0.2. *For a non-toric normal Gizatullin $\mathbb{C}^*$-surface $V = \operatorname{Spec}\mathbb{C}[t][D_+, D_-]$ the following hold.*

1. If (α_*) or (β) is fulfilled then the $\mathbb{C}^*$-action on V is unique up to conjugation in the automorphism group of V and up to inversion $\lambda \mapsto \lambda^{-1}$ in $\mathbb{C}^*$. Moreover the

[2] This is just the Picard group of the smooth part.

given $\mathbb{C}^*$*-action is conjugate to its inverse if and only if for a suitable automorphism* $\psi \in \mathrm{Aut}(\mathbb{A}^1)$

$$\psi^*(D_+) - D_- \text{ is integral and } \quad \psi^*(D_+ + D_-) = D_+ + D_- . \tag{2}$$

2. If (α_+) *or* (β) *holds then up to conjugation there are at most two conjugacy classes of* $\mathbb{A}^1$*-fibrations* $V \to \mathbb{A}^1$*. There is only one such conjugacy class if and only if (2) is fulfilled for some* $\psi \in \mathrm{Aut}(\mathbb{A}^1)$*.*

We notice that for smooth non-toric Gizatullin $\mathbb{C}^*$-surfaces this proves uniqueness of $\mathbb{C}^*$-actions up to conjugation and inversion unless the weights w_i in the boundary zigzag (1) satisfy $w_i = -2 \ \forall i \neq s$ for some s in the range $2 \le s \le n$. In a forthcoming paper we will show that in the latter case, except for the Danilov-Gizatullin surfaces, there always exists a deformation family of pairwise non-conjugate $\mathbb{C}^*$-actions on V. Consequently, for smooth Gizatullin $\mathbb{C}^*$-surfaces the sufficient conditions in Theorem 0.2 are also necessary for the uniqueness of a $\mathbb{C}^*$-action.

Let us survey the content of the various sections. First we review some standard facts on Gizatullin surfaces in Section 1.1 and describe their extended divisors, see Section 1.2. After some preparations in 1.3 we treat in Section 1.4 families of completions of a given Gizatullin surface by zigzags. The main result here is the triviality criterion 1.21, which provides one of the basic tools in the proof of Theorem 0.1. In Section 2 we study possible degenerations of extended divisors in such families. Important is Theorem 2.17, where we give a criterion for when the extended divisor is rigid, i.e. stays constant in a family.

In Section 3 we translate these conditions into the language of DPD presentations. First we recall the description of standard equivariant completions of Gizatullin $\mathbb{C}^*$-surfaces in terms of a DPD presentation according to [FKZ$_2$]. In Theorem 3.24 we give the required criterion for the extended divisor D_{ext} to be distinguished and rigid.

One of our main technical tools is the so called *reconstruction space*. Roughly speaking, this space forms a moduli space for the completions of a given normal surface. In Section 4 we show that this moduli space exists and is isomorphic to an affine space, see Corollary 4.10. This provides a basic ingredient in the proofs of Theorems 0.1 and 0.2 in the final Section 5.

1. Gizatullin surfaces

1.1. **Standard completions of Gizatullin surfaces.**

Let us recall the notion of a standard zigzag [FKZ$_1$].

1.1. Let X be a complete normal algebraic surface. By a *zigzag* on X we mean an SNC divisor[3] D with rational components contained in the smooth part X_{reg}, which has a linear dual graph

$$\Gamma_D : \quad \underset{w_0}{\overset{C_0}{\circ}} \!-\!\!-\!\!-\! \underset{w_1}{\overset{C_1}{\circ}} \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \underset{w_n}{\overset{C_n}{\circ}} \ , \tag{3}$$

where $w_0, \ldots, w_n$ are the weights of Γ_D. We abbreviate this chain by $[[w_0, \ldots, w_n]]$. We also write $[[\ldots, (w)_k, \ldots]]$ if a weight w occurs at k consecutive places. Note that

[3] I.e. a simple normal crossing divisor.

the intersection matrix of a zigzag has at most one positive eigenvalue by the Hodge index theorem. We recall the following notion.

Definition 1.2. ([FKZ$_1$, Definition 2.13 and Lemma 2.17]) A zigzag D is called *standard* if its dual graph Γ_D is one of

$$(4)\quad [[0]]\,,\quad [[0,0]]\,,\quad [[0,0,0]]\quad \text{or}\quad [[0,0,w_2,\ldots,w_n]],\quad \text{where}\quad n\geq 2,\ w_j\leq -2\ \forall j.$$

A linear chain Γ is said to be *semistandard* if it is either standard or one of

$$(5)\qquad [[0,w_1,w_2,\ldots,w_n]],\quad [[0,m,0]]\quad \text{where}\quad m\in\mathbb{Z},\ n\geq 1,\ w_j\leq -2\ \forall j.$$

We note that a standard zigzag $[[0,0,w_2,\ldots,w_n]]$ is unique in its birational class up to reversion

$$(6)\qquad [[0,0,w_2,\ldots,w_n]]\rightsquigarrow[[0,0,w_n,\ldots,w_2]]\,,$$

see Corollary 3.33 in [FKZ$_1$]. A zigzag is called *symmetric* if it coincides with its reversed zigzag.

By definition a *Gizatullin surface* is a normal affine surface V which admits a completion $(\bar V, D)$ with a zigzag D. Such a completion is called *(semi)standard* if D has this property. We need the following facts.

Lemma 1.3. *For a Gizatullin surface V the following hold.*

(a) ([DaGi, Du, FKZ$_1$, Corollary 3.36]*) V admits a standard completion $(\bar V, D)$.*

(b) ([FKZ$_2$, Theorem 2.9(b)]*) If a torus $\mathbb{T}=(\mathbb{C}^*)^m$ acts on V then V admits an equivariant standard completion, which is unique up to reversing the boundary zigzag.*

(c) ([FKZ$_2$, Theorem 2.9(a) and Remark 2.10(1)]*) If $\mathbb{C}_+$ acts on V then V admits an equivariant semistandard completion.*

1.4. The reversion of a zigzag, regarded as a birational transformation of the weighted dual graph, admits the following factorization [FKZ$_1$]. Given $[[0,0,w_2,\ldots,w_n]]$ we can successively move the pair of zeros to the right

$$[[0,0,w_2,\ldots,w_n]]\rightsquigarrow[[w_2,0,0,w_3,\ldots,w_n]]\rightsquigarrow\ldots\rightsquigarrow[[w_2,\ldots,w_n,0,0]]$$

by a sequence of *inner elementary transformations*[4], see Example 2.11(2) in [FKZ$_1$]. The corresponding birational transformation $[[0,0,w_2,\ldots,w_n]]\rightsquigarrow[[w_2,\ldots,w_n,0,0]]$ is non-trivial unless our standard graph is one of $[[0]]$, $[[0,0]]$ or $[[0,0,0]]$.

If $(\bar V, D)$ is a standard completion of a Gizatullin surface V, then reversing the zigzag D by a sequence of inner elementary transformations as explained above we obtain from $(\bar V, D)$ a new completion $(\bar V^\vee, D^\vee)$, which we call the *reverse standard completion.* It is uniquely determined by $(\bar V, D)$. Note that even in the case where the zigzag D is symmetric with dual graph $\neq[[0]]$, $[[0,0]]$, $[[0,0,0]]$, this reverse completion $(\bar V^\vee, D^\vee)$ is not isomorphic to $(\bar V, D)$ under an isomorphism fixing pointwise the affine part V.

[4]By an inner elementary transformation of a weighted graph we mean blowing up at an edge incident to a 0-vertex of degree 2 and blowing down the image of this vertex. We recall that the degree of a vertex in a simple graph is the number of its incident edges.

1.2. **Extended divisors of Gizatullin surfaces.**

1.5. Let V be a Gizatullin surface and $(\bar V, D)$ be a completion of V by a standard zigzag $[[0,0,w_2,\ldots,w_n]]$ with $n\ge 2$ and $w_i\le -2\ \forall i$. We write

$$D = C_0 + \ldots + C_n\,,$$

where the irreducible components C_i are enumerated as in (3). We consider the minimal resolutions of singularities V', $(\tilde V, D)$ of V and $(\bar V, D)$, respectively.

Since $C_0^2 = C_1^2 = 0$, the linear systems $|C_0|$ and $|C_1|$ define a morphism $\Phi=\Phi_0\times\Phi_1 : \tilde V\to \mathbb{P}^1\times\mathbb{P}^1$ with $\Phi_i = \Phi_{|C_i|}$, $i=0,1$. We call it the *standard morphism* associated to the standard completion $(\bar V, D)$ of V. Similarly Φ_0 is referred to as the *standard* $\mathbb{P}^1$*-fibration* of $(\tilde V, D)$.

We note that C_1 is a section of Φ_0 and so the restriction $\Phi_0|V' : V'\to\mathbb{P}^1$ is an $\mathbb{A}^1$-fibration. We can choose the coordinates on $\mathbb{P}^1=\mathbb{A}^1\cup\{\infty\}$ in such a way that

$$C_0=\Phi_0^{-1}(\infty)\,,\qquad \Phi(C_1)=\mathbb{P}^1\times\{\infty\}\quad\text{and}\quad C_2\cup\ldots\cup C_n\subseteq\Phi_0^{-1}(0)\,.$$

The standard morphism Φ contracts the curves C_i for $i\ge 3$ and does not contract C_0, C_1, C_2. By abuse of notation we denote the images of C_0, C_1, C_2 in $\mathbb{P}^1\times\mathbb{P}^1$ by the same letters. The divisor $D_{\text{ext}} = C_0\cup C_1\cup\Phi_0^{-1}(0)$ on $\tilde V$ is called the *extended divisor.*

Remark 1.6. 1. The dual graph of D_{ext} is linear if and only if V is toric [FKZ$_2$, Lemma 2.20].

2. If V carries a $\mathbb{C}^*$-action then we can find an equivariant standard completion $(\bar V, D)$, see Lemma 1.3(b). Since the minimal resolution of singularities is also equivariant, so are $(\tilde V, D)$ and Φ with respect to a suitable $\mathbb{C}^*$-action on $\mathbb{P}^1\times\mathbb{P}^1$, and the divisor D_{ext} is invariant under the $\mathbb{C}^*$-action on $\tilde V$. For $\mathbb{C}^*$-surfaces this divisor was studied systematically in [FKZ$_2$].

3. The morphism $\Phi=\Phi_0\times\Phi_1$ contracts $C_3\cup\ldots\cup C_n$. According to Lemma 2.19 in [FKZ$_2$] it also contracts all exceptional curves of the resolution $V'\to V$, whence descends to a morphism $\bar\Phi=\bar\Phi_0\times\bar\Phi_1 : \bar V\to\mathbb{P}^1\times\mathbb{P}^1$. We also call Φ the standard morphism of $(\bar V, D)$ and $\bar\Phi_0$ the standard $\mathbb{P}^1$-fibration.

We recall the following fact, see [FKZ$_2$, Lemma 2.19].

Lemma 1.7. *With the notation as in 1.5, Φ is birational and induces an isomorphism $\tilde V\backslash\Phi_0^{-1}(0)\cong(\mathbb{P}^1\backslash\{0\})\times\mathbb{P}^1$. In particular, $D_{(e)} := \Phi_0^{-1}(0)$ is the only possible degenerate fiber of the $\mathbb{P}^1$-fibration $\Phi_0 : \tilde V\to\mathbb{P}^1$.*

To exhibit the structure of this extended divisor let us recall some notation from [FKZ$_2$].

1.8. For a primitive dth root of unity ζ and $0\le e<d$ with $\gcd(e,d)=1$ [5] the cyclic group $\mathbb{Z}_d=\langle\zeta\rangle$ acts on $\mathbb{A}^2$ via $\zeta.(x,y)=(\zeta x,\zeta^e y)$. The quotient $V_{d,e}=\mathbb{A}^2/\!/\mathbb{Z}_d$ is a normal affine toric surface. Moreover, any such surface different from[6] $\mathbb{A}^1_*\times\mathbb{A}^1_*$ and $\mathbb{A}^1_*\times\mathbb{A}^1$ arises in this way. Singularities analytically isomorphic to the singular point of $V_{d,e}$ are called cyclic quotient singularities of type (d,e).

[5] In the case $d=1$ this forces $(d,e)=(1,0)$.

[6] Hereafter $\mathbb{A}^1_*=\mathbb{A}^1\setminus\{0\}$.

1.9. We abbreviate by a box $\square$ with rational weight e/m, where $0 < e < m$ and $\gcd(m, e) = 1$, the weighted linear graph

$$\begin{array}{c} C_1 \\ \circ\!\!-\!\!\!-\!\!\!-\!\!\!- \\ -k_1 \end{array} \cdots \begin{array}{c} C_n \\ -\!\!\!-\!\!\!-\!\!\!-\!\!\circ \\ -k_n \end{array} \quad = \quad \begin{array}{c} e/m \\ \square \end{array} \tag{7}$$

with $k_1, \dots, k_n \geq 2$, where

$$m/e = [k_1, \dots, k_n] = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_n}}} \quad .$$

A chain of rational curves (C_i) on a smooth surface with dual graph (7) contracts to a cyclic quotient singularity of type (m, e) [Hi]. It is convenient to introduce the weighted box $\overset{0}{\square}$ for the empty chain. Given extra curves E, F we also abbreviate

$$\overset{E}{\circ}\!-\!\!-\!\!-\!\overset{C_1}{\circ}\!-\!\!-\!\!- \cdots -\!\!-\!\!-\!\overset{C_n}{\circ} \quad = \quad \overset{E}{\circ}\!-\!\!-\!\!-\!\!-\!\overset{e/m}{\square} \quad = \quad \overset{(e/m)^*}{\square}\!-\!\!-\!\!-\!\!-\!\overset{E}{\circ} \tag{8}$$

and

$$\overset{C_1}{\circ}\!-\!\!-\!\!- \cdots -\!\!-\!\!-\!\overset{C_n}{\circ}\!-\!\!-\!\!-\!\overset{F}{\circ} \quad = \quad \overset{e/m}{\square}\!-\!\!-\!\!-\!\!-\!\overset{F}{\circ} \quad = \quad \overset{F}{\circ}\!-\!\!-\!\!-\!\!-\!\overset{(e/m)^*}{\square} \; . \tag{9}$$

The orientation of the chain of curves $(C_i)_i$ in (7) plays an important role. Indeed $[k_n, \dots, k_1] = m/e'$, where $0 < e' < m$, $ee' \equiv 1 \pmod m$, and the box $\square$ marked with $(e/m)^* := e'/m$ corresponds to the reversed chain in (7), see e.g., [Ru]. The chain $[[(-2)_m]]$ will be abbreviated by $\overset{A_m}{\square}$.

Definition 1.10. A *feather* $\mathfrak{F}$ is a linear chain of smooth rational curves with dual graph

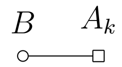
$$\mathfrak{F}: \qquad \overset{B}{\circ}\!-\!\!-\!\!-\!\!-\!\overset{e/m}{\square} \; , \tag{10}$$

where B has self-intersection ≤ -1 and e, m are as before, cf. (8). Note that the box does not contain a (-1)-curve; it can also be empty. The curve B will be called the *bridge curve.*

A collection of feathers $\{\mathfrak{F}_\rho\}$ consists of feathers $\mathfrak{F}_\rho$, $\rho = 1, \dots, r$, which are pairwise disjoint. Such a collection will be denoted by a plus box $\boxplus$. We say that a collection $\{\mathfrak{F}_\rho\}$ is attached to a curve C_i in a chain (3) if the bridge curves B_ρ meet C_i in pairwise distinct points and all the feathers $\mathfrak{F}_\rho$ are disjoint with the curves C_j for $j \neq i$. In a diagram we write in brief

$$\overset{C_i}{\circ}\!-\!\!-\!\!-\!\!-\!\overset{\{\mathfrak{F}_\rho\}}{\boxplus} \quad \text{or, in the case of a single feather,} \quad \overset{C_i}{\circ}\!-\!\!-\!\!-\!\!-\!\overset{\mathfrak{F}}{\square} \; .$$

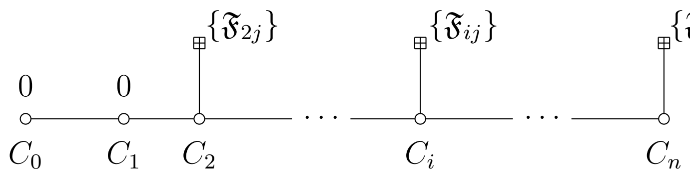
We often draw this diagram vertically, with the same meaning.

An A_k*-feather* 

represents the contractible[7] linear chain $[[-1,(-2)_k]]$. Thus the A_0-feather represents a single (-1)-curve B, while the box is empty.

Let us further exhibit the structure of the extended divisor of a Gizatullin surface according to [Du].

Proposition 1.11. *Let $(\tilde V, D)$ be a minimal SNC completion of the minimal resolution of singularities of a Gizatullin surface V, where $D = C_0 + \ldots + C_n$ is a zigzag as in (3). Then the extended divisor D_{ext} has dual graph*

(11) D_{ext} : 

,

where $\mathfrak{F}_{ij}$ $(1 \le j \le r_i)$ is a collection of feathers attached to the curve C_i, $i \ge 2$. Moreover the surface $\tilde V$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by a sequence of blowups with centers in the images of the components C_i, $i \ge 2$.

Proof. A proof can be found (using different notation) in [Du]. For the convenience of the reader we provide a short argument. First we note that $D_{(e)} = \Phi_0^{-1}(0) \subseteq \tilde V$ is a tree of rational curves, since it is the blowup of a fiber $C_2 = \{0\} \times \mathbb{P}^1 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathfrak{F}_{ij}$, $j = 1, \ldots, r_i$, be the connected components of $D_{\mathrm{ext}} \ominus C_i$[8] that do not contain components of D. Every such connected component contains a unique curve B_{ij}, which meets C_i. The divisor $R_{ij} = \mathfrak{F}_{ij} \ominus B_{ij}$ is then disjoint from D. Since V is affine, R_{ij} contracts to a point in V. Hence it is the exceptional divisor of a minimal resolution of a singular point of V, and so its dual graph contains no (-1)-vertex of degree ≤ 2. On the other hand, the divisor $D_{(e)}$ contracts to C_2. We claim that the dual graph of $\mathfrak{F}_{ij}$ contains no branch point, and its 'bridge vertex' B_{ij} is the only possible (-1)-vertex in $\mathfrak{F}_{ij}$. Let us check this claim by induction on the number of blowdowns in the contraction of $D_{(e)}$ to C_2, or rather of blowups when growing $D_{(e)}$ starting from C_2. Projecting to a surface which appears on some intermediate step of this blowup process, let us assume that

- the image, say D', of the chain $D = C_0 + \ldots + C_n$ is again a linear chain,
- the image $\mathfrak{F}'_{ij}$ of $\mathfrak{F}_{ij}$ is either empty or a connected component of $D'_{\mathrm{ext}} \ominus D'$, where D'_{ext} is the image of D_{ext}. Moreover
- if $\mathfrak{F}'_{ij} \ne \emptyset$ then it contains just one neighbor say B'_{ij} of D' in D'_{ext},
- $R'_{ij} := \mathfrak{F}'_{ij} \ominus B'_{ij}$ is either empty or a minimal resolution of a singular point with a linear dual graph,
- B'_{ij} is a vertex of degree ≤ 2 in the dual graph of D'_{ext}, and the only possible (-1)-curve in $\mathfrak{F}'_{ij}$.

The next blowup must be done at a point of D', which is either a smooth point of D'_{ext}, or a double point of D'_{ext}. Indeed otherwise it would be done at a point of

[7] A graph is contractible if it can be reduced to the graph with a single vertex $[[-1]]$ by a succession of contractions of (-1)-vertices of degree ≤ 2.

[8] For a reduced divisor D on a surface X and an irreducible component C of D, $D_{\mathrm{ext}} \ominus C$ stands for the divisor $D - C$ viewed as a curve on X.

$\mathfrak{F}'_{ij} \ominus D'$, and then clearly R_{ij} cannot be minimal i.e., it would contain a (-1)-curve, which is impossible. Thus all the properties mentioned above are preserved under this blowup.

This implies that the feather $\mathfrak{F}_{ij}$ is a linear chain of the form

$$\mathfrak{F}_{ij}: \quad \overset{B_{ij}}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{R_{ij}}{\square}\,,$$

which yields the desired form of D_{ext}, and also the last assertion. □

Remark 1.12. The collection of linear chains R_{ij} corresponds to the minimal resolution of singularities of V. So V has at most cyclic quotient singularities, cf. [Miy, Ch. 3, Lemma 1.4.4(1)]. Moreover V is smooth if and only if the collection R_{ij} is empty, if and only if every feather $\mathfrak{F}_{ij}$ reduces to a single bridge curve B_{ij}.

1.3. **Simultaneous contractions.** The following lemma is a standard fact in surface theory.

Lemma 1.13. *For a smooth rational surface X and a smooth rational curve C on X with $C^2 = 0$, we have*

$$H^0(X, \mathcal{O}_X(C)) \cong \mathbb{C}^2\,, \qquad H^i(X, \mathcal{O}_X(C)) = 0 \text{ for } i \geq 1\,.$$

Moreover the linear system $|C|$ is base point free and defines a $\mathbb{P}^1$-fibration $\Phi_{|C|} : X \to \mathbb{P}^1$.

A relative version of this result is as follows.

Lemma 1.14. *Let $f : \mathcal{X} \to S$ be a smooth family of rational surfaces over a quasi-projective scheme S with $\mathrm{Pic}(S) = 0$, and let $\mathcal{C}$ be an S-flat divisor in $\mathcal{X}$ such that the fibers $\mathcal{C}_s := f^{-1}(s) \cap \mathcal{C}$ are smooth rational curves of self-intersection 0 in $\mathcal{X}_s := f^{-1}(s)$. Suppose that $R \subset \mathcal{X}$ is a section of f disjoint from $\mathcal{C}$. Then there exists a morphism $\varphi : \mathcal{X} \to \mathbb{P}^1$ such that $\varphi^*(\infty) = \mathcal{C}$ and $\varphi(R) = 0$.*

Proof. In lack of a reference we provide a short proof. Since for every $s \in S$ the curve $\mathcal{C}_s$ has self-intersection 0 in $\mathcal{X}_s$, the cohomology groups $H^i(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}(\mathcal{C}_s))$ vanish for $i \geq 1$. Thus for every coherent sheaf $\mathcal{N}$ on S the higher direct image sheaves $R^i f_*(\mathcal{O}_X(\mathcal{C}) \otimes_{\mathcal{O}_S} \mathcal{N})$ vanish for $i \geq 1$, see e.g. [Ha, 12.10]. Thus $\mathcal{E} = f_*(\mathcal{O}_{\mathcal{X}}(\mathcal{C}))$ is a locally free sheaf of rank 2 on S, and forming $R^0 f_*(\mathcal{O}_{\mathcal{X}}(\mathcal{C}))$ is compatible with restriction to the fiber, i.e. the canonical map

$$\mathcal{E}/\mathfrak{m}_s\mathcal{E} \longrightarrow H^0(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}(\mathcal{C}_s))$$

is bijective, where $\mathfrak{m}_s$ denotes the ideal sheaf of the point $s \in S$ (see [Ha, 12.10 and 3.11]). The inclusion $\mathcal{O}_{\mathcal{X}} \subseteq \mathcal{O}_{\mathcal{X}}(\mathcal{C})$ induces a trivial subbundle $\mathcal{O}_S$ of $\mathcal{E}$ (indeed this is true in each fiber). Since the section R is disjoint from $\mathcal{C}$, the projection $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_R$ extends to a map $\mathcal{O}_{\mathcal{X}}(\mathcal{C}) \to \mathcal{O}_R$. Taking f_* gives a morphism $\mathcal{E} \to f_*(\mathcal{O}_R) \cong \mathcal{O}_S$ which restricts to the identity on $\mathcal{O}_S \subseteq \mathcal{E}$. Thus $\mathcal{E} \cong \mathcal{O}_S \oplus \mathcal{L}$ for some line bundle $\mathcal{L}$ on S. The latter bundle is trivial due to our assumption that $\mathrm{Pic}(S) = 0$. If now σ_0 and σ_1 are sections of $\mathcal{E}$ which correspond to the standard basis of $\mathcal{E} \cong \mathcal{O}_S \oplus \mathcal{O}_S$ then the morphism $[\sigma_0 : \sigma_1] : \mathcal{X} \to \mathbb{P}^1$ has the desired properties. □

The following relative version of Castelnouvo's contractibility criterion is well known[9].

[9] Cf. e.g., [KaZa, Theorem 1.3].

Lemma 1.15. *Let $f : \mathcal{X} \to S$ be a proper smooth family of surfaces and let $\mathcal{C}$ be an S-flat divisor in $\mathcal{X}$ such that the fibers $\mathcal{C}_s := f^{-1}(s) \cap \mathcal{C}$ are smooth rational curves with self-intersection -1 in $\mathcal{X}_s := f^{-1}(s)$. Then there exists a contraction $\pi : \mathcal{X} \to \mathcal{X}'$ of $\mathcal{C}$, and $\mathcal{X}'$ is again flat over S.*

Proof. It is sufficient to treat the case where the base S is affine. In this case there exists an f-ample divisor $\mathcal{D}$ on $\mathcal{X}$ which defines an embedding $\mathcal{X} \hookrightarrow S \times \mathbb{P}^N$ for some N. Then the sheaf $\mathcal{O}_{\mathcal{X}}(\mathcal{D} + k\mathcal{C})$, where $k := \mathcal{D}.\mathcal{C}$, is f-semiample on $\mathcal{X}$ and provides a desired contraction, since this is true in every fiber. □

Lemma 1.16. *Let S be a scheme with $H^1(S, \mathcal{O}_S) = 0$ and $\mathrm{Pic}(S) = 0$. If $f : \mathcal{X} \to S$ is a flat morphism with a section $\sigma : S \to \mathcal{X}$ such that every fiber is isomorphic to $\mathbb{P}^1$, then $\mathcal{X}$ is S-isomorphic to the product $\mathbb{P}^1 \times S$ such that $\sigma(S)$ corresponds to $\{p\} \times S$ for some point $p \in \mathbb{P}^1$.*

Proof. We note first that $R^0 f_*(\mathcal{O}_{\mathcal{X}}) \cong \mathcal{O}_S$ and $R^1 f_*(\mathcal{O}_{\mathcal{X}}) = 0$ since the fibers are isomorphic to $\mathbb{P}^1$. Using the spectral sequence $H^p(S, R^q f_*(\mathcal{O}_{\mathcal{X}})) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and our assumption $H^1(S, \mathcal{O}_S) = 0$ this implies that $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. Letting $\Sigma = \sigma(S)$ we consider the f-ample sheaf $\mathcal{O}_{\mathcal{X}}(\Sigma)$. Its direct image sheaf $\mathcal{E} = f_*(\mathcal{O}_{\mathcal{X}}(\Sigma))$ is locally free of rank 2 and $\mathcal{X} \cong \mathbb{P}(\mathcal{E})$. The sheaf $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\Sigma) \otimes \mathcal{O}_\Sigma$ is a line bundle on $\Sigma \cong S$ and so is trivial, since $\mathrm{Pic}(S) = 0$ by our assumption. Taking the direct image f_* of the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}(\Sigma) \to \mathcal{L} \cong \mathcal{O}_S \to 0$$

yields an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{O}_S \to R^1 f_*(\mathcal{O}_{\mathcal{X}}) = 0\,.$$

Thus $\mathcal{E}$ is an extension of $\mathcal{O}_S$ by $\mathcal{O}_S$ and so can be considered as an element of $\mathrm{Ext}^1_S(\mathcal{O}_S, \mathcal{O}_S) \cong H^1(S, \mathcal{O}_S)$. Since by our assumption the latter group vanishes, this extension splits, i.e., $\mathcal{E} \cong \mathcal{O}_S^2$. Hence $\mathcal{X} \cong \mathbb{P}(\mathcal{E}) = \mathbb{P}^1 \times S$, where by our construction Σ corresponds to $\{p\} \times S$ for some point $p \in \mathbb{P}^1$. □

The following corollary of Lemma 1.16 is well known; the proof is immediate.

Corollary 1.17. *Assume that S as in 1.16 above does not admit non-constant invertible regular functions. Let $\mathcal{C} \to S$ be a flat family of smooth rational curves with a non-empty S-flat subfamily $\mathcal{Z} \subseteq \mathcal{C}$ of reduced effective divisors*[10]. *Then the family $(\mathcal{C}, \mathcal{Z}) \to S$ is trivial i.e., there is an S-isomorphism $h : \mathcal{C} \to \mathbb{P}^1 \times S$ with $h(\mathcal{Z}) = \{P_1, \ldots, P_r\} \times S$, where $P_1, \ldots, P_r$ are points of $\mathbb{P}^1$.*

1.4. **Families of completions of a Gizatullin surface.** In this section we study families of completions of a given Gizatullin surface V. We introduce the notion of a distinguished extended divisor. In Proposition 1.21 we show that any deformation family of completions of a Gizatullin surface over a sufficiently large base is necessarily trivial provided that the extended divisor is distinguished and its dual graph stays constant along the deformation.

1.18. We start with the trivial family $f : \mathcal{V} = V \times S \to S$, where S is a quasiprojective scheme with $\mathrm{Pic}(S) = 0$. We let $(\bar{\mathcal{V}}, \mathcal{D}) \to S$ be a family of completions of $\mathcal{V}$ by a family of standard SNC-divisors $\mathcal{D} = \bigcup_{i=0}^n \mathcal{C}_i$ over S with a fixed dual graph. In other

[10] I.e., a disjoint union of images of several sections $S \to \mathcal{C}$.

words, $\bar{\mathcal{V}} \to S$ is a flat family of complete normal surfaces, $\mathcal{D} \to S$ is a flat subfamily of divisors and for every i, $f : \mathcal{C}_i \to S$ is a flat family of smooth rational curves which form in every fiber a fixed standard zigzag (3). In particular for $i = 0, \ldots, n-1$, $\mathcal{C}_i \cap \mathcal{C}_{i+1}$ are disjoint sections of f.

Since on the affine part our family is trivial, there is a simultaneous minimal resolution of singularities $h : \tilde{\mathcal{V}} \to \bar{\mathcal{V}}$. This means that $\tilde{\mathcal{V}} \to S$ is a smooth family of complex surfaces, which is fiberwise the minimal resolution of singularities of $\bar{\mathcal{V}}$. Clearly $h^{-1}(\mathcal{V}) \cong V' \times S$, where $V' \to V$ is the minimal resolution.

According to 1.14 the components $\mathcal{C}_i$, $i = 0, 1$, define morphisms $\Phi_i = \Phi_{|\mathcal{C}_i|} : \tilde{\mathcal{V}} \to \mathbb{P}^1$ with

$$\Phi_0^{-1}(\infty) = \mathcal{C}_0, \quad \Phi_1^{-1}(\infty) = \mathcal{C}_1 \quad \text{and} \quad \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_n \subseteq \Phi_0^{-1}(0)\,.$$

As in the absolute case, we consider the family of divisors $\mathcal{D}_{(e)} := \Phi_0^{-1}(0)$ and the extended divisor $\mathcal{D}_{\mathrm{ext}} := \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{D}_{(e)}$.

It is convenient to introduce the following subgraphs of the extended divisor D_{ext} as in (11).

1.19. For every $1 \le i \le n$ we let $D_{\mathrm{ext}}^{>i}$ denote the union of all connected components of $D_{\mathrm{ext}} \ominus C_i$ which do not contain C_0. Similarly we let $D_{\mathrm{ext}}^{\ge i}$ be the connected component of $D_{\mathrm{ext}} \ominus C_{i-1}$ that contains C_i.

Obviously, $D_{\mathrm{ext}}^{>i}$ is non-empty for every $1 \le i \le n-1$, while $D_{\mathrm{ext}}^{>n}$ may be empty depending on whether the feather collection $\{\mathfrak{F}_{nj}\}$ in (11) is empty or not.

Definition 1.20. The extended divisor D_{ext} will be called *distinguished* if there is no index i with $3 \le i \le n$ such that $D_{\mathrm{ext}}^{>i}$ is non-empty and contractible.

Proposition 1.21. *Let V be a Gizatullin surface and let $(\bar{\mathcal{V}}, \mathcal{D})$ be a family of standard completions of V over $S = \mathbb{A}^m$ as in 1.18 with a minimal resolution of singularities $(\tilde{\mathcal{V}}, \mathcal{D})$ and extended divisor $\mathcal{D}_{\mathrm{ext}}$. Suppose that at every point $s \in S$ the divisor $\mathcal{D}_{\mathrm{ext},s}$ is distinguished and its dual graph does not depend on $s \in S$. Then the family $(\bar{\mathcal{V}}, \mathcal{D})$ is trivial i.e., there is an isomorphism*[11] *$(\bar{\mathcal{V}}, \mathcal{D}) \cong (\bar{V}, D) \times \mathbb{A}^m$ compatible with the projection to $\mathbb{A}^m$, where $\bar{V} = \bar{\mathcal{V}}_s$ and $D = \mathcal{D}_s$ are the fibers over a point $s \in \mathbb{A}^m$.*

Proof. In the case where D is one of the zigzags $[[0,0]]$ or $[[0,0,0]]$ the map $\Phi := \Phi_0 \times \Phi_1$ (see 1.18) is an isomorphism and the claim is trivial. Otherwise, since the dual graph $\mathcal{D}_{\mathrm{ext},s}$ at each point $s \in S$ is the same, we can find a smooth family of (-1)-curves $\mathcal{E}$ in $\mathcal{D}_{(e)}$. By Lemma 1.15 we can contract $\mathcal{E}$ simultaneously, which results again in a flat family of surfaces together with an induced map to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^m$. Continuing in this way we get a sequence of blowdowns

$$\pi : \tilde{\mathcal{V}} = \mathcal{X}_k \to \mathcal{X}_{k-1} \to \ldots \to \mathcal{X}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^m, \tag{12}$$

where at every step a family of (-1)-curves is blown down. Reading this sequence in the opposite direction, $\tilde{\mathcal{V}}$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^m$ by a sequence of blowups along sections say $\Sigma_i \subseteq \mathcal{X}_i$. Let us show by induction on i that the family $\mathcal{X}_i$ is trivial, i.e. S-isomorphic to $X_i \times \mathbb{A}^m$ for a suitable blowup X_i of $\mathbb{P}^1 \times \mathbb{P}^1$. This yields the desired conclusion, since the triviality of the family $(\tilde{\mathcal{V}}, \mathcal{D})$ implies that of $(\bar{\mathcal{V}}, \mathcal{D})$.

[11]Note that this isomorphism is *not* the identity on V, in general!

In the case $i = 0$ this is evident. If $i = 1$ then we can adjust the coordinates in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^m$ so that the section Σ_1 is contained in $(0,0) \times \mathbb{A}^m$, see Lemma 1.14. Thus the first blowup in (12) takes place at $(0,0) \times \mathbb{A}^m$ and so $\mathcal{X}_1$ is a trivial family.

Assume by induction that we have an S-isomorphism $\mathcal{X}_i \cong X_i \times \mathbb{A}^m$ for some blowup X_i of $\mathbb{P}^1 \times \mathbb{P}^1$, where $i \geq 1$. We let $\mathcal{E}_0 = C_2 \times \mathbb{A}^m \subseteq \mathcal{X}_0$ and let $\mathcal{E}_j \subseteq \mathcal{X}_j$ indicate the exceptional divisor of the jth blowup. If $\mathcal{E}_j^i$ denotes its proper transform in $\mathcal{X}_i$ for $i \geq j$, then by our assumption the family $\mathcal{E}_j^i \to S$ is trivial and S-isomorphic to $E_j^i \times \mathbb{A}^m$.

If the next blowup is inner with center $\Sigma_i = \mathcal{E}_j^i \cap \mathcal{E}_{j'}^i \cong_S (E_j^i \cap E_{j'}^i) \times \mathbb{A}^m$, then also $\mathcal{X}_{i+1}$ is a trivial family. So assume further that the next blowup is outer with center Σ_i contained in $\mathcal{E}_j^i \cong E_j^i \times \mathbb{A}^m$. The section Σ_i is the graph of a map $\sigma_i : \mathbb{A}^m \to E_j^i$ with image contained in $E_j^i \backslash (D'_{\text{ext}} \ominus E_j^i)$, where as before D'_{ext} denotes the image of D_{ext} in X_i. If E_j^i meets two other components of D'_{ext} then σ_i maps $\mathbb{A}^m$ to $\mathbb{P}^1$ with at least 2 points deleted and so must be constant. Hence $\mathcal{X}_{i+1}$ is again a trivial family.

Finally consider the case where the divisor E_j^i meets just one other component of D'_{ext}. According to Proposition 1.11 all blowups in (12) are done at the images of the zigzag D. Thus E_j^i is the image in X_i, say, C'_l, of some component C_l of D. If E_j^i is an exceptional divisor then $l \geq 3$. By our assumption $E_j^i = C'_l$ is an end component of D'_{ext}, and so the image D' of D in X_i is a linear chain with end components C'_0 and C'_l. Therefore C'_l meets a component C'_j with $j < l$. Consequently the divisor $D_{\text{ext}}^{>l}$ is contracted in X_i, hence it is contractible. Since by our assumption D_{ext} is distinguished and $l \geq 3$, this contradicts Definition 1.20.

In the remaining case $l = 2$, $C_2 = E_0^i$ is an end component of D'_{ext}, so $k = 2$, and no blowup was done so far with center at $C_2 \times \mathbb{A}^m$, so $i = 1$. This returns us to the case considered above. □

In the next Sections 2 and 3 we will show that the condition of constancy of the dual graph of $\mathcal{D}_{\text{ext},s}$ in Proposition 1.21 is satisfied under the assumptions of Theorem 0.2. However, in general this condition does not hold as feathers can jump in families of Gizatullin surfaces. We illustrate this below by the example of Danilov-Gizatullin surfaces. In Section 2 we will provide a more thorough treatment of this phenomenon.

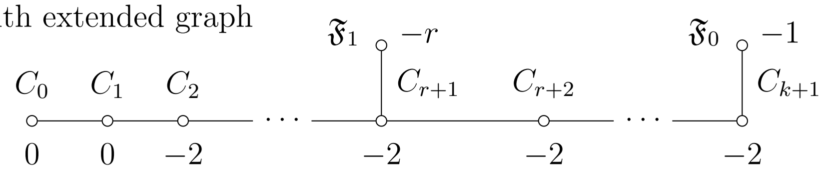
Example 1.22. Recall that a Danilov-Gizatullin surface $V = V_{k+1}$ is the complement of a section say σ in a Hirzebruch surface with self-intersection $\sigma^2 = k+1$. By a theorem of Danilov-Gizatullin [DaGi] the isomorphism class of V_{k+1} depends only on k and not on the choice of σ or of the concrete Hirzebruch surface. This surface V_{k+1} can be completed by the zigzag $[[0, 0, (-2)_k]]$ with components say $C_0, \ldots, C_{k+1}$. According to Proposition 5.14 in [FKZ$_2$], V_{k+1} admits exactly k pairwise non-conjugate $\mathbb{C}^*$-actions. In terms of the DPD presentation (see [FlZa$_1$] or Section 3 below), for a fixed k these $\mathbb{C}^*$-surfaces are given by the pairs of $\mathbb{Q}$-divisors on $C = \mathbb{A}^1$

$$(D_+, D_-) = \left(-\frac{1}{r}[0], -\frac{1}{k+1-r}[1]\right), \qquad r = 1, \ldots, k\,.$$

So any other $\mathbb{C}^*$-action on V_{k+1} is conjugate to one of these.

Given $r \in \{1, \ldots, k\}$ such a $\mathbb{C}^*$-surface $V(r)$ admits an equivariant standard completion $(\bar{V}(r), D)$ with extended graph

(13) $D_{\text{ext}}(r)$:



where the curve C_{r+1} is attractive for the extended $\mathbb{C}^*$-action on $\bar{V}(r)$ (cf. Section 3). Here the bottom line corresponds to the boundary zigzag D, the feather $\mathfrak{F}_1$ consists of a single $(-r)$-curve F_1 attached to the component C_{r+1} and $\mathfrak{F}_0$ represents a single (-1)-curve F_0 attached to C_{k+1}. For $r = k$ both feathers $\mathfrak{F}_0, \mathfrak{F}_1$ are attached to the component C_{k+1}. The standard morphism $\Phi : \bar{V} \to \mathbb{P}^1 \times \mathbb{P}^1$ is equivariant with respect to a suitable $\mathbb{C}^*$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ fixing $(0, 0)$.

We note that the extended divisor $D_{\text{ext}} = D_{\text{ext}}(r)$ is not distinguished (see 1.20); indeed, $D_{\text{ext}}^{>k+1} = \mathcal{F}_0$ is contractible.

Let us construct a family of standard completions of V_{k+1} in which $D_{\text{ext},s}$ jumps from one of these extended graphs to another one, so that r jumps. We restrict for simplicity to the case where $r = k$. Blowing down the contractible divisor $F_0 + C_{k+1}$ in $\bar{V}(k)$ we get a new surface X in which D_{ext} is contracted to a chain $[[0, 0, (-2)_{k-2}, -1, -k + 1]]$ consisting of the images $\bar{C}_0, \ldots, \bar{C}_k, \bar{F}_1$ of $C_0, \ldots, C_k, F_1$, respectively. The affine curve $S := \bar{C}_k \backslash \bar{C}_{k-1}$ is isomorphic to $\mathbb{A}^1$. We let $\mathcal{X}'$ be the blowup of the trivial family $X \times S$ along the graph of the embedding $S \hookrightarrow X$ with exceptional curve $\bar{\mathcal{C}}_{k+1}$ over S. Finally we let $\bar{\mathcal{V}}$ be the blowup of $\mathcal{X}'$ along a section $S \hookrightarrow \bar{\mathcal{C}}_{k+1}$ which does not meet the proper transforms of $\bar{C}_k \times S$ and $\bar{F}_1 \times S$, and we denote its exceptional set by $\mathcal{F}_0$.

The proper transforms

$$\mathcal{C}_0, \ldots, \mathcal{C}_{k+1}, \mathcal{F}_1 \quad \text{of} \quad \bar{C}_0 \times S, \ldots, \bar{C}_k \times S, \bar{\mathcal{C}}_{k+1}, \bar{F}_1 \times S$$

form together with $\mathcal{F}_0$ a family of extended divisors $\mathcal{D}_{\text{ext}}$ in $\bar{\mathcal{V}}$, while $\mathcal{D} = \mathcal{C}_0 \cup \ldots \cup \mathcal{C}_{k+1}$ is a family of zigzags, being all of the same type.

Obviously, the fiber of $(\bar{\mathcal{V}}, \mathcal{D})$ over the point s_0 corresponding to $\bar{C}_k \cap \bar{F}_0$ is $\bar{V}(k)$ with extended divisor $D_{\text{ext}}(k)$, while the fibers over the other points $s \in S \backslash \{s_0\}$ are $\bar{V}(k-1)$ with extended divisor $D_{\text{ext}}(k-1)$. Note that all fibers of the family $\mathcal{V} = \bar{\mathcal{V}} \backslash \mathcal{D} \to S$ are isomorphic to V_{k+1} by the theorem of Danilov-Gizatullin mentioned above.

2. Degenerations of singular fibers in families of $\mathbb{P}^1$-fibrations

As we have seen in Example 1.22, given a family of standard completions of a Gizatullin surface V with the same zigzag as in 1.18, the extended divisors $\mathcal{D}_{\text{ext},s}$ do not necessarily have the same dual graph at each point $s \in S$. In this section we will give a criterion as to when this dual graph stays constant.

2.1. Degenerate fibers of a $\mathbb{P}^1$-fibration. Let us fix the setup.

2.1. Given a surface V, we consider a sequence of blowups

$$\sigma : W = W_m \to W_{m-1} \to \ldots W_1 \to W_0 = V \tag{14}$$

with centers in smooth points on V and in its infinitesimally near points. For $i \geq 1$ we let $E_i \subseteq W_i$ denote the exceptional (-1)-curve of $W_i \to W_{i-1}$. We consider their proper and total transforms $C_i := \hat{E}_i$ and $\tilde{C}_i := E_i^*$ in W, respectively. Clearly the curves C_i (or, equivalently, the effective cycles $\tilde{C}_i$) generate freely the group $\text{Cycl}_1(E)$

of 1-cycles supported on the exceptional set $E = \sum_i C_i$. The intersection form gives a symmetric bilinear pairing on $\mathrm{Cycl}_1(E)$.

In the next lemma we describe all cycles in $\mathrm{Cycl}_1(E)$ with self-intersection -1.

Lemma 2.2. *(a)* $\tilde{C}_i.\tilde{C}_j = -\delta_{ij}$ *for* $1 \le i, j \le m$. *Moreover* $\hat{F}.\tilde{C}_i = 0$ *for every curve* F *of* V.

(b) If C *is a cycle supported in* E *with self-intersection* -1 *then* $C = \pm\tilde{C}_i$ *for some* $i \ge 1$. *In particular the only effective cycles with self-intersection* -1 *are the* $\tilde{C}_i$.

(c) $\tilde{C}_i.C_i = -1$ *and* $\tilde{C}_i.C_j \ge 0$ *for* $i \ne j$.

(d) $\tilde{C}_i$ *and* $\tilde{C}_i - C_i$ *are orthogonal i.e.,* $\tilde{C}_i.(\tilde{C}_i - C_i) = 0$.

Proof. To prove (a) we consider the contraction $\pi_i : W \to W_i$, and we assume that $j \ge i$. If $j > i$ then $\pi_i(\tilde{C}_j)$ is a point and so by the projection formula $\tilde{C}_i.\tilde{C}_j = \pi_{i*}(C_j).E_i = 0$. If $i = j$ then with the same argument $\tilde{C}_i.\tilde{C}_i = E_i.E_i = -1$. The proof of the second part is similar.

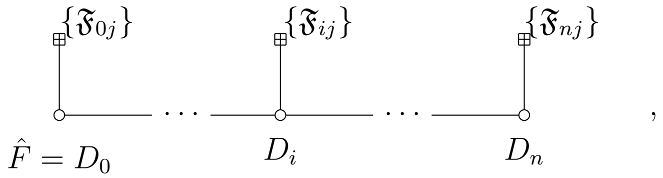
For the proof of (b) we write $C = \alpha_1\tilde{C}_1 + \ldots + \alpha_m\tilde{C}_m$. The self-intersection index $C^2 = -\alpha_1^2 - \ldots - \alpha_m^2$ is equal to -1 if and only if $\alpha_i = \pm 1$ for exactly one i and $\alpha_j = 0$ otherwise.

(c) and (d) follow immediately using the projection formula $\tilde{C}_i.C_j = \pi_i^*(E_i).C_j = E_i.\pi_{i*}(C_j)$. □

To study degenerations of extended divisors as introduced in 1.5, it is convenient to restrict to the piece $D_{(e)} = \Phi_0^{-1}(0)$ instead of the full extended divisor D_{ext}.

2.3. Letting $\pi : V = U \times \mathbb{P}^1 \to U$, where U is a neighbourhood of $0 \in \mathbb{A}^1_{\mathbb{C}}$, we consider a sequence of blowups as in (14) with centers on the fiber $F = \{0\} \times \mathbb{P}^1$ and in infinitesimally near points. We assume that the full fiber $D_{(e)} = \sigma^{-1}(F) = \hat{F} + \sum_i C_i$ has dual graph

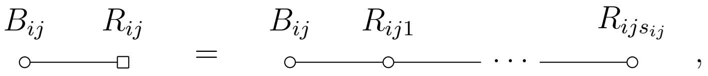
(15) $D_{(e)}$:



,

where at each curve D_i, $0 \le i \le n$, a collection of feathers $\mathfrak{F}_{ij}$ is attached with $1 \le j \le r_i$. Thus each feather $\mathfrak{F}_{ij}$ has dual graph

(16)



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where the box R_{ij} denotes a linear chain of curves R_{ijk} (possibly empty) connected to the bridge curve B_{ij}. We remind that R_{ij} does not contain a (-1)-curve, see Definition 1.10. However, unlike in Section 1 we allow that some of the curves D_i were (-1)-curves. This will be convenient in a later induction argument.

If D_i is one of the curves C_k as considered in 2.1 above then we let $\tilde{D}_i = \tilde{C}_k$. We introduce similarly the effective cycles $\tilde{B}_{ij}$ and $\tilde{R}_{ijk}$. Given an irreducible component H of one of the feathers $\mathfrak{F}_{ij}$, we call a component D_μ of the zigzag D a *mother component* of H if $\tilde{H}.D_\mu = 1$.

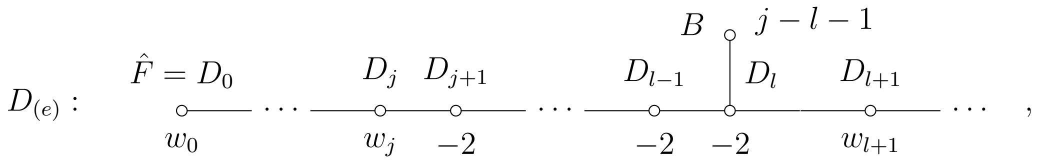
Lemma 2.4. *(a) Every component* H *of* $\mathfrak{F}_{ij}$ *has a unique mother component* D_μ.

(b) $\tilde{H}.C = 0$ *for every component* C *in* $D_{(e)}$ *different from* D_μ, H *and the neighbor of* H *in (16) to the right.*

Proof. Let H be the curve C_k considered in 2.1 so that $\tilde{H} = \pi_k^*(E_k)$, where E_k is the exceptional (-1)-curve created in the blowup $W_k \to W_{k-1}$ and $\pi_k : W \to W_k$ is the contraction as in the proof of 2.2. Since $H = \hat{E}_k$ does not separate the zigzag, the center of $\sigma_k : W_k \to W_{k-1}$ cannot be a double point of the zigzag $\pi_k(D)$. Thus E_k meets a unique component of $\pi_k(D)$. In view of the projection formula $\tilde{H}.D_i = E_k.\pi_k(D_i)$ this proves (a).

To deduce (b), assume that $C \subsetneq D_{(e)}$ is a component different from D_μ, H and satisfying $\tilde{H}.C \neq 0$. Again by the projection formula $\tilde{H}.C = E_k.\pi_k(C) \neq 0$. Since E_k is an at most linear vertex of the dual graph of $\pi_k(D_{(e)})$, this is only possible if $E_k.\pi_k(C) = 1$. As observed before, H does not separate the zigzag D, and so C being different from D_μ must belong to the feather $\mathfrak{F}_{ij}$. Since $\tilde{R}_{ij}$ contains no (-1)-curves, the projection formula forces C to be the neighbor on the right in (16) as claimed in (b). $\square$

Example 2.5. To illustrate these notions let us consider the graph



where $D_{(e)}^{\geq l+1}$ is contractible to a smooth point on D_l. It is easily verified that the mother component of B is D_j.

In the next proposition we collect some important properties of mother components. For a graph $D_{(e)}$ as in (15), similarly as before, $D_{(e)}^{>i}$ denotes the union of all connected components of $D_{(e)} \ominus D_i$ not containing D_0, while $D_{(e)}^{\geq i}$ stands for the connected component of $D_{(e)} \ominus D_{i-1}$ containing D_i.

Proposition 2.6. *(a) Let* D_μ, $\mu = \mu(i,j)$, *be the mother component of* B_{ij}. *Then* $\mu \leq i$ *and* $B_{ij}^2 \geq \mu - i - 1$. *Furthermore,* $B_{ij}^2 \leq -1$ *and* $B_{ij}^2 = -1$ *if and only if* $\mu = i$.

(b) If $\mu < i$ *then the divisors*

$$D_{(e)}^{\geq \mu+1} \ominus \mathfrak{F}_{ij}, \quad D_{(e)}^{>i} \ominus \mathfrak{F}_{ij} \quad \text{and} \quad \mathfrak{F}_{i'j'} \quad \text{with} \quad \mu < i' < i \tag{17}$$

are all contractible inside $D_{(e)}$.

(c) Assume that $B_{i'j'}$ *(where* $(i',j') \neq (i,j)$*) is a further bridge curve with mother component* $D_{\mu'}$. *If* $\mu < i$ *and* $\mu' < i'$ *then the intervals* $[\mu+1, i]$ *and* $[\mu'+1, i']$ *are disjoint.*

Proof. (a) The piece of the zigzag D between $D_{\mu+1}$ and D_i separates the mother component D_μ from $B_{ij} = \hat{E}_k$. Hence π_k contracts this piece. Furthermore, D_μ separates it from $D_0 = \hat{F}$ within D, therefore $\mu \leq i$. Moreover at most $i - \mu$ blowups are done near the images of B_{ij}, so $B_{ij}^2 \geq \mu - i - 1$.

The remaining assertions of (a) are easy and so we leave the proof to the reader.

(b) To show that $D_{(e)}^{\geq\mu+1}\ominus\mathfrak{F}_{ij}$ is contractible, it suffices to verify that $D_{(e)}^{\geq\mu+1}\ominus R_{ij}$ supports the total preimage $\tilde{B}_{ij}$, since then it contracts to $E_k=\pi_k(B_{ij})$ under π_k. As before $E_k=\pi_k(B_{ij})$ represents an at most linear vertex of the dual graph of $\pi_k(D_{(e)})$, where one neighbor is D_μ and the other one (if existent) is the neighbor of B_{ij} in $\mathfrak{F}_{ij}$ to the right in (16). Moreover all components in $D_{(e)}^{\geq\mu+1}\ominus\mathfrak{F}_{ij}$ appear under further blowups with center at $\pi_k(D_\mu)\cap E_k$ and its infinitesimally near points. Hence the assertion follows.

If $D_{(e)}^{>i}\ominus\mathfrak{F}_{ij}$ were not contractible, then contracting successively all (-1)-curves in $D_{(e)}^{\geq\mu+1}\ominus\mathfrak{F}_{ij}$ the vertex D_i of (15) would remain a branching point, which is impossible.

Similarly, if a feather $\mathfrak{F}_{i'j'}$ with $\mu<i'<i$ were not contractible, then contracting successively all (-1)-curves in $D_{(e)}^{\geq\mu+1}\ominus\mathfrak{F}_{ij}$, the vertex $D_{i'}$ of (15) would remain a branching point, which is impossible.

To show (c) we let $B_{ij}=\hat{E}_k$ and $B_{i'j'}=\hat{E}_{k'}$. We may assume that $k'<k$ so that π_k does not contract $B_{i'j'}$. As the divisor $D_{(e)}^{\geq\mu+1}\ominus\mathfrak{F}_{ij}$ is contracted under π_k this implies that $i'\leq\mu$. Hence

$$\mu'<i'\leq\mu<i,$$

proving (c). □

2.2. **Families of rational surfaces: the specialization map.** Let us recall the notion of specialization and generalization map for smooth proper families.

2.7. We consider a proper smooth holomorphic map $\pi:\mathcal{X}\to\mathbb{D}$, where $\mathcal{X}$ is a connected complex manifold and $\mathbb{D}$ stands for the unit disc in $\mathbb{C}$ with center $0\in\mathbb{D}$. By Ehresmann's theorem for any point $s\in\mathbb{D}$ there is a $\mathbb{D}$-diffeomorphism $\mathcal{X}\cong\mathcal{X}_s\times\mathbb{D}$, where $\mathcal{X}_s=\pi^{-1}(s)$. Hence the embedding $\mathcal{X}_s\hookrightarrow\mathcal{X}$ induces an isomorphism in cohomology $H^*(\mathcal{X})\xrightarrow{\cong}H^*(\mathcal{X}_s)$. Composing the isomorphisms

$$H^*(\mathcal{X}_s)\xrightarrow{\cong}H^*(\mathcal{X})\xrightarrow{\cong}H^*(\mathcal{X}_0)$$

we obtain a *specialization map* $\sigma:H^*(\mathcal{X}_s)\xrightarrow{\cong}H^*(\mathcal{X}_0)$; its inverse is called a *generalization map.*

2.8. From now on we assume that the fibers $\mathcal{X}_s$ are complete rational surfaces. Then

$$\mathrm{NS}(\mathcal{X}_s)=\mathrm{Pic}(\mathcal{X}_s)\cong H^2(\mathcal{X}_s;\mathbb{Z}),$$

where $\mathrm{NS}(\mathcal{X}_s)=\mathrm{Div}(\mathcal{X}_s)/\sim$ is the Neron-Severi group of algebraic 1-cycles modulo numerical equivalence. From the exact sequence

$$0=H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}})\to H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}}^\times)\cong\mathrm{Pic}(\mathcal{X})\to H^2(\mathcal{X},\mathbb{Z})\to H^2(\mathcal{X},\mathcal{O}_{\mathcal{X}})=0$$

induced by the exponential sequence we obtain an isomorphism

$$\mathrm{NS}(\mathcal{X})=\mathrm{Pic}(\mathcal{X})\cong H^2(\mathcal{X};\mathbb{Z}),$$

which commutes with restrictions to the fibers that is, with the isomorphisms

$$\mathrm{Pic}(\mathcal{X})\xrightarrow{\cong}\mathrm{Pic}(\mathcal{X}_s)\qquad\text{and}\qquad H^2(\mathcal{X};\mathbb{Z})\xrightarrow{\cong}H^2(\mathcal{X}_s;\mathbb{Z}) \tag{18}$$

induced by the embeddings $\mathcal{X}_s\hookrightarrow\mathcal{X}$. Composing the isomorphisms above leads to

$$\sigma:\mathrm{NS}(\mathcal{X}_s)\xrightarrow{\cong}\mathrm{NS}(\mathcal{X}_0)$$

also called a *specialization map.* Clearly σ is an isometry with respect to the intersection forms.

Lemma 2.9. *For a general point $s \in \mathbb{D}$, the specialization map σ sends the effective cone in* $\mathrm{NS}(\mathcal{X}_s) \otimes \mathbb{Q}$ *into the effective cone in* $\mathrm{NS}(\mathcal{X}_0) \otimes \mathbb{Q}$.

Proof. For an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$, its direct image $R^1\pi_*(\mathcal{L})$ is a coherent sheaf on $\mathbb{D}$, with a torsion located on a discrete set, say, $A(\mathcal{L}) \subseteq \mathbb{D}$. Since $\mathrm{Pic}_0(\mathcal{X}) = 0$ the set $A = \bigcup_{\mathcal{L}\in \mathrm{Pic}(\mathcal{X})} A(\mathcal{L})$ is at most countable.

Picking now a point $s \in \mathbb{D} \setminus A$, for an effective 1-cycle C on $\mathcal{X}_s$ we consider the corresponding invertible sheaf $\mathcal{L}_s = \mathcal{O}_{\mathcal{X}_s}(C)$. By virtue of (18) there exists an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ such that $\mathcal{L}|\mathcal{X}_s = \mathcal{L}_s$. Let t be the coordinate function on $\mathbb{D}$. We consider the cohomology sequence associated to the exact sequence $0 \to \mathcal{L} \stackrel{t-s}{\to} \mathcal{L} \to \mathcal{L}|X_s \to 0$:

$$0 \longrightarrow H^0(\mathcal{X},\mathcal{L}) \xrightarrow{t-s} H^0(\mathcal{X},\mathcal{L}) \xrightarrow{\rho} H^0(\mathcal{X}_s,\mathcal{L}_s) \longrightarrow H^1(\mathcal{X},\mathcal{L}) \xrightarrow{t-s} H^1(\mathcal{X},\mathcal{L}) \ .$$

Since $\mathbb{D}$ is Stein, we have $H^p(\mathcal{X},\mathcal{L}) \cong H^0(\mathcal{X}, R^pf_*(\mathcal{L}))$ for all $p \geq 0$. Since $R^1\pi_*(\mathcal{L})$ has no torsion at s, it follows from the long exact sequence that the restriction map ρ is surjective and so the sections of the sheaf $\mathcal{L}_s$ can be lifted to sections of $\mathcal{L}$. In particular $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\mathcal{C})$ for some effective 1-cycle $\mathcal{C}$ on $\mathcal{X}$ with $\mathcal{C}|\mathcal{X}_s = C$. Hence also $\sigma(C) = \mathcal{C}|\mathcal{X}_0$ is effective. This yields the lemma. □

2.3. **Formal specialization map and jumping feathers.** In this section we study possible degenerations of families of extended divisors. We recall first the geometric setup of Section 1.4.

2.10. Let $V = X \setminus D$ be a Gizatullin surface with a boundary zigzag D. As in Section 1.4 we consider families of standard completions $(\tilde{\mathcal{V}}, \mathcal{D}_s)$, $s \in S$, of a minimal resolution of singularities $V' \to V$ with a corresponding family of extended divisors $(\mathcal{D}_{\mathrm{ext}})_s = (\mathcal{C}_0)_s + (\mathcal{C}_1)_s + (\mathcal{D}_{(e)})_s$. We are interested in degenerations in such families. More precisely, each divisor $(\mathcal{D}_{(e)})_s$ has a dual graph as in (15), *however this graph may depend on $s \in S$.* If $\mathfrak{F}_{ij}(s) = \mathcal{B}_{ij}(s) + \mathcal{R}_{ij}(s)$ denotes the feathers at the point s then clearly the part $\mathcal{R}_s = \sum \mathcal{R}_{ij}(s)$ must be constant being the exceptional set of the resolution of singularities of V. Similarly the dual graph of the boundary zigzag $\mathcal{D}_s \cong D$ stays constant.

Assuming that S is a smooth curve, for a general point $s \in S$ the specialization map

$$\sigma : \mathrm{NS}(\tilde{\mathcal{V}}_s) \xrightarrow{\cong} \mathrm{NS}(\tilde{\mathcal{V}}_{s_0})$$

restricts to an isomorphism

$$\sigma : \mathrm{Cycl}_1((\mathcal{D}_{(e)})_s) \xrightarrow{\cong} \mathrm{Cycl}_1((\mathcal{D}_{(e)})_{s_0})$$

of the corresponding cycle spaces compatible with the intersection forms. In what follows we study this map σ on a formal level.

2.11. Let us consider two modifications $\pi : W \to V$ and $\pi' : W' \to V$ as in 2.3 above, with the same number m of blowups. Moreover assume that on W, W' we have decompositions

$$D_{(e)} = D + \sum \mathfrak{F}_{ij} \qquad \text{and} \qquad D'_{(e)} = D' + \sum \mathfrak{F}'_{ij}$$

as in 2.3 with the same number n of curves D_i, D_i' and with feathers

$$\mathfrak{F}_{ij} = \overset{B_{ij}}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{R_{ij}}{\square} \quad \text{and} \quad \mathfrak{F}'_{ij} = \overset{B'_{ij}}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{R'_{ij}}{\square} ,$$

respectively. We let $G = \mathrm{Cycl}_1(D_{(e)})$ and $G' = \mathrm{Cycl}_1(D'_{(e)})$ be their groups of 1-cycles with generators (C_i) and (C_i') or, equivalently, $(\tilde{C}_i)$ and $(\tilde{C}_i')$, respectively. Suppose that we are given an isomorphism

$$\delta : G \to G'$$

with the following properties:

(i) δ respects the intersection forms.
(ii) δ transforms effective cycles into effective cycles.
(iii) $\delta(D_i) = D_i'$ for all i.
(iv) $\delta(R_{ijk}) = R'_{i'j'k}$ for some i', j', where R_{ijk}, $R'_{i'j'k}$ are the components of R_{ij}, $R'_{i'j'}$, respectively, ordered as in 2.3.

We then call δ a *formal specialization map*, and δ^{-1} a *formal generalization map*.

It is clear from the discussion in 2.10 that any specialization map arising from a degeneration in a family of completions/resolutions of a Gizatullin surface is also a formal specialization map. Indeed (i) and (ii) follow from the construction in view of Lemma 2.9, (iii) follows immediately by the triviality of the family $\mathcal{D} \to S$, and (iv) holds due to the constancy of singularities in the open part $\mathcal{V}_s \cong V$.

We assume in the sequel that δ is a formal specialization map.

The structure of δ can be understood on the level of the generators $\tilde{D}_i$, $\tilde{R}_{ijk}$ and $\tilde{B}_{ij}$ of $G = \mathrm{Cycl}_1((\mathcal{D}_{(e)})_s)$. These generators form an orthogonal basis of G (see Lemma 2.2(a)). The same is true for their images in G'. So according to Lemma 2.2(b)

$$\{\delta(\tilde{D}_i), \delta(\tilde{B}_{ij}), \delta(\tilde{R}_{ijk})\} = \{\tilde{D}_i', \tilde{B}'_{ij}, \tilde{R}'_{ijk}\} . \tag{19}$$

Proposition 2.12. *With the assumptions as before the following hold.*

(a) $\delta(\tilde{D}_i) = \tilde{D}_i'$ and $\delta(\tilde{R}_{ijk}) = \tilde{R}'_{i'j'k}$;
(b) $\delta(\tilde{B}_{ij}) = \tilde{B}'_{i'j'}$;
(c) δ respects the mother components, i.e. if D_μ is the mother component of B_{ij} then D'_μ is the mother component of $B'_{i'j'}$.
(d) Every feather $\mathfrak{F}_{ij} = B_{ij} + R_{ij}$ either stays fixed or jumps to the right under δ, i.e. $\delta(\tilde{B}_{ij}) = \tilde{B}'_{i'j'}$ and $\delta(R_{ij}) = R'_{i'j'}$ with $i' \ge i$.

Proof. To deduce (a) we note that by 2.2(b) $\delta(\tilde{D}_i) = \tilde{C}'$ for some irreducible component C' of $D'_{(e)}$. Using properties (i) and (iii) of δ

$$\tilde{C}'.D_i' = \delta(\tilde{D}_i).D_i' = \tilde{D}_i.\delta^{-1}(D_i') = \tilde{D}_i.D_i = -1.$$

Using Lemma 2.2(c) this implies that $C' = D_i'$. With the same argument it follows that $\delta(\tilde{R}_{ijk}) = \tilde{R}'_{i'j'k}$. Clearly (b) is a consequence of (a) and (19).

(c) follows from the equation

$$\tilde{B}_{ij}.D_\alpha = \delta(\tilde{B}_{ij}).\delta(D_\alpha) = \tilde{B}'_{i'j'}.D'_\alpha$$

and the characterization of mother components given in Lemma 2.4.

(d) By property (ii) δ sends the effective cone of $G = \mathrm{Cycl}_1(D_{(e)})$ into the effective cone of $G' = \mathrm{Cycl}_1(D'_{(e)})$. Moreover, B_{ij}, $B'_{i'j'}$ appear in the cycles $\tilde{B}_{ij}$ and $\delta(\tilde{B}_{ij}) = \tilde{B}'_{i'j'}$, respectively, with coefficient 1. Hence $\delta(B_{ij}) = B'_{i'j'} + \Delta$ with an effective divisor $\Delta = \Delta(i,j)$ which does not contain $B'_{i'j'}$ so that

$$\Delta = \sum_{p\geq 1} \alpha_p D'_p + \sum_{(p,q)\neq(i',j')} \alpha_{pq} B'_{pq} + \sum_{p,q,r} \alpha_{pqr} R'_{pqr}\,, \quad \text{where} \quad \alpha_p,\, \alpha_{pq},\, \alpha_{pqr} \geq 0\,.$$

Suppose that B_{ij} jumps indeed, i.e. $i \neq i'$. Then $D_{i'}.B_{ij} = 0$, hence

$$0 = D'_{i'}.\delta(B_{ij}) = D'_{i'}.B'_{i'j'} + D'_{i'}.\Delta = 1 + D'_{i'}.\Delta.$$

Thus $D'_{i'}.\Delta = -1$ and so $\alpha_{i'} > 0$. It follows that $K := \{p : \alpha_p > 0\}$ contains i'. It is easily seen that $0 \notin K$. We choose $p \in \{0, \ldots, n\} \setminus K$ so that at least one of $p \pm 1$ is in K. Since $\delta(D_i) = D'_i$ $\forall i$ and δ preserves the intersection form, we have

$$D_p.B_{ij} = \delta(D_p).\delta(B_{ij}) = D'_p.B'_{i'j'} + D'_p.\Delta \geq D'_p.\Delta > 0\,. \tag{20}$$

Hence $D_p.B_{ij} = 1$ and so $p = i$. Consequently $K = [i+1, \ldots, n]$ (indeed, $0 \notin K$). Since $i' \in K$ we have $i + 1 \leq i'$. This proves (d). □

2.4. **Rigidity.** In Theorem 2.17 below we give a criterion for the dual graph of the extended divisor D_{ext} to stay constant under any specialization or generalization. We use the following terminology.

Definition 2.13. We say that the divisor $D_{(e)}$ as in 2.3 is *stable under specialization* if for any specialization map $\delta : G = \mathrm{Cycl}_1(D_{(e)}) \to G' = \mathrm{Cycl}_1(D'_{(e)})$ as in 2.11 we have $\delta(B_{ij}) = B'_{ij}$ with a suitable numbering of $B'_{i1}, \ldots, B'_{i,r_i}$. This means that no feather jumps to the right in (15).

Similarly, a divisor $D_{(e)}$ is said to be *stable under generalization* if for any generalization map[12] $\gamma : G = \mathrm{Cycl}_1(D_{(e)}) \to G' = \mathrm{Cycl}_1(D'_{(e)})$ we have $\gamma(B_{ij}) = B'_{ij}$ with a suitable numbering of $B'_{i1}, \ldots, B'_{i,r_i}$. Therefore no feather jumps to the left in (15).

Finally, a divisor $D_{(e)}$, which is stable under both specialization and generalization, is said to be *rigid*. This terminology can be equally applied to the extended divisor $D_{\mathrm{ext}} = C_0 + C_1 + D_{(e)}$.

We have the following fact.

Proposition 2.14. *$D_{(e)}$ is stable under generalization if and only if $B_{ij}^2 = -1$ for all bridge curves B_{ij}.*

Proof. By Proposition 2.12(d) B_{ij} can only jump to the left under generalization so that $\gamma(\tilde{B}_{ij}) = \tilde{B}'_{i'j'}$ with $i' \leq i$. Assume that $B_{ij}^2 = -1$. Then D_i is the mother component of B_{ij}, see Proposition 2.6(a). By virtue of Proposition 2.12(c) D'_i is the mother component of $B'_{i'j'}$. Using again Proposition 2.6(a) $i \leq i'$, hence $i = i'$ and the feather $\mathfrak{F}_{ij}$ stays fixed, as required.

To show the converse we assume that $B_{ij}^2 \leq -2$. By Proposition 2.6(a) then $\mu < i$, where D_μ is the mother component of $\mathfrak{F}_{ij}$. Using Proposition 2.6(b) the divisor $P := D_{(e)}^{\geq \mu+1} \ominus \mathfrak{F}_{ij}$ is contractible. Let F be the contracted divisor $D_{(e)}/P$. Then the image of

[12] i.e., $\delta = \gamma^{-1}$ is a specialization map as in 2.11.

P is the intersection point of the images, say $\bar{B}_{ij}$ and $\bar{D}_\mu$ of B_{ij} and D_μ in F. Moreover $\bar{B}_{ij}^2 = -1$.

Let now q be a point on $\bar{D}_\mu$ different from this intersection point. Rebuilding P at this point q yields a new divisor, say, $D'_{(e)}$. We claim that this procedure provides a non-trivial generalization map $\gamma : \mathrm{Cycl}_1(D_{(e)}) \to \mathrm{Cycl}_1(D'_{(e)})$ or, equivalently, a non-trivial specialization map $\delta : \mathrm{Cycl}_1(D'_{(e)}) \to \mathrm{Cycl}_1(D_{(e)})$.

Obviously the curves in $D_{(e)}$ and $D'_{(e)}$ are in $1-1$-correspondence. Let for a curve C in $D_{(e)}$, C' denote the corresponding curve in $D'_{(e)}$. We define δ by $\delta(C') = C$ for $C' \neq B'_{ij}$ and $\delta(B'_{ij}) := \tilde{B}_{ij}$. Since

$$\tilde{B}_{ij}.C = 0 \quad \text{for } C \neq B_{ij}, D_\mu, \qquad \tilde{B}_{ij}^2 = -1 \quad \text{and} \quad \tilde{B}_{ij}.D_\mu = 1\,,$$

δ is an isometry. Since it maps effective cycles into effective cycles, δ is a specialization map, as required. □

Proposition 2.15. *Assume that a feather $\mathfrak{F}_{ij}$ in (15) jumps to $\mathfrak{F}'_{i'j'}$ under a specialization $\delta : D_{(e)} \to D'_{(e)}$. If $i' > i$ then the following divisors are either empty or contractible:*

(a) $\mathfrak{F}_{kl}$ with $i < k < i'$;

(b) $D_{(e)}^{>i'}$ and $D'^{>i'}_{(e)} \ominus \mathfrak{F}'_{i'j'}$;

(c) $D_{(e)}^{\geq i+1}$ and $D'^{\geq i+1}_{(e)} \ominus \mathfrak{F}'_{i'j'}$.

Proof. (a) follows from Proposition 2.6(b). Indeed, if D_μ is the mother component of B_{ij} the by Proposition 2.12(c) D'_μ is the mother component of $B'_{i'j'}$, so $\mu \leq i < k < i'$. Similarly by the same Proposition 2.6(b), $D'^{>i'}_{(e)} \ominus \mathfrak{F}'_{i'j'}$ is either empty or contractible, as stated in (b).

Now (b) and (c) can be shown by induction on the number of irreducible components of $D_{(e)}$. Let $(\mathrm{b})_m$ and $(\mathrm{c})_m$ be the corresponding statements for divisors with m components. We show below that

(i) $(\mathrm{b})_{m-1}$,$(\mathrm{c})_{m-1} \Rightarrow (\mathrm{b})_m$, and

(ii) $(\mathrm{b})_m$,$(\mathrm{c})_{m-1} \Rightarrow (\mathrm{c})_m$.

To deduce (i) and (ii) we use the following claim.

Claim 1. Suppose that the divisor $D_{(e)}^{>i'}$ is non-empty. Then there exist (-1)-curves C in $D_{(e)}^{>i'}$ and C' in $D'^{>i'}_{(e)}$ with $\delta(C) = C'$, which are contractible in $D_{(e)}$ and $D'_{(e)}$, respectively.

The contractibility of $D_{(e)}^{>i'}$, and then also (i) and (b), follow from this claim by induction on m. Indeed, contracting C, C' in $D_{(e)}$, $D'_{(e)}$, respectively, leads to new divisors, say, $D_{(e)}^\vee$ and $D'^\vee_{(e)}$, where $D'^\vee_{(e)}$ is a specialization of $D_{(e)}^\vee$. By virtue of Proposition 2.6(a) the feather $\mathfrak{F}'_{i'j'}$ is minimal. Hence $\mathfrak{F}_{ij} = \mathfrak{F}_{ij}^\vee$ and $\mathfrak{F}'_{i'j'} = \mathfrak{F}'^\vee_{i'j'}$ are not affected by these contractions and again $\mathfrak{F}_{ij}^\vee$ jumps to $\mathfrak{F}'^\vee_{i'j'}$.

Proof of Claim 1. Assume first that $i' < n$. The divisor $D'^{\geq i'+1}_{(e)}$ is then non-empty and contractible. Hence it contains a (-1)-curve C' representing an at most linear vertex of the dual graph of $D'_{(e)}$. This curve C' can be either $D'_{k'}$ or a bridge $B'_{k'l'}$, where $k' \geq i' + 1$.

In the latter case we let $\tilde{B}'_{k'l'} = \delta(\tilde{B}_{kl})$. Since $(B'_{k'l'})^2 = -1$, by Propositions 2.6(a) and 2.12(c) $D_{k'}$ and $D'_{k'}$ are the mother components of B_{kl} and $B'_{k'l'}$, respectively.

Hence $k \ge k'$. Since under specialization a feather can only jump to the right (see Proposition 2.12(d)), we have $k = k'$. Therefore again by Proposition 2.6(a), $B_{kl}^2 = -1$ and the curves $C = B_{kl}$, $C' = B'_{k'l'}$ are as desired. Indeed, in view of Lemma 2.2(b), $B_{kl} = \tilde{B}_{kl}$, $B'_{k'l'} = \tilde{B}'_{k'l'}$ and so by Proposition 2.12(b), $\delta(B_{kl}) = B'_{k'l'}$.

In the former case $C = D_{k'}$ is again a (-1)-curve, since δ respects the intersection forms and $C' = \delta(C)$. If $D_{k'}$ is at most linear vertex of the dual graph then the curves C, C' are as desired. Otherwise $D_{k'}$ is a branch point of the dual graph while $D'_{k'}$ is not. So there is a feather $\mathfrak{F}_{k'l}$ at $D_{k'}$ which jumps to the right under δ. Thus $k' < n$, and we can repeat the consideration using induction on k'.

Suppose further that $i' = n$. Since by our assumption $D_{(e)}^{>n}$ is non-empty, there is a non-empty feather, say, $\mathfrak{F}_{nl}$ at D_n. This feather stays fixed under δ i.e., $\delta(\tilde{B}_{nl}) = \tilde{B}'_{nl'}$. Moreover, since $\mu < n$, by virtue of Proposition 2.6(c) D_n and D'_n are the mother components of $\mathfrak{F}_{nl}, \mathfrak{F}'_{nl'}$, respectively. Similarly as above, this implies that $C = B_{nl}$, $C' = B'_{nl'}$ are (-1)-curves with $\delta(C) = C'$, as desired. This proves the claim.

The proof of (ii) proceeds in a similar way. Because of $(b)_m$ we may assume that $D_{(e)}^{>i'}$ and $D'^{>i'}_{(e)} \ominus \mathfrak{F}'_{i'j'}$ are empty since otherwise we can contract them inside $D_{(e)}$, $D'_{(e)}$, respectively, and use induction on m as before. Similarly due to (a) we may suppose that both $D_{(e)}$ and $D'_{(e)}$ have no feathers at components D_k, D'_k with $i < k < i'$.

Now the induction step can be done due to the following

Claim 2. Under the assumption as above there are (-1)-curves C in $D_{(e)}^{\ge i+1}$ and C' in $D'^{\ge i+1}_{(e)} \ominus \mathfrak{F}'_{i'j'}$, with $\delta(C) = C'$, which are contractible in $D_{(e)}$, $D'_{(e)}$, respectively.

Proof of Claim 2. These divisors in our case consist of the linear strings $[D_{i+1}, \ldots, D_{i'}]$ and $[D'_{i+1}, \ldots, D'_{i'}]$, respectively. It is enough to show that there is a (-1)-curve in one of these linear strings, and then similarly as above there is also the second one.

Let as before D_μ be the mother component of the bridge curve B_{ij}. Then D'_μ is the mother component of $B'_{i'j'}$. If $\mu = i (< i')$ then by Proposition 2.6(b) the non-empty divisor $D'^{\ge i+1}_{(e)} \ominus \mathfrak{F}'_{i'j'} = [D'_{i+1}, \ldots, D'_{i'}]$ is contractible and so the result follows. If $\mu < i$ then again by Proposition 2.6(b) the divisor $D_{(e)}^{>i} \ominus \mathfrak{F}_{ij}$ is contractible, and also its connected component $D_{(e)}^{\ge i+1} = [D_{i+1}, \ldots, D_{i'}]$ is. Hence again we are done, and so the proof is completed. $\square$

The following fact is in a sense a converse to Proposition 2.15.

Proposition 2.16. *Suppose that, for two indices i, i' with $0 \le i < i' \le n$, each one of the following divisors is either empty or contractible:*

(a) the feathers $\mathfrak{F}_{kl}$ with $i < k < i'$;

(b) the divisor $D_{(e)}^{>i'}$;

(c) the divisor $D_{(e)}^{\ge i+1}$.

Then any feather $\mathfrak{F}_{ij}$ jumps to a feather $\mathfrak{F}'_{i'j'}$ under a suitable specialization.

Proof. The proof is similar to that of Proposition 2.14. Contracting first $D_{(e)}^{>i'}$ and then the remaining part, say, P of $D_{(e)}^{\ge i+1}$, we rebuild P blowing up at the intersection point of D_i and $\mathfrak{F}_{ij}$ and its infinitesimally near points. After that we rebuild $D_{(e)}^{>i'}$ at points of $D_{i'}$ different from the intersection point with the new feather $\mathfrak{F}'_{i'j'}$. We leave the details to the reader. $\square$

Now we are ready to formulate our main rigidity criterion. This enables us in the next section to check rigidity for Gizatullin $\mathbb{C}^*$-surfaces satisfying one of the conditions (α_+), (α_*) or (β_+), (β_*) of Theorem 0.2.

Similarly as in 1.20 we call a divisor $D_{(e)}$ *distinguished* if there is no index i with $1 \le i \le n$ such that $D_{(e)}^{>i}$ is non-empty and contractible.

Theorem 2.17. *A distinguished divisor $D_{(e)}$ is rigid provided that all its bridges B_{ij} are (-1)-curves and one of the following conditions is satisfied.*

(i) $D_{(e)}^{>n} \neq \emptyset$.

(ii) If for some i, $0 \le i < n$, the feather collection $\{\mathfrak{F}_{ij}\}$ is non-empty then the divisor $D_{(e)}^{\ge i+1}$ is not contractible.

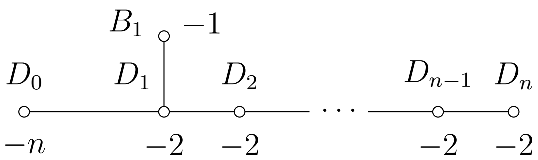
Proof. By Proposition 2.14 $D_{(e)}$ is stable under generalization. Suppose on the contrary that a feather $\mathfrak{F}_{ij}$ jumps to $\mathfrak{F}'_{i'j'}$ under a specialization, where $i < i' \le n$. By Proposition 2.15(c) $D_{(e)}^{\ge i+1}$ is contractible and so (ii) is violated. Similarly, by Proposition 2.15(b) $D_{(e)}^{>i'}$ is contractible. Since $D_{(e)}$ is distinguished and $i' + 2 \ge 3$, this is only possible if $i' = n$ and $D_{(e)}^{>n} = \emptyset$. Thus (i) is violated as well, proving the theorem. $\square$

We finish this section with several examples of rigid or non-rigid divisors.

Examples 2.18. 1. Consider the Gizatullin $\mathbb{C}^*$-surface V defined by the following pair of $\mathbb{Q}$-divisors on $\mathbb{A}^1$ (see Section 3.1):

$$(D_+, D_-) = \left(\frac{1}{n}[0] - [1],\ -\frac{1}{n}[0]\right) .$$

According to Proposition 3.10 below its standard completion has degenerate fiber with dual graph

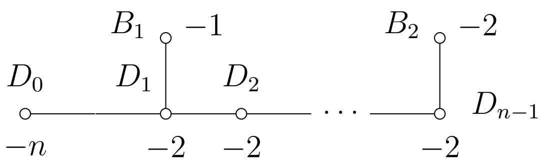
$D_{(e)}$:


.

Using Propositions 2.14 and 2.15 the divisor $D_{(e)}$ is rigid i.e., stable under specialization or generalization.

2. Let us revisit the standard completion of a Danilov-Gizatullin surface $V = V_n$ with $n = k + 1 \ge 3$ (see 1.22), which has extended divisor (13). The feather $\mathfrak{F}_1$ has mother component C_2. By Proposition 2.14 it can jump to C_2 under a suitable generalization, but also to any other component C_i, $i \ge 2$, using Proposition 2.16.

3. Let $D_{(e)}$ be the divisor

$D_{(e)}$:


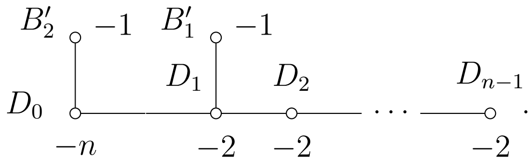
.

Again this is the dual graph of the degenerate fiber in a standard completion of a Gizatullin $\mathbb{C}^*$-surface $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ with

$$(D_+, D_-) = \left(\frac{1}{n}[0] - [1],\ -\frac{1}{n-1}[0]\right)$$

(cf. Example 1 above). According to Proposition 2.15, $D_{(e)}$ does not admit a nontrivial specialization. The mother component of B_2 is D_0. Hence by Proposition 2.14, $D_{(e)}$ admits a generalization into the divisor

$D'_{(e)}$:



It is interesting to note that $D'_{(e)}$ does no longer correspond to a $\mathbb{C}^*$-surface (see Proposition 3.10 below).

3. Extended divisors of $\mathbb{C}^*$-surfaces

In this section we examine as to when a Gizatullin $\mathbb{C}^*$-surface has a distinguished or rigid extended divisor. Our main criterion is Theorem 3.24 below. We review first some basic facts about $\mathbb{C}^*$-surfaces.

3.1. **DPD presentation for $\mathbb{C}^*$-surfaces.**

3.1. A normal affine surface $V = \operatorname{Spec} A$ endowed with an effective $\mathbb{C}^*$-action is called a $\mathbb{C}^*$-*surface*. Such a surface can be elliptic, parabolic or hyperbolic. On a non-toric elliptic or parabolic $\mathbb{C}^*$-surface V the $\mathbb{C}^*$-action is unique up to conjugation in the automorphism group $\operatorname{Aut}(V)$ and inversion in $\mathbb{C}^*$ [FlZa$_3$, Corollary 4.3]. Moreover by Corollaries 3.23, 4.4 and Theorem 4.5 in [FlZa$_2$] for any non-toric Gizatullin $\mathbb{C}^*$-surface V, the $\mathbb{C}^*$-action on V is hyperbolic. Therefore to deduce Theorem 0.2 it is enough to restrict to hyperbolic $\mathbb{C}^*$-surfaces.

3.2. A simple and convenient description for elliptic and parabolic $\mathbb{C}^*$-surfaces in terms of the associated gradings on the coordinate rings was elaborated by Dolgachev, Pinkham and Demazure. It was extended to the hyperbolic case in [FlZa$_1$], where this construction was called a *DPD presentation*.

Namely, any hyperbolic $\mathbb{C}^*$-surface V can be presented as

$$V = \operatorname{Spec} A, \qquad \text{where} \quad A = A_0[D_+, D_-] = A_0[D_+] \oplus_{A_0} A_0[D_-]$$

for a pair of $\mathbb{Q}$-divisors (D_+, D_-) on a smooth affine curve $C = \operatorname{Spec} A_0$ satisfying the condition $D_+ + D_- \le 0$. Here

$$A_0[D_\pm] = \bigoplus_{k \ge 0} H^0(C, \mathcal{O}_C(\lfloor kD_\pm \rfloor))u^{\pm k} \subseteq \operatorname{Frac}(A_0)[u, u^{-1}],$$

where $\lfloor D \rfloor$ stands for the integral part of a divisor D and u is an independent variable. A posteriori, $u \in \operatorname{Frac}(A_0) \otimes_{A_0} A$ and $\deg(u) = 1$. One can change u by multiplying it by a function $\varphi \in \operatorname{Frac}(A_0)$; then $D_\pm$ will be replaced by $D'_\pm = D_\pm \pm \operatorname{div} \varphi$.

We say that two pairs (D_+, D_-) and (D'_+, D'_-) are *equivalent* if $D'_\pm = D_\pm \pm \operatorname{div} \varphi$ for a rational function φ on C. By Theorem 4.3(b) in [FlZa$_1$] two hyperbolic $\mathbb{C}^*$-surfaces $V = \operatorname{Spec} A_0[D_+, D_-]$ and $V' = \operatorname{Spec} A_0[D'_+, D'_-]$ are equivariantly isomorphic over $C = \operatorname{Spec} A_0$ if and only if the pairs (D_+, D_-) and (D'_+, D'_-) are equivalent.

3.3. The embedding $A_0 \hookrightarrow A_0[D_+, D_-]$ induces an orbit map $\pi : V \to C$. The fixed points on a hyperbolic $\mathbb{C}^*$-surface V are all isolated, attractive in one and repelling in the other direction. The numerical characters of these singular points are precized in the next result.

Lemma 3.4. *(*[FlZa$_1$, Theorem 4.15]*) For a point $p \in \mathbb{A}^1$ we let*

$$D_+(p) = -\frac{e^+}{m^+} \text{ and } D_-(p) = \frac{e^-}{m^-} \text{ with } \gcd(e^\pm, m^\pm) = 1 \text{ and } \pm m^\pm > 0\,. \tag{21}$$

Then the following hold.

(a) If $D_+(p) + D_-(p) = 0$ then $\pi^{-1}(p) \cong \mathbb{C}^*$ is a fiber of multiplicity $m := m^+ = -m^-$ which contains no singular point of V.

(b) If $D_+(p) + D_-(p) < 0$ then the fiber $\pi^{-1}(p)$ in V consists of two orbit closures $O^\pm \cong \mathbb{A}^1$ of multiplicity $\pm m^\pm$ in the fiber $\pi^{-1}(p)$ meeting in a unique point p'. Moreover V has a cyclic quotient singularity of type (Δ, e) at p', where

$$\Delta = \Delta(p') = -\begin{vmatrix} e^+ & e^- \\ m^+ & m^- \end{vmatrix} = m^+m^-(D_+(p) + D_-(p)) > 0\,, \tag{22}$$

and e with $0 \le e < \Delta$ is defined by

$$e = e(p') \equiv \begin{vmatrix} a & e^- \\ b & m^- \end{vmatrix} \mod \Delta \quad \textit{if} \quad \begin{vmatrix} a & e^+ \\ b & m^+ \end{vmatrix} = 1\,.$$

For instance, if $D_\pm(p)$ are both integral and $k = -(D_+(p) + D_-(p)) > 0$ then V has an A_{k-1}-singularity at p'. We also need the following observation, see [FlZa$_2$, Theorem 4.5] and [FKZ$_2$, Lemma 4.2(b)].

Lemma 3.5. *For a $\mathbb{C}^*$-surface $V = \operatorname{Spec} A_0[D_+, D_-]$ the following hold.*

(a) V is a Gizatullin $\mathbb{C}^*$-surface if and only if $A_0 \cong \mathbb{C}[t]$ and $\operatorname{supp}\{D_\pm\} \subseteq \{p_\pm\}$ for some points $p_\pm \in \mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$.

(b) V is toric if and only if $A_0 \cong \mathbb{C}[t]$ and up to equivalence (D_+, D_-) is the divisor $(\frac{-e^+}{m^+}[p_0], \frac{e^-}{m^-}[p_0])$, for some point $p_0 \in \mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$.

3.2. **Completions of $\mathbb{C}^*$-surfaces.** We let $V = \operatorname{Spec} A_0[D_+, D_-]$ be a normal affine $\mathbb{C}^*$- surface. We review here some facts on equivariant completions of V; for proofs we refer the reader to [FKZ$_2$].

Lemma 3.6. *V admits an equivariant normal completion $(\bar X, \bar D)$ with the following properties.*

1. (Cf. [FKZ$_2$, Proposition 3.8 and Remark 3.9(4)]) *The orbit map $V \to C = \operatorname{Spec} A_0$ extends to a $\mathbb{P}^1$-fibration $\pi : \bar X \to \bar C$, where $\bar C$ is the smooth completion of C.*

2. $\bar D$ has exactly two horizontal components $\bar C_\pm$, which are sections of π, where $\bar C_+$ is repelling and $\bar C_-$ is attractive.

3. (Cf. [FlZa$_2$, 3.10 and Proposition 4.18]) *For $D_+(p) + D_-(p) = 0$ the fiber $\bar O_p = \pi^{-1}(p) \cong \mathbb{P}^1$ has multiplicity $m^+ = -m^-$, where $m^\pm$ are as in (21).*

4. (Cf. [FKZ$_2$, 3.10 and Proposition 3.13(d)] and [FlZa$_2$, Proposition 4.18]) *If $D_+(p) + D_-(p) < 0$ then*[13] *the fiber $\pi^{-1}(p)$ consists of two orbit closures $\bar O_p^\pm \cong \mathbb{P}^1$ of multiplicity $\pm m^\pm$ meeting in a unique point p' (cf. Lemma 3.4(b)). Moreover $\bar O_p^\pm$ have self-intersection indices $\frac{m^\mp}{\Delta m^\pm}$, respectively.*

[13] with the notation as in Lemma 3.4.

In general, $\bar{D}$ can contain singular points of $\bar{X}$. Let $\rho : \tilde{X} \to \bar{X}$ denote the minimal resolution of singularities of $\bar{X}$ and $\tilde{D} := \rho^{-1}(\bar{D})$. The $\mathbb{C}^*$-action on $\bar{X}$ then lifts to $\tilde{X}$.

Lemma 3.7. (See [FKZ$_2$, Proposition 3.16]) *Let $\tilde{\pi} : \tilde{X} \to \bar{C}$ be the induced $\mathbb{P}^1$-fibration and let $\tilde{C}_\pm$ be the proper transforms of $\bar{C}_\pm$ on $\tilde{X}$. Then the following hold.*

(a) $(\tilde{X}, D)$ is an SNC completion of the minimal resolution of V' of V. Moreover, $\tilde{C}_\pm^2 = \deg\lfloor D_\pm \rfloor$.

(b) If $D_+(p) + D_-(p) < 0$ then the fiber $\tilde{\pi}^{-1}(p)$ together with $\tilde{C}_\pm$ has dual graph

$$\begin{array}{ccccccccccccc} \tilde{C}_+ & & \{D_+(p)\} & & \tilde{O}_p^+ & & (e/\Delta)^* & & \tilde{O}_p^- & & \{D_-(p)\}^* & & \tilde{C}_- \\ \circ & \text{—} & \square & \text{—} & \circ & \text{—} & \square & \text{—} & \circ & \text{—} & \square & \text{—} & \circ \end{array} \quad , \tag{23}$$

where $\tilde{O}_p^\pm$ with $(\tilde{O}_p^\pm)^2 = \lfloor \frac{m^\mp}{\Delta m^\pm} \rfloor$ are the proper transforms of $\bar{O}_p^\pm$, respectively, and at least one of them is a (-1)-curve [14].

(c) If $D_+(p) + D_-(p) = 0$ then the fiber $\tilde{\pi}^{-1}(p)$ together with $\tilde{C}_\pm$ has dual graph

$$\begin{array}{ccccccccc} \tilde{C}_+ & & \{D_+(p)\} & & \tilde{O}_p & & \{D_-(p)\}^* & & \tilde{C}_- \\ \circ & \text{—} & \square & \text{—} & \circ & \text{—} & \square & \text{—} & \circ \end{array} \quad , \tag{24}$$

where the proper transform $\tilde{O}_p$ of $\bar{O}_p$ is a (-1)-curve.

Remark 3.8. 1. If $D_\pm(p) \in \mathbb{Z}$ and $-(D_+(p) + D_-(p)) = \Delta > 0$, then by Lemma 3.7(b) V has an $A_{\Delta-1}$-singularity at p' and the graph (23) is

$$\begin{array}{ccccccccc} \tilde{C}_+ & & \tilde{O}_p^+ & & A_{\Delta-1} & & \tilde{O}_p^- & & \tilde{C}_- \\ \circ & \text{—} & \circ & \text{—} & \square & \text{—} & \circ & \text{—} & \circ \\ & & -1 & & & & -1 & & \end{array} \quad . \tag{25}$$

3.3. **Extended divisors of Gizatullin $\mathbb{C}^*$-surfaces.**

3.9. In this section we let $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ denote a Gizatullin $\mathbb{C}^*$-surface. By Lemma 3.5(a) $\operatorname{supp}\{D_+\} \subseteq \{p_+\}$, $\operatorname{supp}\{D_-\} \subseteq \{p_-\}$, and the orbit map $\tilde{\pi} : \tilde{X} \to \bar{C} = \mathbb{P}^1$ is defined by the linear system $|F_\infty|$, where F_∞ denotes the fiber $\tilde{\pi}^{-1}(\infty)$ over the point $\{\infty\} = \mathbb{P}^1 \setminus \mathbb{A}^1$. Furthermore by Lemma 3.7 the boundary zigzag $\tilde{D}$ of V in $\tilde{X}$ has dual graph

$$\tilde{D} : \quad \begin{array}{ccccccccc} \{D_+(p_+)\}^* & & \tilde{C}_+ & & F_\infty & & \tilde{C}_- & & \{D_-(p_-)\} \\ \square & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \square \\ & & & & 0 & & & & \end{array} \quad . \tag{26}$$

A standard equivariant completion $\tilde{V}$ of the resolution V' of V can be obtained from $\tilde{X}$ by moving the zero weight in (26) to the left via elementary transformations [FKZ$_1$]. More precisely, to obtain the standard zigzag $D = C_0 + \ldots + C_n$ from (26) one has to perform first a sequence of elementary transformations at F_∞ until $\tilde{C}_+$ becomes a 0-curve. At this step the self-intersection index of the image C_s of $\tilde{C}_-$ becomes equal to $w_s = \deg(\lfloor D_+ \rfloor + \lfloor D_- \rfloor)$. By moving the two resulting neighboring zeros to the left

[14] See 1.9 for the notation $(e/\Delta)^*$.

via a sequence of elementary transformations (which contracts in general the curve $\tilde{C}_+$ and does not affect $C_s^2 = w_s$) one gets a completion $(\tilde{V}, D)$ of V' by the zigzag

$$D:\quad \underset{0}{\overset{C_0}{\circ}} - \underset{0}{\overset{C_1}{\circ}} - \overset{\{D_+(p_+)\}^*}{\square} - \underset{w_s}{\overset{C_s}{\circ}} - \overset{\{D_-(p_-)\}}{\square}\ . \tag{27}$$

This zigzag is standard as soon as $w_s \le -2$. Indeed, all curves in the boxes labelled $\{D_+(p_+)\}^*$ and $\{D_-(p_-)\}$ have weight ≤ -2. The elementary transformations as above result in a birational morphism $\tilde{X} \dashrightarrow \tilde{V}$, which is the identity on V'.

By abuse of notation we keep the same symbols $\tilde{O}_p$, $\tilde{O}^-_{p\pm}$ in both completions $\tilde{X}$ and $\tilde{V}$, cf. (23). Note that the self-intersection indices $(\tilde{O}^-_{p\pm})^2$, $\tilde{O}_p^2$ are the same in $\tilde{X}$ and in $\tilde{V}$.

To describe the resulting extended graph it is convenient to introduce *admissible* feather collections $\{\mathfrak{F}_\rho\}_{\rho\ge 1}$; see [FKZ$_2$]. By this we mean that all but at most one feather $\mathfrak{F}_\rho$ are A_k-feathers. Further, a curve on a $\mathbb{C}^*$-surface is called *parabolic* if it is pointwise fixed. For the next result, we refer the reader to Proposition 5.8 in [FKZ$_2$] and its proof.

Proposition 3.10. *With the notations as above, the resolution of singularities $V' \to V$ of a normal affine Gizatullin $\mathbb{C}^*$-surface V admits an equivariant SNC completion $(\tilde{V}, D)$ with extended graph*

$$D_{\mathrm{ext}}:\quad \begin{array}{ccccccccc} &&&&&& \boxplus\ \{\mathfrak{F}_\rho\}_{\rho\ge1} && \square\ \mathfrak{F}_0\\ &&&&&& | && | \\ \overset{C_0}{\circ} & - & \overset{C_1}{\circ} & - & \square & - & \overset{C_s}{\circ} & - & \square \\ 0 && 0 && \{D_+(p_+)\}^* && w_s && \{D_-(p_-)\} \end{array} \tag{28}$$

and with boundary zigzag D represented by the bottom line in (28). Here $w_s = \deg(\lfloor D_+\rfloor + \lfloor D_-\rfloor)$, $\mathfrak{F}_0$ is a single feather (possibly empty), $\{\mathfrak{F}_\rho\}_{\rho\ge1}$ is an admissible feather collection with all $\mathfrak{F}_\rho$, $\rho \ge 2$, being A_k-feathers, and $C_s = \tilde{C}_-$ is an attractive parabolic component. Moreover the following hold:

(a) The feather collection $\{\mathfrak{F}_\rho\}_{\rho\ge1}$ is empty if and only if V is a toric surface. If V is non-toric[15] *then $w_s \le -2$ and consequently $(\tilde{V}, D)$ is a standard completion of V'.*

(b) If $p_+ \ne p_-$ then $(\tilde{O}^-_{p_-})^2 = -1$ and the feathers

$$\mathfrak{F}_0:\quad \overset{\tilde{O}^-_{p_-}}{\circ} \longrightarrow \overset{(e/\Delta)(p_-)}{\square} \qquad \textit{and} \qquad \mathfrak{F}_1:\quad \overset{\tilde{O}^-_{p_+}}{\circ} \longrightarrow \overset{(e/\Delta)(p_+)}{\square} \tag{29}$$

are contained in the fibers over p_- and p_+, respectively, as described in (23).

(c) If $p_+ = p_- =: p$ then the $\mathfrak{F}_\rho$ are A_{k_ρ}-feathers $\forall \rho \ge 1$. The feather $\mathfrak{F}_0$ is empty if and only if $D_+(p) + D_-(p) = 0$. Otherwise it is as in (29) with $p_- = p$ and $(\tilde{O}^-_{p_-})^2 = \lfloor \frac{m^+}{\Delta m^-}\rfloor$.

Proof. By virtue of Lemma 2.20 in [FKZ$_2$], V is toric if and only if the extended divisor D_{ext} is linear. This yields the first assertion in (a). Thus by Proposition 5.8 in [FKZ$_2$], only the second assertion in (a) and the first one in (b) need to be proved.

[15]However, see Remark 3.11(4) below.

Assuming that $w_s = 0$ it is easily seen that $(D_+, D_-) \sim (0,0)$, $D = [[0,0,0]]$ and $\mathfrak{F}_\rho = \emptyset\ \forall \rho \geq 0$. But then $V \cong \mathbb{A}^1 \times \mathbb{C}^*$ is a toric surface. If further $w_s = -1$ then necessarily $p_+ = p_-$ and $\lfloor D_+(q) \rfloor + \lfloor D_-(q) \rfloor = 0$ for any point q different from $p = p_+ = p_-$. Since $D_+(q) + D_-(q) \leq 0$ it follows that $D_+(q) = -D_-(q)$ are integral for $q \neq p$. Passing to an equivalent pair of divisors we may suppose that D_+ and D_- are both supported at p. Hence again V is toric by Lemma 4.2(b) in [FKZ$_2$][16].

Finally, the equality $(\tilde{O}^-_{p_-})^2 = -1$ in (b) follows from Lemma 3.7(b). Indeed, as $D_+(p_-) \in \mathbb{Z}$ we have $m^+ = m^+(p_-) = 1$ and so $(\tilde{O}^-_{p_-})^2 = \lfloor \frac{m^+}{\Delta m^-} \rfloor = -1$. ☐

Remarks 3.11. 1. One can also move the zeros in (26) to the right. In the case where $w_s \leq -2$ this yields a second standard completion with the boundary zigzag reversed. However, in this completion $(\tilde{V}^\vee, D^\vee)$ the parabolic component is repelling, and it becomes attractive when the given $\mathbb{C}^*$-action is replaced by the inverse one via the automorphism $t \mapsto t^{-1}$ of $\mathbb{C}^*$. The extended dual graph D_{ext} in (28) is uniquely determined by the requirement that it corresponds to an equivariant standard completion of V' with attractive parabolic component.

2. If V is smooth then every feather in (28) consists of a single irreducible curve, see 1.12. A more detailed description can be found in [FKZ$_2$, Corollary 5.10]. If for instance $p_+ \neq p_-$ or one of the fractional parts $\{D_+\}$, $\{D_-\}$ vanishes then, up to passing to an equivalent pair of $\mathbb{Q}$-divisors,

$$(D_+, D_-) = \left(-\frac{1}{k}[p_+]\ , -\frac{1}{l}[p_-] - D_0 \right) \qquad \text{with} \quad k, l \geq 1\,,$$

where $D_0 = \sum_{\rho=2}^t [p_\rho]$ is a reduced integral divisor on $C \cong \mathbb{A}^1$ so that all points p_ρ are pairwise distinct and different from $p_\pm$. Thus in (28) the boxes adorned $\{D_+(p_+)\}^*$ and $\{D_-(p_-)\}$ are just A_{k-1}- and A_{l-1}-boxes, which represent chains of (-2)-curves $[[(-2)_{k-1}]]$ and $[[(-2)_{l-1}]]$, respectively.

3. Contracting the exceptional curves in $\tilde{V}$ corresponding to the singularities in the affine part V we obtain a standard completion $(\bar{V}, D)$ of V.

4. For a toric Gizatullin surface it may happen that $w_s = -1$, take e.g. $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ with $(D_+, D_-) = \left(-\frac{1}{2}[0],\ \frac{1}{3}[0]\right)$. The boundary zigzag as in Proposition 3.10 is now $[[0,0,-2,-1,-3]]$, which has standard form $[[0,0]]$. Thus $V \cong \mathbb{A}^2$.

Let us compute more generally the standard boundary zigzag of an arbitrary affine toric surface $V = V_{d,e} = \mathbb{A}^2/\mathbb{Z}_d$ (see 1.8), where $0 \leq e < d$ and $\gcd(e,d) = 1$.

Lemma 3.12. *The toric surface $V_{d,e}$ admits a standard completion with boundary zigzag*

$$D: \quad \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\frac{d-e}{d}}{\square}\,. \tag{30}$$

Moreover, the reverse zigzag $D^\vee$ is given by $\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\frac{d-e'}{d}}{\square}$ *, where e' is the unique number with $0 \leq e' < d$ and $ee' \equiv 1 \mod d$. In particular, the standard boundary of a toric surface is symmetric if and only if $e^2 \equiv 1 \mod d$.*

[16] Cf. Claim (α) in the proof of Proposition 5.8 in [FKZ$_2$].

Proof. Using Lemmas 3.4 and 3.5(b), $V_{d,e} \cong \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ with $D_+ = 0$ and $D_- = \frac{d}{e-d}[0]$. According to Proposition 3.10 the standard boundary has dual graph

$$\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{\lfloor \frac{d}{e-d} \rfloor}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{\{\frac{d}{e-d}\}}{\square}\ .$$

A simple computation gives

$$\overset{\lfloor \frac{d}{e-d} \rfloor}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{\{\frac{d}{e-d}\}}{\square} \quad = \quad \overset{\frac{d-e}{d}}{\square}\ .$$

Finally, the form of $D^\vee$ follows from 1.9. □

Remark 3.13. 1. The form of $D^\vee$ reflects the well known fact that $V_{d,e} \cong V_{d',e'}$ if and only if $d = d'$ and either $e = e'$ or $ee' \equiv 1 \mod d$, see e.g. [FlZa$_1$, Remark 2.5].

2. Due to the lemma, the toric surface $V_{d,e}$ is uniquely determined by its standard boundary zigzag.

For later use we give a criterion as to when a $\mathbb{C}^*$-action is equivalent to its inverse.

Lemma 3.14. *For a $\mathbb{C}^*$-surface $V = \operatorname{Spec} A_0[D_+, D_-]$ over $C = \operatorname{Spec} A_0$, the associated hyperbolic $\mathbb{C}^*$-action Λ on V and its inverse action Λ^{-1} are conjugate in the automorphism group* $\operatorname{Aut}(V)$ *if and only if there exists an automorphism $\psi \in \operatorname{Aut}(C)$ such that*

(i) $\psi^*(D_+ + D_-) = D_+ + D_-$ and
(ii) $\psi^*(D_-) - D_+$ is a principal divisor.

Proof. Inverting the $\mathbb{C}^*$-action results in interchanging the components $A_0[D_+]$ and $A_0[D_-]$ of the graded algebra $A_0[D_+, D_-]$ or, equivalently, in interchanging the divisors D_+ and D_- (see Section 3.1). Thus the inverse action Λ^{-1} corresponds to the $\mathbb{C}^*$-surface $V^\vee = \operatorname{Spec} A_0[D_-, D_+]$ over C. By Theorem 4.3(b) in [FlZa$_1$], the actions Λ and Λ^{-1} are conjugate in the group $\operatorname{Aut} V$ if and only if the $\mathbb{C}^*$-surfaces (V, Λ) and $(V^\vee, \Lambda^{-1})$ are equivariantly isomorphic, if and only if there is an automorphism, say, ψ of C such that the pairs (D_+, D_-) and $(\psi^*(D_-), \psi^*(D_+))$ are equivalent i.e.,

$$D_+ + D_0 = \psi^*(D_-) \qquad \text{and} \qquad D_- - D_0 = \psi^*(D_+)$$

for some principal divisor D_0 on C. The first of these equalities yields (ii), and taking their sum gives (i). □

Remarks 3.15. 1. Suppose that $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ is a $\mathbb{C}^*$-surface over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. Then condition (ii) in Lemma 3.14 is equivalent to $\psi^*(\{D_+\}) = \{D_-\}$. In particular, if the divisor $D_+ - D_-$ is integral then (i) and (ii) are automatically satisfied with $\psi = \mathrm{id}$.

2. We have seen in Remark 3.11(1) that changing the $\mathbb{C}^*$-action of a Gizatullin $\mathbb{C}^*$-surface $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ by the automorphism $t \mapsto t^{-1}$ of $\mathbb{C}^*$ amounts to reversing the standard zigzag. So *if the $\mathbb{C}^*$-action on V is conjugate to its inverse then the standard zigzag D of V is symmetric.*

3. Note however that for a Gizatullin $\mathbb{C}^*$-surface with a symmetric standard boundary zigzag the $\mathbb{C}^*$-action is not conjugate to its inverse, in general. A simple example is given by the toric surface $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-] \cong \mathbb{A}^2$ with $(D_+, D_-) = \left(-\frac{1}{2}[0],\ \frac{1}{3}[0]\right)$, see Remark 3.11(4). This pair does not satisfy condition (ii) of Lemma 3.14 although its standard boundary zigzag is equal to $[[0, 0]]$ and so is symmetric.

3.4. **A rigidity criterion.** In Theorem 3.24 below we show that under the assumptions (α_+) and (β) of Theorem 0.2, the standard divisor (15) is distinguished and rigid. Moreover, if (α_*) holds then this divisor is rigid after possibly interchanging D_+ and D_-.

3.16. We begin by recalling the assumptions (α_+), (α_*) and (β) of Theorem 0.2.

(α_+) $\operatorname{supp}\{D_+\}\cup\operatorname{supp}\{D_-\}$ is empty or consists of one point, say, p satisfying either $D_+(p)+D_-(p)=0$ or

$$D_+(p)+D_-(p)\le -\max\left(\frac{1}{m^{+2}},\frac{1}{m^{-2}}\right), \tag{31}$$

where $\pm m^\pm$ is the minimal positive integer such that $m^\pm D_\pm(p)\in\mathbb{Z}$.

(α_*) $\operatorname{supp}\{D_+\}\cup\operatorname{supp}\{D_-\}$ is empty or consists of one point p, where

$$D_+(p)+D_-(p)\le -1 \quad\text{or}\quad \{D_+(p)\}\ne 0\ne\{D_-(p)\}.$$

(β) $\operatorname{supp}\{D_+\}=\{p_+\}$ and $\operatorname{supp}\{D_-\}=\{p_-\}$ for two different points p_+,p_-, where

$$D_+(p_+)+D_-(p_+)\le -1 \quad\text{and}\quad D_+(p_-)+D_-(p_-)\le -1\,. \tag{32}$$

Lemma 3.17. *For a point $p\in\mathbb{A}^1$ with $(D_++D_-)(p)<0$ the following hold.*

(a) $\tilde O_p^\pm$ in (23) is a (-1)-curve if and only if $(D_++D_-)(p)\le -1/(m^\pm)^2$. In particular, both $\tilde O_p^+$ and $\tilde O_p^-$ in (23) are (-1)-curves[17] *if and only if (31) is fulfilled.*

(b) If $\min(\{D_+(p)\},\{D_-(p)\})=0$ then (31) is equivalent to

$$D_+(p)+D_-(p)\le -1\,. \tag{33}$$

Proof. We let as before $D_\pm(p)=e^\pm/m^\pm$ with $\gcd(e^\pm,m^\pm)=1$, $m^+,-m^-\ge 1$ and

$$\Delta=\Delta(p)=m^+m^-(D_+(p)+D_-(p))\ge 1\,.$$

(a) follows from the equalities $(\tilde O_p^\pm)^2=\lfloor\frac{m^\mp}{\Delta m^\pm}\rfloor$, see Lemma 3.7(b). Indeed,

$$\left\lfloor\frac{m^\mp}{\Delta m^\pm}\right\rfloor=-1 \iff \frac{m^\mp}{\Delta m^\pm}\ge -1 \iff \frac{m^\mp}{m^\pm}\ge -\Delta \iff \frac{-1}{(m^\pm)^2}\ge (D_++D_-)(p).$$

To show (b), after interchanging D_+ and D_-, if necessary, and passing to an equivalent pair of divisors, which does not affect our assumptions, we may suppose that $D_+(p)=0$. Thus $m^-\le -1$ and $m^+=1$ and so

$$\max\left(\frac{1}{m^{+2}},\frac{1}{m^{-2}}\right)=\max\left(1,\frac{1}{m^{-2}}\right)=1\,.$$

Now (b) follows. □

For a Gizatullin $\mathbb{C}^*$-surface $V=\operatorname{Spec}\mathbb{C}[t][D_+,D_-]$ we let $V^\vee=\operatorname{Spec}\mathbb{C}[t][D_-,D_+]$, and we denote by D_{ext}, $D_{\text{ext}}^\vee$ the corresponding extended divisors of the equivariant standard completions as in Proposition 3.10.

Lemma 3.18. *(a) All bridges B_ρ of the feathers $\mathfrak{F}_\rho$, $\rho\ge 0$, are (-1)-curves in both divisors D_{ext} and $D_{\text{ext}}^\vee$ if and only if (α_+) or (β) holds.*

(b) If (α_*) is fulfilled then all bridges B_ρ are (-1)-curves in at least one of these divisors.

[17] Anyway, at least one of these is a (-1)-curve, see Lemma 3.7(b).

Proof. Assume first that $p_+ = p_- = p$. By Proposition 3.10(c) the $\mathfrak{F}_\rho$ are A_{k_ρ}-feathers $\forall \rho \ge 1$. Hence the corresponding bridges are (-1)-curves. If $D_+(p) + D_-(p) = 0$ then again by Proposition 3.10(c), $\mathfrak{F}_0 = \emptyset$ and we are done. If $D_+(p) + D_-(p) < 0$ then by Lemma 3.17 the remaining bridges $\tilde{O}_p^\pm$ of the feather $\mathfrak{F}_0$ in both D_{ext} and $D_{\mathrm{ext}}^\vee$ are (-1)-curves if and only if (31) holds, as claimed in (a). Anyhow, according to Lemma 3.7(b) at least one of $\tilde{O}_p^\pm$ is a (-1)-curve, hence (b) follows as well in this case.

Suppose further that $p_+ \ne p_-$. By Proposition 3.10(b) the bridge $\tilde{O}_{p_-}^-$ of the feather $\mathfrak{F}_0$ in D_{ext} is a (-1)-curve and, symmetrically, the bridge $\tilde{O}_{p_+}^+$ of the feather $\mathfrak{F}_0$ in $D_{\mathrm{ext}}^\vee$ is a (-1)-curve. Thus by Lemma 3.17 the bridge $\tilde{O}_{p_+}^-$ of the feather $\mathfrak{F}_1$ in D_{ext}[18] is a (-1)-curve if and only if the first inequality in (32) is fulfilled. Similarly the bridge $\tilde{O}_{p_-}^+$ of the feather $\mathfrak{F}_1$ in $D_{\mathrm{ext}}^\vee$ is a (-1)-curve if and only if the second inequality in (32) is satisfied. The other bridges are as well (-1)-curves due to the fact that the feather collection $\{\mathfrak{F}_\rho\}$ is admissible and $\mathfrak{F}_1$ is the only potential non-A_k-feather, see Proposition 3.10. This implies (a) in this case. □

Remark 3.19. Switching D_+ and D_- amounts to interchanging D_{ext} and $D_{\mathrm{ext}}^\vee$. So replacing the given $\mathbb{C}^*$-action by its inverse one can achieve, if necessary, that the conclusion of Lemma 3.18(b) holds for the model with an attractive parabolic component.

Definition 3.20. Suppose that $\operatorname{supp}\{D_+\} \subseteq \{p_+\}$ and $\operatorname{supp}\{D_-\} \subseteq \{p_-\}$ with (not necessarily distinct) points $p_\pm$. By the *tail* of the extended divisor (28) we mean the subgraph

$$L = L_{s+1} = \overset{\{D_-(p_-)\}}{\square}\!\!-\!\!\!-\!\!\!-\!\!\overset{\mathfrak{F}_0}{\square} \quad = \quad \overset{C_{s+1}}{\circ}\!-\!\!-\cdots-\!\!-\!\overset{C_n}{\circ}\!-\!\!-\!\overset{\tilde{O}_{p_-}^-}{\circ}\!-\!\!-\!\overset{(e/\Delta)(p_-)}{\square}\;, \tag{34}$$

cf. (28), (29), and by a *subtail* a subgraph of L of the form

$$L_t = \overset{C_t}{\circ}\!-\!\!-\cdots-\!\!-\!\overset{C_n}{\circ}\!-\!\!-\!\overset{\tilde{O}_{p_-}^-}{\circ}\!-\!\!-\!\overset{(e/\Delta)(p_-)}{\square} \tag{35}$$

with $s+1 \le t \le n$.

Lemma 3.21. *If $D_+(p_-) + D_-(p_-) \ne 0$ then the tail L is contractible if and only if $\{D_+(p_-)\} = 0$. In particular, if $p_+ \ne p_-$ then L is contractible*[19].

Proof. Suppose first that L is contractible. By Lemma 3.7(b) the fiber $\tilde{\pi}^{-1}(p)$ with $p := p_-$ has dual graph

$$\overset{\{D_-(p)\}}{\square}\!-\!\!-\!\overset{\tilde{O}_p^-}{\circ}\!-\!\!-\!\overset{e/\Delta}{\square}\!-\!\!-\!\overset{\tilde{O}_p^+}{\circ}\!-\!\!-\!\overset{\{D_+(p)\}^*}{\square} \quad = \quad \overset{L}{\square}\!-\!\!-\!\overset{\tilde{O}_p^+}{\circ}\!-\!\!-\!\overset{\{D_+(p)\}^*}{\square}\;, \tag{36}$$

where we use the notations of *loc.cit.*. If L is contractible then contracting it in the fiber (36) leads to the divisor $\overset{A}{\circ}\!-\!\!-\!\overset{\{D_+(p)\}^*}{\square}$, where A denotes the image of $\tilde{O}_p^+$ and all the weights in the box adorned $\{D_+(p)\}^*$ are ≤ -2. This divisor has to be contractible to a smooth fiber $[[0]]$, which is only possible if the box is empty.

[18]See (29).

[19]Cf. [FKZ$_2$, Proposition 5.8].

Conversely, if $\{D_+(p)\} = 0$ then by 3.6(4) and Lemma 3.7(b) $\tilde{O}_p^+$ has multiplicity 1 in the fiber (36), hence the rest of it, which is L, can be contracted to a smooth point. □

Lemma 3.22. *(a) If $p_+ \neq p_-$ and $(D_+ + D_-)(p_-) \leq -1$ then none of the subtails L_t with $t \geq s+2$ is contractible. The same holds if (α_+) is satisfied.*
(b) If $p_+ \neq p_-$ and $0 > (D_+ + D_-)(p_-) > -1$ then the subtail L_{s+2} is contractible.
(c) If $p_+ = p_- =: p$, $(D_+ + D_-)(p) \neq 0$ and (31) is not satisfied then either $\tilde{O}_p^-$ is not a (-1)-curve or the subtail L_{s+2} is contractible.

Proof. If in (a) $(D_+ + D_-)(p_-) = 0$ then $\mathfrak{F}_0 = \emptyset$ and so every non-empty subtail of L is minimal and hence non-contractible. Otherwise $\mathfrak{F}_0 \neq \emptyset$, and under the assumptions of (a) Lemma 3.17(a) implies $(\tilde{O}_{p_-}^+)^2 = -1$. If a proper subtail L_t of L were contractible then, while contracting the fiber (36) with $p = p_-$ to $[[0]]$, at least one component neighboring $\tilde{O}_{p_-}^+$ would be contracted. Hence the image of $\tilde{O}_{p_-}^+$ would have self-intersection ≥ 0 and so it must be the full fiber. This contradicts the assumption that $t \geq s+2$ and so (a) holds.

(b) In this case $(\tilde{O}_{p_-}^+)^2 \leq -2$, see Lemma 3.17(a). If (b) does not hold then contracting L, C_{s+1} must be contracted before the subtail L_{s+2} is contracted. It follows that there is a proper contractible subchain, say, P of L which contains the piece $[C_{s+1}, \ldots, C_n, \tilde{O}_{p_-}^-]$. Contracting P in the full fiber (36) leads to a linear chain

$$\underset{-1}{\circ}\!\!-\!\!\!-\!\!\underset{E_1}{\circ}\!\!-\!\!\!-\cdots-\!\!\!-\!\!\underset{E_s}{\circ}\!\!-\!\!\!-\!\!\underset{\tilde{O}_{p_-}^+}{\circ}\,, \tag{37}$$

where $[E_1, \ldots, E_s]$ is a subchain of the box labelled by e/Δ. However, since all curves in $[E_1, \ldots, E_s, \tilde{O}_{p_-}^+]$ have self-intersection ≤ -2, (37) cannot be blown down to $[[0]]$, which gives a contradiction.

(c) By Lemma 3.17(a) one of the curves $\tilde{O}_p^\pm$ is not a (-1)-curve. Thus, if $\tilde{O}_p^-$ is a (-1)-curve then $(\tilde{O}_{p_-}^+)^2 \leq -2$. Arguing as in (b) it follows that the subtail L_{s+2} is contractible. □

Lemma 3.23. *Suppose that (α_+) or (β) holds. Then the divisors D_{ext}, $D_{\text{ext}}^\vee$ are both distinguished*[20]*.*

Proof. Since the conditions (α_+) and (β) are symmetric in D_+, D_-, it suffices to show that D_{ext} is distinguished. If for some i with $3 \leq i \leq s$ the divisor $D_{\text{ext}}^{>i} = D_{(e)}^{>i-2}$ were contractible (cf. (28)) then after contracting $D_{(e)}^{>i-2}$ inside $D_{(e)}$ we would obtain as dual graph

$$\overset{C_2}{\underset{w_2}{\circ}}\!\!-\!\!\!-\cdots-\!\!\!-\!\!\overset{C_{i-1}}{\underset{w_{i-1}}{\circ}}\!\!-\!\!\!-\!\!\overset{C_i}{\underset{-1}{\circ}}\,, \qquad \text{where} \qquad w_j \leq -2 \quad \forall j = 2, \ldots, i-1\,. \tag{38}$$

However, $D_{(e)}$ can be contracted to $[[0]]$ while (38) cannot, a contradiction. Thus it is enough to consider the divisors $D_{\text{ext}}^{>i}$ with $i \geq s+1$.

If (α_+) or (β) holds then by Lemma 3.22(a) the divisors $D_{\text{ext}}^{>i}$ are not contractible for all $i = s+1, \ldots, n$. Therefore D_{ext} is distinguished. □

[20]See Definition 1.20.

Theorem 3.24. *If $V = \operatorname{Spec}\mathbb{C}[t][D_+, D_-]$ is a Gizatullin $\mathbb{C}^*$-surface then the following hold.*

(1) If (α_+) or (β) is fulfilled then both divisors $D_{\rm ext}$, $D_{\rm ext}^\vee$ are distinguished and rigid.

(2) If (α_*) holds then at least one of the divisors $D_{\rm ext}$, $D_{\rm ext}^\vee$ is rigid.

Proof. (1) Since the conditions (α_+) and (β) are stable under interchanging D_+ and D_-, it is enough to consider the extended divisor $D_{\rm ext}$ for the standard completion of V. By Lemmas 3.18(a) and 3.23 $D_{\rm ext}$ is distinguished and all its bridges are (-1)-curves. In particular, no feather can jump to the left, see Proposition 2.14.

If the feather collection $\{\mathfrak{F}_\rho\}_{\rho\ge 1}$ as in (28) is empty then also no feather can jump to the right, so $D_{\rm ext}$ is rigid. Moreover, $D_{\rm ext}$ is rigid if one of the conditions (i), (ii) of Theorem 2.17 is fulfilled.

Suppose further that $\{\mathfrak{F}_\rho\}_{\rho\ge 1}\ne\emptyset$ but 2.17(i) fails. Then in (28) $s<n$ and $D_{\rm ext}^{>n}=\emptyset$. In particular $\mathfrak{F}_0=\emptyset$, and so by Proposition 3.10(b,c) $p_+=p_-=:p$ and $D_+(p)+D_-(p)=0$. Since $s<n$ and $\mathfrak{F}_0=\emptyset$ the tail $L=D_{\rm ext}^{\ge s+1}$ is non-empty and contains only curves of self-intersection ≤ -2. Thus L cannot be contractible and so 2.17(ii) holds, whence (1) follows.

(2) In view of (1) we have to consider only the case that $\{D_+(p)\}\ne 0$, $\{D_-(p)\}\ne 0$ and $D_+(p)+D_-(p)\ne 0$. By Lemma 3.18(b), after interchanging $D_\pm$ if necessary, the bridge curves of the extended divisor $D_{\rm ext}$ are all (-1)-curves. In particular, no feather can jump to the left. According to Lemma 3.21 the tail L is not contractible and so condition (c) in Proposition 2.15 is violated. Thus none of the feathers $\mathfrak{F}_\rho$ $(\rho\ge 1)$ can jump to the right and so $D_{\rm ext}$ is rigid, as required. $\square$

Remark 3.25. 1. It is worthwhile to remark that Theorem 3.24(1) is sharp. More precisely, let us establish the following.

(a) If neither (α_*) nor (β) are satisfied then none of the divisors $D_{\rm ext}$, $D_{\rm ext}^\vee$ is rigid. If $\operatorname{supp}\{D_+\}\cup\operatorname{supp}\{D_-\}$ consists of two distinct points and (β) is violated then at least one of them is not distinguished.

(b) If $p_+=p_-=p$ and (α_+) fails then none of the divisors $D_{\rm ext}$, $D_{\rm ext}^\vee$ is at the same time distinguished and rigid.

Proof. Let us first deduce (a) in the case $p_+\ne p_-$. This means that $\{D_+(p_+)\}$, $\{D_-(p_-)\}\ne 0$ while one of the two numbers $(D_++D_-)(p_\pm)$ is >-1. By symmetry it suffices to show that $D_{\rm ext}$ is non-rigid.

If $(D_++D_-)(p_+)>-1$ then the bridge $\tilde O_{p_+}^-$ of the feather $\mathfrak{F}_1$ for V (cf. (29)) has self-intersection ≤ -2 (see Lemma 3.17(a)) and so $D_{\rm ext}$ is non-rigid. If $(D_++D_-)(p_-)>-1$ then by Lemmas 3.21 and 3.22(b) the tail $L=L_{s+1}$ in (34) and its subtail L_{s+2} are both contractible. In other words, the divisors $D_{\rm ext}^{>s+1}$ and $D_{\rm ext}^{\ge s+1}$ are both contractible. Thus by Proposition 2.16 with $i=s$ and $i'=s+1$ any feather $\mathfrak{F}_\rho=\mathfrak{F}_{s,\rho}$, $\rho\ge 1$, can jump to a feather $\mathfrak{F}'_{s+1,\rho'}$ under a suitable specialization, and again $D_{\rm ext}$ is non-rigid. Moreover it is non-distinguished. By interchanging D_+ and D_-, if necessary, the assumption $(D_++D_-)(p_-)>-1$ is satisfied. This proves the second assertion in (a).

The proof of (a) in the case $p_+=p_-$ is similar and left to the reader.

To deduce (b) assume that $p_+=p_-=p$. As (α_+) is not satisfied we have $D_+(p)+D_-(p)\ne 0$ while (31) does not hold. By symmetry it is enough to show that the divisor $D_{\rm ext}$ cannot be distinguished and rigid at the same time. By Lemma 3.22(c) either $\tilde O_p^-$

is not a (-1)-curve, or the subtail L_{s+2} is contractible. In the first case D_{ext} is not rigid while in the second one it is not distinguished. $\square$

Theorem 3.24 and Remark 3.25 imply the following.

Corollary 3.26. *(a) Under the assumptions of Theorem 3.24 suppose additionally that* $\operatorname{supp}\{D_+\} \cup \operatorname{supp}\{D_-\}$ *consists of at most one point. Then at least one of the divisors* D_{ext}, $D^\vee_{\text{ext}}$ *is rigid if and only if* (α_*) *holds. Moreover the following are equivalent:*

- *both* D_{ext}, $D^\vee_{\text{ext}}$ *are distinguished and rigid;*
- *at least one of them is;*
- (α_+) *is fulfilled.*

(b) In the case where $\operatorname{supp}\{D_+\} \cup \operatorname{supp}\{D_-\}$ *consists of two distinct points, the following are equivalent:*

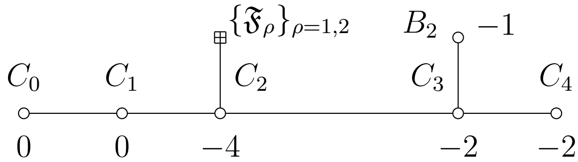
- *both* D_{ext}, $D^\vee_{\text{ext}}$ *are distinguished;*
- *at least one of them is rigid;*
- *both of them are distinguished and rigid;*
- (β) *is fulfilled.*

The condition that D_{ext}, $D^\vee_{\text{ext}}$ are both distinguished is also necessary in order that (β) were fulfilled. Indeed, for $(D_+, D_-) = \left(-\frac{3}{2}[p_+], -\frac{1}{2}[p_-]\right)$ the divisor D_{ext} is distinguished, while $D^\vee_{\text{ext}}$ is not and both of them are non-rigid.

In the next example we exhibit two smooth Gizatullin surfaces completed by the same zigzag, such that one of them is a $\mathbb{C}^*$-surface, whereas the second one does not admit a $\mathbb{C}^*$-action, even after any logarithmic deformation keeping the divisor at infinity fixed.

Example 3.27. There exists a smooth Gizatullin $\mathbb{C}^*$-surface, say V_0, with boundary zigzag $[[0, 0, -4, -2, -2]]$, see Example 4.7.3 in [FKZ$_2$]. To construct a second Gizatullin surface, say V, let us consider the following configuration D_{ext} in a suitable blowup $\bar V \to Q = \mathbb{P}^1 \times \mathbb{P}^1$:

D_{ext} :



,

where the map $\Phi : \bar V \to Q$ is given by the linear systems $|C_0|$ and $|C_1|$ and the feathers $\mathfrak{F}_1$ and $\mathfrak{F}_2$ consist of two single (-1)-bridges. Inspecting Proposition 3.10 we see that this extended divisor D_{ext} does not correspond to a Gizatullin $\mathbb{C}^*$-surface.

By Proposition 2.14 the divisor D_{ext} is stable under generalization. However, due to Proposition 2.16 it does admit a nontrivial specialization. Namely, any of the feathers $\mathfrak{F}_\rho$ can jump to C_3 or to C_4. Using Proposition 2.6(c) under such a specialization the dual graph of D_{ext} still has at least two branching vertices and so, cannot correspond to a Gizatullin $\mathbb{C}^*$-surface, see Proposition 3.10.

Thus indeed the surface $V = \bar V \backslash D$ with $D = C_0 + \ldots + C_4$ cannot be deformed to one with a $\mathbb{C}^*$-action.

4. The reconstruction space

Given a Gizatullin surface, any two SNC completions are related via a birational transformation which we call a *reconstruction*. Let us denote by γ the corresponding combinatorial transformations of the weighted dual graphs of the boundary divisors. The main result of this section (Corollary 4.10) states that the space of all geometric reconstructions of a pair (X, D) with a given combinatorial type γ has a natural structure of an affine space $\mathbb{A}^m$ for some m.

4.1. **Reconstructions of boundary zigzags.** We use in the sequel the following terminology from [FKZ$_1$].

Definition 4.1. Let Γ and Γ' be weighted graphs. A *combinatorial reconstruction* or simply *reconstruction* of Γ into Γ' consists in a sequence

$$\gamma : \quad \Gamma = \Gamma_0 \overset{\gamma_1}{\dashrightarrow} \Gamma_1 \overset{\gamma_2}{\cdots\cdots} \cdots \overset{\gamma_n}{\dashrightarrow} \Gamma_n = \Gamma' ,$$

where each arrow γ_i is either a blowup or a blowdown. The graph Γ' is called the *end graph* of γ. The inverse sequence $\gamma^{-1} = (\gamma_n^{-1}, \ldots, \gamma_1^{-1})$ yields a reconstruction of Γ' with end graph Γ. Reconstructions can be composed: if γ is a reconstruction of Γ with end graph Γ' and γ' is a reconstruction of Γ' with end graph Γ'', then the sequence (γ, γ') gives a reconstruction of Γ into Γ''.

A reconstruction γ is called *admissible* if it only involves

- blowdowns of at most linear vertices;
- inner blowups i.e., blowups at edges;
- outer blowups done at end vertices i.e., vertices of degree ≤ 1.

Thus an admissible reconstruction does not change the number of branch points of the graph and their degrees.

4.2. We let (X, D) and (Y, E) be two pairs consisting of smooth complete surfaces and SNC divisors on them. Similarly as in the combinatorial setting we can speak about a reconstruction $\tilde{\gamma}$ of (X, D) into (Y, E) meaning a sequence of blowups and blowdowns

$$\tilde{\gamma} : \quad X = X_0 \overset{\tilde{\gamma}_1}{\dashrightarrow} X_1 \overset{\tilde{\gamma}_2}{\dashrightarrow} \cdots \overset{\tilde{\gamma}_n}{\dashrightarrow} X_n = Y \quad ,$$

performed on D and on its subsequent total transforms. We say that $\tilde{\gamma}$ *is of type* γ if γ is the corresponding reconstruction of the dual graph Γ_D into Γ_E. Clearly the complements $X \backslash D$ and $Y \backslash E$ are isomorphic under the birational transformation $\tilde{\gamma} : X \dashrightarrow Y$.

A reconstruction $\tilde{\gamma}$ will be called *linear* if there is a domination

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

such that the total transform of D is a linear chain of rational curves.

The next fact follows immediately from Proposition 2.9 in [FKZ$_1$].

Proposition 4.3. *For any two standard completions (X, D) and (Y, E) of a Gizatullin surface V there exists an admissible reconstruction of (X, D) into (Y, E).*

Proposition 4.4. *Let $\gamma : \Gamma \dashrightarrow \Gamma'$ be an admissible reconstruction as in 4.1 between two linear chains Γ, Γ', and let $D \subseteq X$ be an SNC divisor with dual graph Γ. Then there exists a linear reconstruction $\tilde{\gamma} : (X, D) \dashrightarrow (Y, E)$ of type γ.*

Proof. Using induction on the length n of γ we may assume that for the shorter reconstruction $\gamma' : \Gamma = \Gamma_0 \dashrightarrow \ldots \dashrightarrow \Gamma_{n-1}$ there exists already a linear reconstruction

$$\tilde{\gamma}' : X = X_0 \dashrightarrow \ldots \dashrightarrow X_{n-1}$$

of type γ'. Thus X, X_{n-1} are dominated by a blowup Z_{n-1} such that the total transform D' of D in Z_{n-1} is linear. Since γ is admissible the last transform γ_n can be either a blowdown, an inner blowup or an outer blowup at an end vertex, see 4.1.

If γ_n is a blowdown then blowing down the corresponding curve in X_{n-1} gives a morphism $\tilde{\gamma}_n : X_{n-1} \to Y$. Obviously $\tilde{\gamma} = (\tilde{\gamma}', \tilde{\gamma}_n)$ is a reconstruction of type γ dominated by $Z := Z_{n-1}$ and so is linear. The same construction works in the case where γ_n is an inner or an outer blowup dominated by the contraction $\Gamma_{D'} \to \Gamma_D$.

We let G denote the total transform of D in X_{n-1}. If γ_n is an inner blowup which is not dominated by the contraction $\Gamma_{D'} \to \Gamma_D$ then we perform an additional blowup $\tilde{\gamma}_n : X_{n-1} \dashrightarrow Y$ at the corresponding double point of G. This is dominated by the corresponding inner blowup $Z_{n-1} \dashrightarrow Z$. Hence Z provides a linear domination of both X and Y, as desired.

Similarly, if γ_n is an outer blowup at an end vertex, say, v_i of Γ_{n-1} which is not dominated by the contraction $\Gamma_{D'} \to \Gamma_D$ then necessarily the proper transform v_i' of v_i in $\Gamma_{D'}$ is also an end vertex. In this case we perform additionally an outer blowup $Z_{n-1} \dashrightarrow Z$ at a point of the corresponding irreducible component G_i' of D' which is not a double point of D'. This yields a linear domination Z of both X and Y, as required. Now the proof is completed. □

4.2. **Symmetric reconstructions.**

Definition 4.5. A reconstruction of a graph Γ is called *symmetric* if it can be written in the form (γ, γ^{-1}). Clearly for a symmetric reconstruction the end graph is again Γ.

We have the following results on symmetric reconstructions.

Proposition 4.6. *(a) We let (X, D) and (Y, E) be two standard completions of a normal Gizatullin surface $V \not\cong \mathbb{A}^1 \times \mathbb{C}^*$. After replacing, if necessary, (X, D) by its reversion $(X^\vee, D^\vee)$ there exists a symmetric reconstruction of (X, D) into (Y, E).*

(b) Let X be a normal surface and D be a complete SNC divisor on X with dual graph Γ. Given an admissible symmetric reconstruction $\gamma = (\tau, \tau^{-1}) : \Gamma \dashrightarrow \Gamma$, there is a reconstruction of (X, D) into itself of type γ.

Proof. (a) By Proposition 4.3 there exists an admissible reconstruction $\tilde{\gamma} : (X, D) \dashrightarrow (Y, E)$ of type, say, γ. Using again Proposition 4.4 we can find a linear reconstruction $\tilde{\eta} : (X', D') \dashrightarrow (X, D)$ of type $\eta := \gamma^{-1}$, where (X', D') is another standard completion of V. Thus the composition $(\tilde{\eta}, \tilde{\gamma}) : (X', D') \dashrightarrow (Y, E)$ of type (γ^{-1}, γ) is symmetric. We note that our standard zigzags are different from $[[0, 0, 0]]$ since $V \not\cong \mathbb{A}^1 \times \mathbb{C}^*$. As follows from Proposition 3.4 in [FKZ$_1$], any linear reconstruction of a standard zigzag different from $[[0_{2k+1}]]$ is either the identity or the reversion. Thus $(X', D') = (X, D)$ or $(X', D') = (X^\vee, D^\vee)$.

(b) Clearly there is a reconstruction $\tilde{\tau}$ of (X, D) of type τ. Then $\tilde{\gamma} = (\tilde{\tau}, \tilde{\tau}^{-1})$ has the desired properties. This completes the proof. □

4.3. **Moduli space of reconstructions.** In this subsection we show that the reconstructions of a given type form in a natural way a moduli space.

Definition 4.7. Let $f : \mathcal{X} \to S$ be a flat family of normal surfaces and $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_r \subseteq \mathcal{X}$ be a family of SNC divisors sitting in the smooth part of f. We assume that $\mathcal{D}_i \to S$ is a smooth family of curves for every i and that the fiber $\mathcal{D}(s)$ forms an SNC divisor with the same dual graph Γ in each fiber $\mathcal{X}_s$. If γ is a reconstruction of Γ as in Definition 4.1, then a reconstruction of $\mathcal{X}/S$ of type γ is a sequence

$$\tilde{\gamma} : \quad \mathcal{X} = \mathcal{X}_0 \xrightarrow{\tilde{\gamma}_1} \mathcal{X}_1 \xrightarrow{\tilde{\gamma}_2} \cdots \xrightarrow{\tilde{\gamma}_n} \mathcal{X}_n \quad ,$$

where at each step $\mathcal{X}_{i+1}$ is either the blowup of $\mathcal{X}_i$ in a section $\iota : S \hookrightarrow \mathcal{X}_i$ or a blowdown of a family of (-1)-curves $\mathcal{C} \subseteq \mathcal{X}_i$ such that fiberwise $\tilde{\gamma}$ is of type γ.

In the next result we show that the set of all reconstructions of X of type γ has a natural structure of a smooth scheme. It is convenient to formulate this result in a relative setup.

With the notations as in Definition 4.7, if $S' \to S$ is a morphism of algebraic $\mathbb{C}$-schemes and $\tilde{\gamma}$ is a reconstruction of $\mathcal{X}/S$ of type γ then by a base change $S' \to S$ we obtain a reconstruction $\tilde{\gamma}'$ of $\mathcal{X} \times_S S'/S'$. This defines a set valued functor R_γ on the category of S-schemes that assigns to an S-scheme S' the set of all reconstructions of type γ of $\mathcal{X} \times_S S'/S'$.

Proposition 4.8. *With Γ and $\mathcal{X}/S$ as in Definition 4.7 the functor R_γ is representable. The latter means that there exists an S-scheme $\mathcal{R} = \mathcal{R}_\gamma$ of finite type over S and a universal reconstruction in $R_\gamma(\mathcal{R})$:*

$$\tilde{\gamma}_u : \quad \mathcal{X}_0 := \mathcal{X} \times_S \mathcal{R} \xrightarrow{\tilde{\gamma}_{u1}} \mathcal{X}_1 \xrightarrow{\tilde{\gamma}_{u2}} \cdots \xrightarrow{\tilde{\gamma}_{un}} \mathcal{X}_n$$

such that for every S-scheme S' and every reconstruction $\tilde{\gamma} \in R_\gamma(S')$ there is a unique S-morphism $g : S' \to \mathcal{R}$ satisfying $\tilde{\gamma} = g^*(\tilde{\gamma}_u)$. Moreover $\mathcal{R}$ is smooth over S.

Proof. Let us first assume that γ consists of a single blowdown or an inner blowup of Γ. We claim that in these cases $\mathcal{R} := S$ is the required moduli space. The universal family $\tilde{\gamma}_u$ is constructed as follows. If γ is the blowdown of the vertex corresponding to the component $\mathcal{D}_\rho$ of $\mathcal{D}$, then $\mathcal{D}_\rho$ is a family of (-1)-curves and so can be blown down via a map $\tilde{\gamma} : \mathcal{X} \to \mathcal{X}'$ so that $\mathcal{X}' \to S$ is a flat family, see Lemma 1.15. It is clear that $\tilde{\gamma}_u := \tilde{\gamma}$ is in this case the universal reconstruction of type γ.

Similarly, suppose that γ is the blowup of the edge joining the two vertices which correspond to $\mathcal{D}_\rho$ and $\mathcal{D}_\tau$. In particular $\mathcal{D}_\rho \cap \mathcal{D}_\tau$ is a section of $\mathcal{X} \to S$. Blowing up this section leads to a morphism $\tau : \mathcal{X}' \to \mathcal{X}$, and the composed map $\mathcal{X}' \to \mathcal{X} \to S$ is flat. It is easy to check that in this case $\tilde{\gamma}_u := \tau^{-1} \in R_\gamma(S)$ is the universal reconstruction of type γ.

We assume further that γ is an outer blowup in a vertex of Γ which corresponds to $\mathcal{D}_\rho$. The complement

$$\mathcal{R} := \mathcal{D}_\rho \backslash \bigcup_{\tau \neq \rho} \mathcal{D}_\tau$$

is then smooth over S, and the fiber product $\mathcal{X}_\mathcal{R} := \mathcal{X} \times_S \mathcal{R} \to \mathcal{R}$ is a flat family of normal surfaces which has a canonical section given by the diagonal embedding $\mathcal{R} \hookrightarrow$

$\mathcal{X}_{\mathcal{R}}$. The blowup $\tau : \mathcal{X}' \to \mathcal{X}_{\mathcal{R}}$ of this section provides again a universal reconstruction $\tilde{\gamma}_u := \tau^{-1} \in R_\gamma(S)$ of type γ.

To build up the reconstruction space for an arbitrary sequence $\gamma = (\gamma_1, \dots, \gamma_n)$ as in Definition 4.7 we proceed by induction on n. Assume that there is a universal reconstruction space $\mathcal{R}'$ for the sequence $\gamma' := (\gamma_1, \dots, \gamma_{n-1})$ of length $n-1$. Thus the universal reconstruction $\tilde{\gamma}'_u$ of type γ' consists in a sequence

$$\tilde{\gamma}'_u : \quad \mathcal{X}'_0 = \mathcal{X} \times_S \mathcal{R}' \xrightarrow{\tilde{\gamma}'_{u1}} \mathcal{X}'_1 \xrightarrow{\tilde{\gamma}'_{u2}} \cdots \xrightarrow{\tilde{\gamma}'_{un-1}} \mathcal{X}'_{n-1}$$

as in Definition 4.7. Let $\mathcal{D}' \subseteq \mathcal{X}'_{n-1}$ be the total transform of $\mathcal{D} \times_S \mathcal{R}'$ so that the dual graph of $\mathcal{D}'$ is Γ_{n-1}. Now $\gamma_n : \Gamma_{n-1} \dashrightarrow \Gamma_n$ is a reconstruction of length 1. Hence by the first part of the proof there exists a universal reconstruction space $\mathcal{R}$ for $\mathcal{X}'_{n-1}/\mathcal{R}'$, where the universal reconstruction is a birational transformation

$$\tilde{\gamma}_{un} : \mathcal{X}_{n-1} := \mathcal{X}'_{n-1} \times_{\mathcal{R}'} \mathcal{R} \dashrightarrow \mathcal{X}_n .$$

Combining the universal properties of $\mathcal{R}'$ and $\mathcal{R}$ it follows that $\mathcal{R}$ together with

$$\tilde{\gamma}_u : \quad \mathcal{X}_0 = \mathcal{X} \times_S \mathcal{R} \xrightarrow{\tilde{\gamma}_{u1}} \mathcal{X}_1 := \mathcal{X}'_1 \times_{\mathcal{R}'} \mathcal{R} \xrightarrow{\tilde{\gamma}_{u2}} \cdots \xrightarrow{\tilde{\gamma}_{un-1}} \mathcal{X}_{n-1} \xrightarrow{\tilde{\gamma}_{un}} \mathcal{X}_n ,$$

where $\tilde{\gamma}_{ui} := \tilde{\gamma}'_{ui} \times_{\mathcal{R}'} \mathrm{id}_{\mathcal{R}}$, forms the required universal reconstruction of type γ.

Finally let us show that $\mathcal{R}$ is smooth over S. Using the iterative construction of $\mathcal{R}$ it is sufficient to show this for a reconstruction $\gamma : \Gamma \dashrightarrow \Gamma_1$ of length 1. But the latter is immediate from the first part of the proof. □

In the case where the reconstruction is admissible we get the following important information on the structure of $\mathcal{R}$.

Proposition 4.9. *Let Γ, γ and $\mathcal{X}/S$ be as in Definition 4.7. We let Γ_i denote the dual graph of the total transform $\mathcal{D}^{(i)}$ of $\mathcal{D}$ in $\mathcal{X}_i$, and we assume that the following conditions are fulfilled:*

(i) $H^1(S, \mathcal{O}_S) = 0$ *and* $\mathrm{Pic}(S) = 0$.
(ii) Γ *is connected, and for every i the graph Γ_i is not reduced to a point.*
(iii) γ *is admissible.*

Then the reconstruction space $\mathcal{R} = \mathcal{R}_\gamma$ is isomorphic to $S \times \mathbb{A}^m$ for some $m \in \mathbb{N}$.

Proof. Let us first consider the case where the reconstruction $\gamma : \Gamma \to \Gamma_1$ has length 1. If γ is a blowdown or an inner blowup we have $\mathcal{R} = S$, hence the assertion is obvious. If γ is an outer blowup then by our assumption it is performed in an end vertex of Γ. The corresponding component of $\mathcal{D}$, say, $\mathcal{D}_\rho$ meets exactly one other component, say, $\mathcal{D}_\tau$. The intersection $\Sigma := \mathcal{D}_\rho \cap \mathcal{D}_\tau$ is a section of the $\mathbb{P}^1$-bundle $\mathcal{D}_\rho \to S$. Thus by Lemma 1.16 $\mathcal{D}_\rho \to S$ is S-isomorphic to the product $S \times \mathbb{P}^1$ so that the section corresponds to $S \times \{\infty\}$. Since $\mathcal{R} = \mathcal{D}_\rho \backslash \mathcal{D}_\tau$ by our construction, we conclude that $\mathcal{R}$ is S-isomorphic to $S \times \mathbb{A}^1$.

In the general case we proceed by induction. We consider $\gamma' = (\gamma_1, \dots, \gamma_{n-1})$ and the universal reconstruction space $\mathcal{R}'$ over S of combinatorial type γ'. By induction hypothesis $\mathcal{R}'$ is S-isomorphic to $S \times \mathbb{A}^{m'}$. Since $\mathcal{R} = \mathcal{R}_\gamma$ is the universal reconstruction of γ_n with respect to[21] $\mathcal{X}_{n-1} \times_S \mathcal{R}'/\mathcal{R}'$, from the first part of the proof we obtain that $\mathcal{R} \cong \mathcal{R}'$ or $\mathcal{R} \cong \mathcal{R}' \times \mathbb{A}^1$, proving the result. □

[21]See the proof of Proposition 4.8.

Propositions 4.8 and 4.9 lead to the following corollary.

Corollary 4.10. *Let X be a normal surface, and let D be an SNC divisor in X_{reg} with dual graph Γ. Given a reconstruction γ of Γ, the set $\mathcal{R}_\gamma$ of all reconstructions of X of type γ has a natural structure of a smooth scheme. Moreover if γ is admissible then $\mathcal{R}_\gamma \cong \mathbb{A}^m$ for some $m \geq 0$.*

5. Applications

Here we prove Theorems 0.1 and 0.2 on the uniqueness of $\mathbb{C}^*$- and $\mathbb{C}_+$-actions. The proofs are based on the results of the previous sections and on Theorem 5.2 below, which states that a standard completion of a Gizatullin surface with a distinguished and rigid extended divisor D_{ext} is up to reversion (see 1.4) unique.

5.1. **The main technical result.** To formulate our result let us first fix the notations.

5.1. Let V be a non-toric Gizatullin surface and let $(\bar{V}, D)$ and $(\bar{V}', D')$ be standard completions of V. We also consider the minimal resolutions of singularities V', $(\tilde{V}, D)$, $(\tilde{V}', D')$ of V, $(\bar{V}, D)$ and $(\bar{V}', D')$, respectively. As in 1.5 we let

$$\Phi = \Phi_0 \times \Phi_1 : \tilde{V} \to \mathbb{P}^1 \times \mathbb{P}^1 \quad \text{and} \quad \Phi' = \Phi'_0 \times \Phi'_1 : \tilde{V}' \to \mathbb{P}^1 \times \mathbb{P}^1$$

denote the standard morphism and D_{ext}, D'_{ext} the extended divisors.

Reversing the zigzag $D' = [[0, 0, w'_2, \ldots, w'_n]]$ by a sequence of inner elementary transformations provides the standard completion $(\bar{V}'^{\vee}, D'^{\vee})$, see 1.4.

Theorem 5.2. *Assume that the extended divisor D_{ext} of $(\tilde{V}, D)$ is distinguished and rigid. After replacing $(\bar{V}', D')$ by $(\bar{V}'^{\vee}, D'^{\vee})$ if necessary there is an isomorphism $f : \bar{V} \to \bar{V}'$ with $f(D) = D'$.*

Note that this isomorphism is *not* the identity on the affine part V, in general.

Proof. Replacing $(\bar{V}', D')$ by $(\bar{V}'^{\vee}, D'^{\vee})$ if necessary, by Proposition 4.6(a) there is a reconstruction $\tilde{\gamma}'$ from $(\tilde{V}, D)$ to $(\tilde{V}', D')$ of type, say γ, which is admissible and symmetric. Thus $\tilde{\gamma}'$ can be considered as a point in the reconstruction space $\mathcal{R} = \mathcal{R}_\gamma \cong \mathbb{A}^m$, see Corollary 4.10. By Proposition 4.6(b) there is also a reconstruction $\tilde{\gamma}$ of $(\tilde{V}, D)$ of type γ into itself. Let

$$\tilde{\gamma}_u : \quad \mathcal{X}_0 = \tilde{V} \times \mathcal{R} \xrightarrow{\tilde{\gamma}_{u1}} \mathcal{X}_1 \xrightarrow{\tilde{\gamma}_{u2}} \cdots \xrightarrow{\tilde{\gamma}_{un-1}} \mathcal{X}_{n-1} \xrightarrow{\tilde{\gamma}_{un}} \mathcal{X}_n$$

be the universal reconstruction of combinatorial type γ and consider the family $\tilde{\mathcal{V}} := \mathcal{X}_n$ together with the total transform $\mathcal{D}$ of $D \times \mathcal{R}$ in $\tilde{\mathcal{V}}$. Thus $(\tilde{\mathcal{V}}, \mathcal{D})$ is a family of completions of V' over the reconstruction space $\mathcal{R}$ as considered in Proposition 1.21. Moreover, by construction the completions $(\tilde{V}, D)$ and $(\tilde{V}', D')$ are the fibers over the points $\tilde{\gamma}$, $\tilde{\gamma}' \in \mathcal{R}$, respectively.

Let now $\mathcal{D}_{\text{ext}}$ be the family of extended divisors of $(\tilde{\mathcal{V}}, \mathcal{D})$. Its fiber over $\tilde{\gamma}$ is D_{ext} and so is rigid. Hence the family of extended divisors $\mathcal{D}_{\text{ext}}$ has the same dual graph over each point of $\mathcal{R}$. By Proposition 1.21 the family $(\tilde{\mathcal{V}}, \mathcal{D})$ is trivial and so there is an isomorphism $(\tilde{V}, D) \times \mathcal{R} \cong (\tilde{\mathcal{V}}, \mathcal{D})$. Restricting it to the fiber over $\tilde{\gamma}'$ gives an isomorphism $\tilde{f} : (\tilde{V}, D) \to (\tilde{V}', D')$ that induces an isomorphism $f : \bar{V} \to \bar{V}'$ with the desired property. $\square$

In particular, in the situation of Theorem 5.2 it follows that the extended divisors $D_{\rm ext}$, $D'_{\rm ext}$, considered as schemes via their reduced structures, are isomorphic at least after reversion, if necessary. It is important to note that this holds even without the assumption that $D_{\rm ext}$ is distinguished:.

Proposition 5.3. *With the notations as in 5.1, assume that the extended divisor $D_{\rm ext}$ of $(\tilde{V}, D)$ is rigid. After replacing $(\bar{V}', D')$ by $(\bar{V}'^{\vee}, D'^{\vee})$, if necessary, the corresponding extended divisors are isomorphic as reduced curves under an isomorphism $\tilde{f} : D_{\rm ext} \to D'_{\rm ext}$ with $\tilde{f}(D) = D'$ preserving the weights.*

Proof. As in the proof above the family of extended divisors $\mathcal{D}_{\rm ext}$ has the same dual graph over each point of $\mathcal{R} \cong \mathbb{A}^m$. Since the fibers of $\mathcal{D}_{\rm ext}$ are trees of rational curves with at least 2 components, the result is immediate from Corollary 1.17. □

5.2. **Uniqueness of $\mathbb{C}^*$-actions.** In Theorem 5.4 below we deduce part (1) of Theorem 0.2.

Theorem 5.4. *Let $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ be a non-toric normal Gizatullin surface satisfying one of the following two conditions.*

(α_*) $\operatorname{supp}\{D_+\} \cup \operatorname{supp}\{D_-\}$ *is empty or consists of one point, say p, where*

$$D_+(p) + D_-(p) \le -1 \quad \text{or} \quad \{D_+(p)\} \ne 0 \ne \{D_-(p)\}.$$

(β) $\operatorname{supp}\{D_+\} = \{p_+\}$ *and* $\operatorname{supp}\{D_-\} = \{p_-\}$ *for two distinct points p_+, p_-, where*

$$D_+(p_+) + D_-(p_+) \le -1 \quad \text{and} \quad D_+(p_-) + D_-(p_-) \le -1\,.$$

Then the $\mathbb{C}^*$-action on V is unique, up to conjugation in the group $\operatorname{Aut}(V)$ and up to inversion $\lambda \mapsto \lambda^{-1}$ in $\mathbb{C}^*$. Moreover the given $\mathbb{C}^*$-action is conjugate to its inverse if and only if there is an automorphism $\psi : \mathbb{A}^1 \to \mathbb{A}^1$ such that

$$\psi^*(D_-) - D_+ \text{ is integral and } \psi^*(D_+ + D_-) = D_+ + D_-\,.$$

Proof. Let $\Lambda, \Lambda' : \mathbb{C}^* \times V \to V$ be two $\mathbb{C}^*$-actions on V, where Λ is the given one. We consider the corresponding equivariant standard completions $(\bar{V}, D)$ and $(\bar{V}', D')$ of V. After reversing the first one, if necessary, its extended divisor $D_{\rm ext}$ is rigid according to Theorem 3.24. Applying Proposition 5.3, after reversing $(\bar{V}', D')$, if necessary, the extended divisors $D_{\rm ext}$ and $D'_{\rm ext}$ are isomorphic. Since by Proposition 5.12 in [FKZ$_2$] and its proof a non-toric Gizatullin $\mathbb{C}^*$-surface is uniquely determined by its extended divisor, the first part follows. The second one is a consequence of Lemma 3.14. □

Applying Theorem 5.4 to smooth Gizatullin $\mathbb{C}^*$-surfaces, we obtain the following

Corollary 5.5. *If $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ is a non-toric smooth Gizatullin $\mathbb{C}^*$-surface, then its $\mathbb{C}^*$-action is uniquely determined up to conjugation and inversion unless its standard zigzag is*

$$[[0, 0, (-2)_{s-2}, w_s, (-2)_{n-s}]]\,, \quad \text{where} \quad w_s \le -2,\ n \ge 4 \text{ and } 2 \le s \le n\,. \tag{39}$$

Proof. Suppose that $\operatorname{supp}\{D_+\} \subseteq \{p_+\}$ and $\operatorname{supp}\{D_-\} \subseteq \{p_-\}$. If $p_+ = p_- =: p$ and $\{D_+(p)\} \ne 0 \ne \{D_-(p)\}$ then by Theorem 5.4 the $\mathbb{C}^*$-action is unique up to conjugation and inversion. Otherwise either $p_+ \ne p_-$ or one of the fractional parts $\{D_+\}$, $\{D_-\}$ vanishes. Anyhow the smoothness of V implies the desired form (39) of the dual graph of D, see Remark 3.11(2). □

5.3. **Uniqueness of $\mathbb{A}^1$-fibrations.**

5.6. In this subsection we consider a normal Gizatullin surface V with a fixed standard completion $(\bar{V}, D)$, where $D = [[0, 0, w_2, \ldots, w_n]]$ is a zigzag with irreducible components $C_0, \ldots, C_n$. As usual the linear system $|C_0|$ defines an $\mathbb{A}^1$-fibration $\Phi_0 : V \to \mathbb{A}^1$. Reversion as in 1.4 provides the standard completion $(\bar{V}^\vee, D^\vee)$ so that $D^\vee$ has irreducible components $C_0^\vee, \ldots, C_n^\vee$ with self-intersections $[[0, 0, w_n, \ldots, w_2]]$. The linear system $|C_0^\vee|$ defines a second $\mathbb{A}^1$-fibration $\Phi_0^\vee : V \to \mathbb{A}^1$, which we call the *reverse fibration*. We say that two $\mathbb{A}^1$-fibrations $\varphi, \varphi' : V \to \mathbb{A}^1$ are *conjugate* if $\varphi' = \beta \circ \varphi \circ \alpha$ for some automorphisms α of V and β of $\mathbb{A}^1$.

In Theorem 5.10 below we give a partial answer to the following problem.

Problem 5.7. *Suppose that V is not a Danilov-Gizatullin surface. Is then every $\mathbb{A}^1$-fibration $\varphi : V \to \mathbb{A}^1$ conjugate to one of the standard $\mathbb{A}^1$-fibrations $\Phi_0, \Phi_0^\vee$?*

A complete answer is known in the case of normal Gizatullin surface with a finite divisor class group [DR]. The problem above is actually equivalent to the uniqueness problem for $\mathbb{C}_+$-actions on V in the sense of (3) and (4) below. Let us recall some standard facts concerning $\mathbb{C}_+$-actions.

5.8. (1) ([Re]) If $\mathbb{C}_+$ acts on an affine algebraic $\mathbb{C}$-scheme $V = \operatorname{Spec} A$ then the associated derivation ∂ on A is locally nilpotent, i.e. for every $f \in A$ we can find $n \in \mathbb{N}$ such that $\partial^n(f) = 0$. Conversely, given a locally nilpotent $\mathbb{C}$-linear derivation $\partial : A \to A$ the map $\varphi : \mathbb{C}_+ \times A \to A$ with $\varphi(t, f) := e^{t\partial} f$ defines an action of $\mathbb{C}_+$ on V.

(2) (See e.g., [ML_1, Zai]) Assume that A as in (1) above is a domain and let $\partial \in \operatorname{Der}_{\mathbb{C}} A$ be a locally nilpotent derivation of A. Then the subalgebra $\ker \partial = A^{\mathbb{C}_+} \subseteq A$ is algebraically and factorially closed, or inert[22], in A, and the field extension $\operatorname{Frac}(\ker \partial) \subseteq \operatorname{Frac} A$ has transcendence degree 1. Moreover for any $u \in \operatorname{Frac} A$ with $u\partial(A) \subseteq A$, the derivation $u\partial \in \operatorname{Der}_{\mathbb{C}} A$ is locally nilpotent if and only if $u \in \operatorname{Frac}(\ker \partial)$.

If A as in (1) above is normal then the ring of invariants $A^{\mathbb{C}_+}$ is normal too. If $\dim A \leq 3$ then by a classical result of Zariski [Zar] $A^{\mathbb{C}_+}$ is finitely generated and $C = \operatorname{Spec} A^{\mathbb{C}_+}$ is the algebraic quotient $V//\mathbb{C}_+$. Thus the orbit map $V \to C$ provides an $\mathbb{A}^1$-fibration.

(3) Conversely if a normal affine surface V admits an $\mathbb{A}^1$-fibration $V \to C$ over a smooth affine curve C, then there exists a non-trivial regular $\mathbb{C}_+$-action on V along this fibration. It is unique up to multiplication of an infinitesimal generator ∂ with an element $u \in \operatorname{Frac}(\ker \partial)$ as in (2) above.

(4) As mentioned in the introduction, every normal affine surface V which is not a Gizatullin surface admits at most one $\mathbb{A}^1$-fibration over $\mathbb{A}^1$, see [BML].

We restrict in the sequel to $\mathbb{A}^1$-fibrations on Gizatullin surfaces. Let us provide several examples of such fibrations.

Example 5.9. 1. Let $V = \mathbb{C}[t][D_+, D_-]$ be a Gizatullin $\mathbb{C}^*$-surface. Taking in 5.6 an equivariant standard completion the $\mathbb{A}^1$-fibrations Φ_0, $\Phi_0^\vee$ on V are equivariant with respect to suitable $\mathbb{C}^*$-actions on $\mathbb{A}^1$. By Proposition 3.25 in [FlZa_2], they are given by

[22]The latter means that $ab \in \ker \partial \Rightarrow a, b \in \ker \partial$.

two homogeneous elements

$$v_+ : V \to \mathbb{A}^1 \quad \text{and} \quad v_- : V \to \mathbb{A}^1 \tag{40}$$

of positive and negative degree, respectively. Moreover, by *loc. cit.* any other $\mathbb{A}^1$-fibration $\varphi : V \to \mathbb{A}^1$ compatible with the $\mathbb{C}^*$-action on V is equal to v_+ or v_-.

2. The toric surface $V_{d,e} = \mathbb{A}^2/\!/\mathbb{Z}_d$ (see 1.8) admits many hyperbolic $\mathbb{C}^*$-actions. Indeed, for any coprime integers a, b the action $t.(x, y) := (t^a x, t^b y)$, $t \in \mathbb{C}^*$, on $\mathbb{A}^2$ descends to V, and in the case where $ab < 0$ it is hyperbolic. Up to a twist, the $\mathbb{A}^1$-fibrations $v_\pm : V \to \mathbb{A}^1$ are induced by the projections $(x, y) \mapsto x$, $(x, y) \mapsto y$, respectively.

3. Let now $V = V_{k+1}$ be a Danilov-Gizatullin surface, see [FKZ$_2$], section 5.3. According to *loc.cit.*, Corollary 5.16(b) V carries at least $\lfloor \frac{k+1}{2} \rfloor$ pairwise non-conjugate $\mathbb{A}^1$-fibrations $V_{k+1} \to \mathbb{A}^1$.

The following theorem is the main result of this subsection.

Theorem 5.10. *Let V be a Gizatullin surface with a distinguished and rigid extended divisor*[23] *D_{ext}. Then every $\mathbb{A}^1$-fibration $\varphi : V \to \mathbb{A}^1$ is conjugate to one of Φ_0, $\Phi_0^\vee$.*

Before starting the proof, let us make the following observation.

5.11. Consider a semistandard completion[24] $(\bar{V}', D')$ of a Gizatullin surface V, where $D' = C_0' + \ldots + C_n'$ and $(C_0')^2 = 0$. Then the linear system $|C_0'|$ defines a morphism $\Phi_0' : \bar{V}' \to \mathbb{P}^1$ which restricts to an $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$.

Conversely, we claim that *any $\mathbb{A}^1$-fibration $\varphi : V \to \mathbb{A}^1$ is induced by the standard $\mathbb{A}^1$-fibration of a suitable standard completion $(\bar{V}, D)$ of V.* Indeed, given an $\mathbb{A}^1$-fibration $\varphi : V \to \mathbb{A}^1$, there exists an effective $\mathbb{C}_+$-action on V along this fibration, see 5.8(3). By virtue of Lemma 1.3(c) one can find an equivariant semistandard completion $(\bar{V}', D')$ of V such that φ extends to a morphism $\varphi' : \bar{V}' \to \mathbb{P}^1$. Performing a sequence of elementary transformations with centers at the fiber C_0' of φ', one can reach a standard completion, say, $(\bar{V}, D)$ of V, where this time $D = C_0 + \ldots + C_n$ with $C_0^2 = C_1^2 = 0$. The morphism $\Phi_0 : \bar{V} \to \mathbb{P}^1$ defined by the linear system $|C_0|$ restricts again to $\varphi : V \to \mathbb{A}^1$.

Proof of Theorem 5.10. We let as in 5.6 $(\bar{V}, D)$ denote the standard completion of V with standard $\mathbb{A}^1$-fibration Φ_0, and we let $(\bar{V}', D')$ denote another such standard pair with standard morphism as in 5.11 inducing the given fibration $\varphi : V \to \mathbb{A}^1$.

Since by our assumption the extended divisor D_{ext} is distinguished and rigid, Theorem 5.2 applies. By this theorem, $(\bar{V}, D)$ is isomorphic to one of the pairs $(\bar{V}', D')$, $(\bar{V}'^\vee, D'^\vee)$ or, equivalently, $(\bar{V}', D')$ is isomorphic to one of $(\bar{V}, D)$, $(\bar{V}^\vee, D^\vee)$. In particular φ is conjugate to Φ_0 or $\Phi_0^\vee$ under this isomorphism. □

The following lemma shows that the extended divisor is uniquely determined by φ.

Lemma 5.12. *Let $(\bar{V}, D)$ and $(\bar{V}', D')$ be two standard completions of the same Gizatullin surface V. If the associated $\mathbb{A}^1$-fibrations $\Phi_0, \Phi_0' : V \to \mathbb{A}^1$ are conjugate then there is an isomorphism $f : D_{\mathrm{ext}} \to D'_{\mathrm{ext}}$ of the corresponding extended divisors (regarded as reduced curves) with $f(D) = D'$, which preserves the weights.*

[23]This is fulfilled for instance if the assumptions of Theorem 2.17 hold.

[24]See 1.2.

Proof. We may assume that the automorphism of V which conjugates Φ_0 and Φ'_0 extends to a birational map $\tilde f : \tilde V \dashrightarrow \tilde V'$ of the minimal resolutions of $\bar V$, $\bar V'$ with $\Phi'_0 \circ \tilde f = \Phi_0$. If $D = C_0 + \ldots + C_n$ and $D' = C'_0 + \ldots + C'_n$ then clearly $\tilde f$ is regular at the points of $C_1 \backslash (C_0 \cup C_2)$. Performing elementary transformations on $\tilde V$ with centers at C_0, if necessary, we may suppose that $\tilde f$ is biregular along C_0, so that $\tilde f^{-1}$ is also regular along $(C'_0 \cup C'_1) \backslash C'_2$. Contracting the divisors[25] $C_2 + \ldots + C_n$ and $C'_2 + \ldots + C'_n$ on the surfaces $\tilde V$ and $\tilde V'$ to singular points p, p', respectively, yields two normal surfaces W and W'. Moreover $\tilde f$ induces a birational map $\bar f : W \to W'$ which is an isomorphism outside p, p'. By the Riemann extension theorem $\bar f$ is actually an isomorphism. Then also $\tilde f$, obtained from $\bar f$ via minimal resolution of singularities, is an isomorphism. Hence $\tilde f$ induces an isomorphism of the boundaries and the extended divisors of the two completions. Since $(C'_1)^2 = 0$, also $C_1^2 = 0$ and so the standard zigzag D remains the same under the elementary transformations above. Now the lemma follows. □

Let us apply these results to a $\mathbb{C}^*$-surface $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$. In this case we choose in 5.6 the equivariant standard completion $(\bar V, D)$ so that Φ_0 and $\Phi_0^\vee$ are equivariant. The next result yields part (2) of Theorem 0.2.

Corollary 5.13. *We let $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ be a Gizatullin $\mathbb{C}^*$-surface. If one of the conditions (α_+), (β) of 3.16 is fulfilled, then the following hold.*

(1) Every $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ is conjugate to one of Φ_0 or $\Phi_0^\vee$.

(2) Assume furthermore that V is non-toric. The $\mathbb{A}^1$-fibrations Φ_0, $\Phi_0^\vee$ are then conjugate if and only if $\{D_+(p_+)\} = \{D_-(p_-)\}$ and the divisor $D_+ + D_-$ is stable under an automorphism of $\mathbb{A}^1$ interchanging p_+ and p_-. In the latter case up to conjugation there is only one $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$.

Proof. By Theorem 3.24 under our assumptions the extended divisor D_{ext} is distinguished and rigid. So (1) follows directly from Theorem 5.10. To deduce (2), assume first that $\{D_+(p_+)\} = \{D_-(p_-)\}$ and $D_+ + D_- = \psi^*(D_+ + D_-)$ for an appropriate automorphism $\psi \in \operatorname{Aut}(\mathbb{A}^1)$ interchanging p_+ and p_-. By Lemma 3.14 the $\mathbb{C}^*$-surfaces $\operatorname{Spec} A_0[D_+, D_-]$ and $\operatorname{Spec} A_0[D_-, D_+]$ with $A_0 = \mathbb{C}[t]$ are isomorphic. This isomorphism interchanges the fibrations v_+ and $\psi \circ v_-$ as in Example 5.9(1). Hence Φ_0, $\Phi_0^\vee$ are conjugate.

Suppose now that Φ_0, $\Phi_0^\vee$ are conjugate. By Lemma 5.12 there is an isomorphism of extended divisors $f : D_{\text{ext}} \to D'_{\text{ext}}$ as reduced curves with $f(D) = D'$ preserving the weights. According to Proposition 5.12 in [FKZ₂] and its proof the $\mathbb{C}^*$-surfaces $\operatorname{Spec} A_0[D_+, D_-]$ and $\operatorname{Spec} A_0[D_-, D_+]$ are equivariantly isomorphic. Now the assertion follows from Lemma 3.14. □

As a particular case we obtain the following result, which was proved in the smooth case by Daigle [Dai] and Makar-Limanov [ML_2].

Corollary 5.14. *Let V be a normal surface in $\mathbb{A}^3_{\mathbb{C}}$ with equation $xy = P(t)$, where $P(t) \neq 0$ is a polynomial. Then every $\mathbb{A}^1$-fibration on V is conjugate to $x : V \to \mathbb{A}^1$.*

Proof. According to Example 4.10 in [FlZa_2], V admits a DPD presentation $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$ with integral divisors $D_+ = 0$ and $D_- = -\operatorname{div}(P)$. Thus condition

[25] Both of them have negatively definite intersection forms.

(α_+) is fulfilled and so the result follows from Corollary 5.13(1,2) in virtue of Remark 3.15. $\square$

Let us finally examine $\mathbb{A}^1$-fibrations of affine toric surfaces.

Proposition 5.15. *The toric surface $V_{d,e} \cong \mathbb{A}^2/\!/\mathbb{Z}_d$ (see 1.8) admits at most 2 conjugacy classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$. Moreover, there is only one such conjugacy class if and only if $e^2 \equiv 1 \mod d$.*

Proof. The DPD presentation of $V_{d,e}$ considered in the proof of Lemma 3.12 satisfies (α_+). Applying Corollary 5.13 gives the first part. To prove the second assertion, we assume first that $e^2 \equiv 1 \mod d$. Using the notations of Example 5.9(1), (2) the affine fibrations Φ_0, $\Phi_0^\vee$ are induced by the projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$. Because of our assumption the map $h : (x, y) \mapsto (y, x)$ satisfies $h(\zeta.(x, y)) = \zeta^e.(y, x)$. Hence h induces an automorphism $\bar{h}$ on the quotient $V_{d,e}$ that interchanges these projections and thus also Φ_0 and $\Phi_0^\vee$.

Conversely assume that the $\mathbb{A}^1$-fibrations Φ_0, $\Phi_0^\vee$ are conjugate in $\mathrm{Aut}(V)$. According to Lemma 5.12 the standard zigzag D of V is symmetric. Due to Lemma 3.12 D and the reversed zigzag $D^\vee$ are given by

$$D: \quad \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\frac{d-e}{d}}{\square}\,, \qquad D^\vee: \quad \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\frac{d-e'}{d}}{\square}\,,$$

where $0 \le e, e' < d$ and $ee' \equiv 1 \mod d$, cf. 1.9. Hence D and $D^\vee$ are equal if and only if $e = e'$ or, equivalently, $e^2 \equiv 1 \mod d$. $\square$

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