THE SQUARE ROOT OF ANY $c > 0$ EXISTS IN $\mathbb{R}$

Let $c > 0$. Then $\sqrt{c}$ exists in $\mathbb{R}$.

Proof.

1) We first show that in general, for any two positive real numbers

$$x < y \quad \Leftrightarrow x^2 < y^2$$

This is because $y^2 - x^2 = (y - x)(y + x)$ and $y + x > 0$, implying that both sides of the equation have the same sign (here we used the axioms of compatibility for the order relation).

This also proves that if the square root exists, it is unique.

Define $A = \{p \in \mathbb{Q} \,|\, p > 0\,, \quad p^2 < c\}$

Notice that $A \neq \emptyset$ because if $c > 1$ then $1 \in A$ and if $c < 1$ then there exists a rational $p > 0$, $p < c$ and then $p^2 < c^2 < c$. If $c = 1$ then $1^2 = 1$ and there is nothing to prove.

2) $\forall\, p \in A \qquad (c + 1)^2 > c > p^2$ implies that $\forall\, p \in A \qquad c + 1 > p$ so A has an upper bound equal to $c + 1$. This implies $a = \sup A$ is a real number.

<u>We shall show that $a^2 = c$.</u>

3) If $a^2 > c$, then there exists $r > 0$ such that $\forall\, p \in A \qquad a - r \geq p$.

You should verify that any $r = \frac{a^2 - c}{q + a}$, where q is such that $q^2 \geq c$ satisfies our requirements. This implies that $a - r$ is also an upper bound of A, yet $a - r < a$. This is a contradiction, so $a^2 \leq c$.

4) If $a^2 < c$ then we shall find $r' > 0$ such that $(a + r')^2 < c$.

You should verify that $r' = -r$ is a good choice. Notice that now $r < 0$, which makes $r' > 0$. Between a and $a + r'$ there exists $p \in \mathbb{Q}$. Then

$$a^2 < p^2 < (a + r')^2 < c \quad \Rightarrow \quad p \in A$$

which contradicts $a = \sup A$.

5) In $\mathbb{R}$ only one of

$$a^2 < c \qquad a^2 = c \qquad a^2 > c$$

is true and we have shown that the first and last lead to a contradiction. Hence $a^2 = c$.

Can you prove using similar arguments that $\sqrt[n]{c}$ exists for n a natural number $n \geq 1$? (problems 17-18 in section 1.1 in Fitzpatrick).