LEFT - AND RIGHT SIDE CONTINUITY

This solves problem 4 in 3.1.

Continuity to the right. Let $f : D \to \mathbb{R}$ and $x_0 \in D$. Let $D_+ = D \cap [x_0, \infty)$. If f is continuous at x_0 as a function on D_+ we say it is right-continuous at x_0.

This definition is equivalent to

$$\text{(Right)} \qquad \forall (x_n) \subseteq D,\, x_n \geq x_0,\, x_n \to x_0 \text{ implies } f(x_n) \to f(x_0)$$

Continuity to the left. Let $f : D \to \mathbb{R}$ and $x_0 \in D$. Let $D_- = D \cap (-\infty, x_0]$. If f is continuous at x_0 as a function on D_- we say it is left-continuous at x_0.

This definition is equivalent to

$$\text{(Left)} \qquad \forall (x_n) \subseteq D,\, x_n \leq x_0,\, x_n \to x_0 \text{ implies } f(x_n) \to f(x_0)$$

These definitions coincide with the familiar concepts from elementary calculus.

Theorem. The function f is continuous at x_0 if and only if it is both right- and left continuous at x_0.

Proof.
$\Rightarrow$ is simple, because f continuous means

$$\forall (x_n) \subseteq D,\, x_n \to x_0 \text{ implies } f(x_n) \to f(x_0)$$

in other words, without any specification on whether $x_n \geq x_0$ or $x_n \leq x_0$, so it is valid for both.

$\Leftarrow$ Let $(x_n) \subseteq D$, $x_n \to x_0$. Suppose $f(x_n)$ does not converge to $f(x_0)$. Then
$\exists \epsilon > 0 \quad \forall N \quad \exists n \geq N \quad |f(x_n) - f(x_0)| \geq \epsilon$.
This implies that either there exist infinitely many terms of the sequence with $x_n \in D_+$, or infinitely many with $x_n \in D_-$ such that $|f(x_n) - f(x_0)| \geq \epsilon$. In either case, there exists a subsequence x_{n_k}, $k \geq 1$, with all terms in only one of D_+ or D_-, such that $x_{n_k} \to x_0$ as $k \to \infty$ and $|f(x_n) - f(x_0)| \geq \epsilon$. But such a sequence violates one of the conditions $(Right)$ or $(Left)$ from above. Contradiction: the function must be continuous at x_0.

Examples

1. $f(x) = x$, $x < 0$ and $f(x) = x + 1$ if $x \geq 0$ is not continuous at $x_0 = 0$ because f is discontinuous on $D_- = (-\infty, 0]$ since $f(0) = 1$ and not 0 as it should be according to the left hand side component.

2. Let g, h be continuous functions on $D = \mathbb{R}$. Then $f(x) = g(x)$, $x < x_0$ and $f(x) = h(x)$ if $x \geq x_0$ is continuous at x_0 if and only if $g(x_0) = h(x_0)$.

3. Let g, h be continuous functions on $D = \mathbb{R}$. Then $f(x) = g(x)$, $x \geq x_0$ and $f(x) = h(x)$ if $x > x_0$ is continuous at x_0 if and only if $g(x_0) = h(x_0)$.