PROBLEM SET 0

*__1.__ Show that, in general, the direct image does not commute with intersection and complementation (for subsets of the domain); more precisely check $f(A \cup B)$ vs $f(A) \cup f(B)$, the intersection and complement. Give simple examples/counterexamples.

2. Prove that the existence of the Cartesian product as defined in the text is equivalent to the Axiom of Choice.

Axiom of choice: Let $(A_i)_{i \in I}$ be an arbitrary collection of nonempty sets. There exists $A \subseteq \cup_{i \in I} A_i$ such that $\forall i \in I$ we have $A \cap A_i$ has exactly one element.

*__3.__ Prove that simple induction and strong induction are equivalent.

Hint: Simple induction step $\Rightarrow$ strong induction step so all sets satisfying simple induction satisfy strong induction. The other way around, define $S' = \{n | \{1, 2, \ldots, n\} \subseteq S\}$ and proceed fom there.

*__4.__ Review the construction of $\mathbb{Z}$ with equivalence clases. Justify with details that addition of equivalence classes of $\mathbb{N} \times \mathbb{N}$, with the relation used to define $\mathbb{Z}$, is consistent (sums of representatives of classes add to the representative of the summation class) and that $0 = \widehat{(m, m)}$, $m \in \mathbb{N}$. Here the zero on the left hand side is the additive neutral element of $\mathbb{Z}$.

*__5.__ Prove that the triangle inequality in the definition of the Euclidean norm on $\mathbb{R}^n$, $n \geq 1$ is equivalent to the Cauchy-Schwarz inequality.

6. Show that the principle of mathematical induction is equivalent to the fact that $\mathbb{N}$ is well ordered. *A set is well ordered when any nonempty subset has a minimal element, i.e. an element in the subset less or equal than all other elements in the subset.*

7. State clearly the mening of the following relations between cardinal numbers and then prove them.

1) Sketch the proof that $|\mathbb{N}| \times |\mathbb{N}| = |\mathbb{N}|$, using the second diagonal construction, giving the exact form of the bijection.

*2) Show that a countable union of countable sets is countable.

3) Prove that the rationals in $(0,1)^2$, and then the positive rationals, are countable, with the observation that

$$\mathbb{Q}_+ = \cup_{i=1}^{\infty}\{x = \frac{m}{n} \,|m, n \in \mathbb{N}\ ,\ n \leq i\}\,.$$

*4) Prove that $|C|^{|A|\cdot|B|} = |C|^{|A|}\cdot|C|^{|B|}$. Here $|C|^{|A|} = |C^A|$ and $C^A = \{f \,|\, f : A \to C\}$.

5) $|\{$the set of all sequences of natural numbers$\}| = \mathfrak{c}$

*6) What is the cardinality of the set of sequences of reals? Use cardinal arithmetic like in 4).

*7) $\mathfrak{c}\cdot\mathfrak{c} = \mathfrak{c}$

8. 1) Prove that the existence of the supremum (i.e. completeness of $\mathbb{R}$) implies the Archimedean principle (state it).

2) Prove the existence and uniqueness of the integer part $[x]$ of a real number x, i.e.

"there exists a unique $m \in \mathbb{Z}$, $m \leq x < m+1$" This m is denoted $[x]$.

3) Show that if $0 < a < b$ and $b - a > 1$ then $\exists m \in \mathbb{N}$ such that $a < m < b$.

4) Show that between any two real numbers there is a rational number.

*__9.__ Let ℓ^∞ be the set of sequences $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$ of real numbers with $||\mathbf{a}|| = \sup_{i\geq 1}|a_i| < \infty$.

1) Show that $||\cdot||$ is a norm, and then $d(\mathbf{a},\mathbf{b}) = ||\mathbf{a}-\mathbf{b}||$ is a metric on ℓ^∞.

2) Show that the set C of sequences of zeros and ones is uncountable and the distance between any two distinct elements of C is one.

3) If $B \subseteq \ell^\infty$ is countable, then B cannot be dense in ℓ^∞. (In other words, ℓ^∞ is not separable).

*__10.__ Show that $\mathbb{R}^n$ has the property that, if

$$\mathcal{V}_0 = \{B(x,r)|x = (x_1,\ldots,x_n),\ r > 0 \ \text{are all rational}\}\,,$$

then

$D \subseteq \mathbb{R}^n$ is open iff $\forall\, a \in D \quad \exists V \in \mathcal{V}_0$ such that $a \in V \subseteq D$.