LIMIT POINTS AND CONVERGENCE

Definition. A number $a$ is said a limit point of $(x_n)$ if there exists a subsequence of $(x_n)$ convergent to $a$.

Proposition. If $(x_n)$ is convergent, then any subsequence is convergent to the same limit.

In other words, if $(x_n)$ converges to $x$, then $x$ is the only limit point of $(x_n)$.

Let $L$ be the set of all limit points of $(x_n)$. If the sequence is bounded, then $L$ is bounded and has both a maximum value and a minimum value. More precisely, inf $L$ and sup $L$ belong to $L$: there exist subsequences converging to each of them.

All statements below are mini-exercises if you want to test yourself.

Examples.
$x_n \to x$ then $L = \{x\}$.
$x_n = (-1)^n$, $L = \{-1, 1\}$ just two points
$x_n = \sin(\frac{n\pi}{p})$, $p$ positive integer will have a finite number of limit points depending on $p$.
$x_n = \{pn\}$, where $\{x\} = x - [x]$ is the fractional part of $x$: $L$ has a finite number of values if $\rho \in \mathbb{Q}$ and $L = [0, 1]$ if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ i.e. irrational.

Let $x_n, y_n$ be two convergent sequences, to $x$, respectively $y$. Construct $z_n = x_n$ when $n$ is even and $z_n = y_n$ when $n$ is odd. Show that 1) if $x = y$ then $L = \{x\}$ and $(z_n)$ is convergent; 2) if $x \neq y$ then $L = \{x, y\}$ and $(z_n)$ is not convergent.

Consequence. A sequence is not convergent (i.e. divergent) if and only if:
1) either it is unbounded,
2) or it is bounded but has two convergent subsequences converging to different limits.

If we allow the possibility of infinite limit points in the extended real numbers $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ then inf $L$ and sup $L$ always exist, possibly with infinite values, with no assumption on the boundedness of the sequence.

Formulas for $\limsup$ and $\liminf$. An alternative formula is
$$\sup L = \limsup_{n \to \infty} x_n = \inf_{k \geq 0} \sup_{n \geq k} x_n$$
for $\limsup$ and
\[
\inf L = \lim \inf_{n \to \infty} x_n = \sup_{k \geq 0} \inf_{n \geq k} x_n
\]

for \( \liminf \).

To better understand these formulas, we notice that they are defined in terms of regular sup and inf of sets (in this case the range of values of sequences).

Fix an index \( k \). Consider \( b_k = \sup_{n \geq k} x_n \) taken over all \( n = k, k+1, k+2, \ldots \). This is a new sequence indexed by \( k \geq 0 \). Since it is a supremum, the smaller the set, the smaller the supremum. This implies that \( b_{k+1} \leq b_k \). Now we take the limit of this non-increasing sequence. This is \( \sup L \), the limit superior of \( (x_n) \).

Other properties.
The set \( L \) is a closed set, i.e. any convergent sequence of points in \( L \) has limit in \( L \).

A sequence is bounded above if and only if \( \sup L < \infty \).

A sequence is bounded below if and only if \( \inf L > -\infty \).

A sequence is bounded if both \( \inf L \) and \( \sup L \) are real numbers (i.e. finite).

A sequence has limit if \( \inf L = \sup L \). The limit is equal to their common value.

A sequence is convergent if it is bounded and \( \inf L = \sup L \).

For any \( \epsilon > 0 \), there are at most finitely many terms outside the interval

\( (\inf L - \epsilon, \sup L + \epsilon) \).

Remark. Let \( R = \{x_n | n \in \mathbb{N}\} \) be the range of the sequence. Let \( M > 0 \) such that \( |x_n| \leq M \).

Then

\[-M \leq \inf R \leq \inf L \leq \sup L \leq \sup R \leq M\]

but all inequalities may be strict! The reason is that we can change the maximum (minimum) value of a sequence by modifying only one term, while the limit points depend only on behavior for infinitely many terms.

If the sequence is bounded nondecreasing, then \( \min R \) exists and \( \max R \) may or may not exist (give examples), and \( \sup R = \sup L = \inf L \).

Other properties.

\[
\lim \sup_{n \to \infty} (-x_n) = -\lim \inf_{n \to \infty} x_n
\]

\[
c > 0, \quad \lim \sup_{n \to \infty} (cx_n) = c \lim \sup_{n \to \infty} x_n
\]
\[ \lim \sup_{n \to \infty} (x_n + y_n) \leq \lim \sup_{n \to \infty} x_n + \lim \sup_{n \to \infty} y_n \]

\[ \lim \inf_{n \to \infty} (x_n + y_n) \geq \lim \inf_{n \to \infty} x_n + \lim \inf_{n \to \infty} y_n \]

These are consequences of the formulas (easy to prove) for the supremum:

\[ \sup_{n} (x_n + y_n) \leq \sup_{n} x_n + \sup_{n} y_n \]

\[ \sup_{n} (-x_n) = - \inf_{n} x_n \]

How to use \( \lim \sup \) and \( \lim \inf \). Usually we apply this form of the squeeze theorem.

A non-negative sequence \((u_n)\) converges to zero if and only if \( \lim \sup_{n \to \infty} u_n = 0 \).

**Example.** New proof of the Cesaro theorem.

If \( x_n \to x \in \mathbb{R} \), then

\[ \sigma_n = \frac{\sum_{k=1}^{n} x_k}{n} \to x \]

as well.

**Proof.** We know that

\[ \forall \epsilon > 0 \exists N = N_\frac{x}{2} \forall n \geq N_\frac{x}{2} \quad |x_n - x| < \frac{x}{2} \]

which implies (like before) that for a given \( \epsilon \),

\[ \forall n \geq N_\frac{x}{2} \quad |\sigma_n - x| \leq \frac{2MN_\frac{x}{2}}{n} + \frac{\epsilon}{2}. \]

We take the \( \lim \sup_{n \to \infty} \) on the left hand side for \( u_n = |\sigma_n - x|, \) denoting it by \( \sup L \) and then the \( \lim \sup_{n \to \infty} \) on the right hand side. On the right hand side only the first term depends on \( n \), so

\[ \sup L = \lim \sup_{n \to \infty} |\sigma_n - x| \leq \lim \sup_{n \to \infty} \left( \frac{2MN_\frac{x}{2}}{n} + \frac{\epsilon}{2} \right) \]

\[ = 2MN_\frac{x}{2} \lim \sup_{n \to \infty} \left( \frac{1}{2} \right) + \lim \sup_{n \to \infty} \left( \frac{\epsilon}{2} \right) \]

\[ = 2MN_\frac{x}{2} \cdot 0 + \frac{\epsilon}{2} = \frac{\epsilon}{2}. \]

But \( \sup L \geq 0 \) since \( |\sigma_n - x| \geq 0 \). Now for any \( \epsilon \)

\[ 0 \leq \sup L \leq \frac{\epsilon}{2} \]

which implies \( \sup L = 0 \), so \( \lim_{n \to \infty} |\sigma_n - x| = 0 \) which is equivalent to \( \lim_{n \to \infty} \sigma_n = x \).
Cauchy sequences.

A sequence \((x_n)\) has the Cauchy property if
\[
\forall \epsilon > 0 \ \exists N = N_\epsilon \ \forall m, n \geq N_\epsilon \quad |x_m - x_n| < \epsilon.
\]

**Theorem.** A sequence has the Cauchy property if and only if it is convergent.

**Remark.** The importance of the Cauchy property is to characterize a convergent sequence without using the actual value of its limit, but only the relative distance between terms.

**Proof.** Cauchy \(\Rightarrow\) convergent.

A Cauchy sequence is bounded. Pick \(\epsilon = 1\) and \(N_1\) the corresponding rank. Choose \(m = N_1\) and \(n \geq N_1\). Then \(|x_n - x_{N_1}| \leq 1 \ \forall n \geq N_1\). Then \(M = \max\{|x_{N_1}| + 1, |x_0|, |x_1|, \ldots, |x_{N_1-1}|\}\). It follows that it has a convergent subsequence \((x_{n_k})\). Let \(x\) be its limit. It is always true that
\[
|x_n - x| < |x_n - x_{n_k}| + |x_{n_k} - x|.
\]
For a given \(\epsilon > 0\), choose \(N_{\epsilon/2}\) from the Cauchy property. Since \(n_k\) is strictly increasing, \(n_k \geq k\) and so \(n_k \geq N_{\epsilon/2}\) as soon as \(k \geq N_{\epsilon/2}\).

Thus \(\forall n, k \geq N_{\epsilon/2}, \ |x_n - x_{n_k}| < \frac{\epsilon}{2}\).

We denote the ranks of the subsequence with \(k\) and \(K\). Pick \(K = K_{\epsilon/2}\) the rank in the definition of convergence of the subsequence such that \(\forall k \geq K_{\epsilon/2}, \ |x_{n_k} - x| < \frac{\epsilon}{2}\).

Choose an index \(k \geq N'_{\epsilon} = \max\{N_{\epsilon/2}, K_{\epsilon/2}\}\). For all \(n \geq N'_{\epsilon}\), we have
\[
|x_n - x_{n_k}| < \frac{\epsilon}{2} \text{ and } |x_{n_k} - x| < \frac{\epsilon}{2}.
\]
This means that for a given \(\epsilon\) we found a rank \(N = N'_{\epsilon}\) such that \(\forall n \geq N'_{\epsilon}\) we have \(|x_n - x| < \epsilon\).
But this is definition of convergence to \(x\).

**Convergent \(\Rightarrow\) Cauchy.** This is much easier. Choose \(N_{\epsilon}^{Cauchy} = N_{\epsilon/2}\) and we are done.

**Applications.**

1) If \((x_n)\) has at least two limit points \(a\) and \(b\), then the Cauchy property shows that the sequence is divergent (show the details).

2) A series \(x_n = \sum_{k=1}^{n} \frac{c_k}{k^2}\) with \((c_k)\) bounded is a Cauchy sequence, and thus convergent, even though we cannot apriori say anything about the limit.

3) Fixed point theorem. Let \(f : [0, 1] \to \mathbb{R}\) have the property \(|f(x) - f(y)| \leq c|x - y|, \ c \in [0, 1]\) for all \(x, y \in [0, 1]\). Then the equation \(f(x) = x\) has exactly one solution \(x \in [0, 1]\). Hint: we’ll do this problem in Chapter 3 (Continuity). It is based on the construction \(x_n = f(x_{n-1}), \ n \geq 1, \) with \(x_0\) arbitrary in \([0, 1]\).

4) Cauchy sequences provide an alternative construction of the real numbers, by identifying numbers with classes of Cauchy sequences with the same limit.