# TRIGONOMETRIC FUNCTIONS - GEOMETRIC APPROACH

There are several ways to define the trigonometric functions rigorously: using power series (later in the text), differential equations (in Chapter 5) and geometric. Only the power series give a fully independent construction. Sine and Cosine, seen as a pair of solutions of a second order ODE, depend on a theorem about the existence and uniqueness of solutions of linear differential equations in two dimensions.

The last construction is natural, going back to elementary geometry. It assumes the notion of arc length on the circle, which will remain purely intuitive, even though it coincides with the one developed in Calculus.

**1.** The circle has a length equal to $\pi$ times its diameter. The ratio $\pi$ does not need to be known. The circle centered at $(0,0)$ with radius one is the *unit circle*.

**2.** For any number $\theta$, we write $\theta = 2\pi n + \theta_0$, where $n = [\theta/2\pi]$, the integer part giving the number of times we need to wind around the unit circle in order to reach an arc length equal to $\theta$, starting from $(1,0)$. The rule is:

- we wind counterclockwise (*trigonometric* sense) if $\theta \geq 0$
- we wind clockwise (*anti-trigonometric* sense) if $\theta < 0$

**3.** Define

$$C(\theta) = x(\theta_0)\,, \qquad S(\theta) = y(\theta_0)$$

where $(x,y)$ are the coordinates of the point on the circle we stop at when winding around to obtain an arc length of $\theta$. Notice that

$$0 \leq \theta_0 < 2\pi\,.$$

These stand for $C = \cos$ and $S = \sin$.

**4.** Establish the periodicity

$$C(\theta + 2\pi) = C(\theta)\,, \qquad S(\theta + 2\pi) = S(\theta)$$

and the symmetries

$$C(-\theta) = C(\theta)\,, \qquad S(-\theta) = -S(\theta)$$
$$C(\theta + \pi) = -C(\theta)\,, \qquad S(\theta + \pi) = -S(\theta)$$
$$C(\frac{\pi}{2} - \theta) = S(\theta)\,, \qquad S(\frac{\pi}{2} - \theta) = -C(\theta)$$

which show that it is enough to know the functions on $[0,\pi]$.

**5.** Use the formula for the area of a triangle to show that

$$\mathit{Area} = \frac{1}{2} \cdot \text{product of adjacent sides} \cdot S(\theta)\,, \qquad \theta = \text{angle between sides}$$

**6.** Draw a line through the vertex $A$ of a triangle $\triangle ABC$ and denote $D$ the point it intersects the opposite side. Let $\theta_1$ and $\theta_2$ the angles formed at the same vertex. Using areas show

$$S(\theta_1 + \theta_2) = S(\theta_1)C(\theta_2) + C(\theta_1)S(\theta_2)$$

and the corresponding relation for $C$.

**7.** The relations **7** can be extended to any angles.

**8.** Using periodicity and symmetries, show continuity.

**9.** Calculate

$$\lim_{\theta \to 0} \frac{S(\theta)}{\theta} = 1$$

using areas and the squeeze theorem.

**10.** Show that $S' = C$ and $C' = -S$ with $C(0) = 1$, $S(0) = 0$.

**11.** Prove that functions satisfying **10** on $\mathbb{R}$ are unique.

**11.** Define all other trig functions and using the theorems about inverse functions, determine their derivatives.