Midterm # 1.

MTH433

Advanced Calculus

six

Do ~~five~~ of the following problems.
Show your work.

P 1. a) Negate the logical proposition using quantifiers

P: " For all real x and y, if $x^2 < y^2$, then $x < y$." Is P true or false?

$\exists x \; \exists y \; (x^2 < y^2) \wedge (x \geq y)$ is the negation

P false. $x = 1$ $y = -2$ counterexample

b) Let $f : A \to B$, $g : B \to C$. Show that if $g \circ f$ surjective, then g surjective.

Let $c \in C$ then: $g \circ f$ surj $\Rightarrow \exists \; a \in A$ s.t.

$c = g \circ f(a) = g(f(a))$. But $f(a) \in B$ let's denote $b = f(a) \in B$. Then $\exists \; b \in B$ s.t.

$g(b) = c$.

c) State the Archimedean principle.

$\forall x \in \mathbb{R} \quad \exists \; n \in \mathbb{N} \quad x < n$.

P 2. Prove that there exists a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

We construct $f : \mathbb{N} \to \mathbb{Z}$ bijective.

If $n = 2m$, $m \geq 1$ (even)

we set $f(n) = -m = -\frac{n}{2}$.

If $n = 2m-1$, $m \geq 1$ (odd)

we set $f(n) = m-1 = \frac{n+1}{2} - 1 = \frac{n-1}{2}$.

In this way:

(A) f surjective. Let $k \in \mathbb{Z}$. If $k < 0$ then

$-\frac{n}{2} = k \iff n = -2k > 0$ ✓

If $k = 0$ it is true that $f(1) = 0$ $n = 1$ (odd)

If $k > 0$ then $\frac{n-1}{2} = k \iff n = 2k+1$.

These are all odd numbers $n \geq 3$.

(B) f injective. $f(n_1) = f(n_2)$.

$\Rightarrow$ both are positive or both are negative, or both zero.

If both positive, then they are odd

and so $\frac{n_1 - 1}{2} = \frac{n_2 - 1}{2} \Rightarrow n_1 = n_2$

If both are negative, then they are even

and so $-n_1/2 = -n_2/2 \Rightarrow n_1 = n_2$

If both are zero then $n_1 = n_2 = 1$.

P 3. Find all numbers $n \in \mathbb{N}$ such that $2^n > n^2 + n$. Justify your answer by induction.

- We notice that $n_0 = 5$ satisfies the inequality. This is our verification step.
- $P(n) \Rightarrow P(n+1)$. We would like to show

$$2^{n+1} > (n+1)^2 + (n+1) = n^2 + 3n + 2$$

Since $2^n > n^2 + n \Rightarrow 2 \cdot 2^n = 2^{n+1} > 2n^2 + 2n$

This would be enough if $2n^2 + 2n \geq n^2 + 3n + 2$

$\Leftrightarrow n^2 - n - 2 \geq 0$. This has roots:

$$\frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases} \qquad (n+1)(n-2) \geq 0$$

for $n \geq 2$. So for sure it is satisfied when $n \geq n_0 = 5 \geq 2$.

We proved that $P(n)$ true for $n \geq 5$.

$n = 1, 2, 3, 4$ will be checked one by one.

$P(1)$ $2 > 2$ false

$P(2)$ $4 > 6$ false

$P(3)$ $8 > 12$ false

$P(4)$ $16 > 20$ false

Answer $\{n \in \mathbb{N} \mid n \geq 5\}$

P 4. a) Determine

$$\sup\{\frac{a}{2a+1} \mid a \in A\},$$

where $A \subseteq \mathbb{N}$ and A unbounded. Is this a maximum?

$a \in A \subseteq \mathbb{N} \Rightarrow a \geq 1 > 0$ So $\frac{a}{2a+1} < \frac{1}{2}$

because $2a < 2a+1$. So $\frac{1}{2}$ is an upper bound.

$\frac{1}{2}$ = supremum. <u>Proof</u> Let $\varepsilon > 0$ and $\frac{1}{2} - \varepsilon$.

$\Rightarrow \exists a \in A \quad \frac{a}{2a+1} > \frac{1}{2} - \varepsilon$. Why? This is

equivalent to $\varepsilon > \frac{1}{2} - \frac{a}{2a+1} = \frac{1}{2(2a+1)} \Leftrightarrow 2\varepsilon > \frac{1}{2a+1}$

$\Leftrightarrow a > \frac{1}{2}\left[\frac{1}{2\varepsilon} - 1\right]$ If there would be no such $a \in A$, then

$\forall a \in A, \quad a \leq \frac{1}{2}\left[\frac{1}{2\varepsilon} - 1\right] = M$ so A bounded.

false. $\Rightarrow \boxed{\frac{1}{2} = \sup}$

b) Prove the limit

$$\frac{1}{\sqrt{3n^4+1}} \to 0$$

with the ϵ - N_ϵ definition of convergence.

$$\frac{1}{\sqrt{3n^4+1}} < \varepsilon \Leftarrow \frac{1}{\varepsilon} < \sqrt{3n^4+1} \Leftarrow \frac{1}{\varepsilon^2} - 1 < 3n^4$$

$$\Leftarrow \frac{1}{\varepsilon^2} < 3n^4 \Leftarrow \sqrt[4]{\frac{1}{3\varepsilon^2}} < n.$$

$\forall \varepsilon \quad \exists N = N_\varepsilon = \left[\sqrt[4]{\frac{1}{3\varepsilon^2}}\right] + 1$ s.t. $\forall n \geq N_\varepsilon$

$-\varepsilon < 0 < \frac{1}{\sqrt{3n^4+1}} < \varepsilon$. <u>Done</u>.

P 5. Justify if the limits exist and calculate them. You may use all theorems about limits without proof.

$$a) \quad \lim_{n\to\infty} n(\sqrt{n^2+3}-n)$$

$$b) \quad \lim_{n\to\infty} \frac{n^2+n-1}{2^n} = 0$$

$$c) \quad \lim_{n\to\infty} \sqrt[n]{n^3-n} = 1.$$

(a) $n \dfrac{(\sqrt{n^2+3}+n)(\sqrt{n^2+3}-n)}{\sqrt{n^2+3}+n} = n \cdot \dfrac{3}{\sqrt{n^2+3}+n} =$

$= \dfrac{3}{\sqrt{1+\frac{3}{n^2}}+\frac{1}{n}}$. Since $x_n \equiv 3$ const $x_n \to 3$

$y_n = \sqrt{1+\frac{3}{n^2}} \to \sqrt{1} = 1$

$z_n = \frac{1}{n} \to 0$

$y_n + z_n \to 1+0 \neq 0$ Then

$$\lim_{n\to\infty} \frac{x_n}{y_n+z_n} = \frac{3}{1+0} = 3$$ ✓

(b) $a_{n+1}/a_n = \dfrac{(n+1)^2+(n+1)-1}{n^2+n-1} \cdot \dfrac{2^n}{2^{n+1}} = \dfrac{n^2+3n+1}{n^2+n-1} \cdot \dfrac{1}{2}$

$\to \frac{1}{2} < 1$ so $\lim\limits_{n\to\infty} a_n = 0$

(c) $\sqrt[n]{n^2} \le \sqrt[n]{n^3-n} \le \sqrt[n]{n^3} = [\sqrt[n]{n}]^3$ The left hand side true

$[\sqrt[n]{n}]^2 =$

if $n^2 \geq n+1$ always true $n \geq 2$

The Squeeze thm + $\lim \sqrt[n]{n} = 1$ Shows (c) $\mapsto$ 1 ✓

P 6. Let $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing, bounded below, and then find its limit.

· We notice that $x_1 \geqslant 2$ and

$$x_{n+1} = 1 + \underbrace{\sqrt{x_n - 1}}_{+} \geqslant 2 \text{ since } \sqrt{x_n - 1} \geqslant 1 .$$

So $x_n \geq 2 \ \forall n \geqslant 1$. Then the expression $\sqrt{x_n - 1}$ makes sense and (x_n) bdd below by 2.

<u>Decreasing</u> $x_2 = 1 + \sqrt{x_1 - 1} \leq x_1$

$\iff \sqrt{x_1 - 1} \leq x_1 - 1 \iff (x_1 - 1) \leq (x_1 - 1)^2$

$\iff 1 \leq x_1 - 1$ true since $x_1 \geqslant 2$.

This is the verification step $x_1 \geqslant x_2$.

$P(n)$: $x_{n+1} < x_n$, $n \geqslant 1$

$P(n+1)$: $x_{n+2} < x_{n+1}$.

Since $x_{n+1} < x_n$ we have

$$1 + \sqrt{x_{n+1} - 1} < 1 + \sqrt{x_n - 1}$$

$\Rightarrow$ $P(n+1)$ true.

$(x_n) \downarrow$ and bdd below by 1 $\Rightarrow$ convergent.

Let $\lim_{n \to \infty} x_n = x$. Then $\lim_{n \to \infty} x_{n+1} = x$ and $\lim_{n \to \infty} 1 + \sqrt{x_n - 1} = 1 + \sqrt{x - 1}$.

$x = \sqrt{x-1} + 1 \Rightarrow x - 1 = \sqrt{x-1} \Rightarrow x - 1 = 0$ or $x - 1 = 1$ only $x = 2$

$\boxed{x_n \geqslant 2 \ \forall n}$ is possible since

P 7. You get one point for correct answer (Yes or No) and one point for a brief justification of the answer, like just the name of the theorem/result implying it. True or false:

T 1) There exists a bijection $f : \mathbb{Z} \to \mathbb{Q}$.

both denumerable sets

T 2) Between any two real numbers, there exists a rational number.

$\mathbb{Q}$ dense in $\mathbb{R}$ $\Leftarrow$ Archimedean pple.

T 3) The sum of two rationals is rational.

$\mathbb{Q}$ is a field. Closed to $+$, $\cdot$ etc.

F 4) The sum of two irrationals is irrational.

$\underbrace{1-\sqrt{2}} + \sqrt{2} = 1$

T 5) The square root of any prime is irrational.

Same pf as for $\sqrt{2}$. Must have an even exponents!

F 6) For any function $f : A \to B$, $f \circ f^{-1}(E) = E$ when $E \subseteq B$.

$f^{-1}(E)$ by def. are $a \in A$ $f(a) \in E$ so $\subseteq$ true.

$f: \mathbb{R} \to \mathbb{R}$ If f not surjective. $f(x) = x^2$. $f \circ f^{-1}([-1,1]) = [0,1]$

F 7) There exists a smallest positive number.

$\forall a > 0 \quad 0 < \frac{a}{2} < a$

F 8) Any bounded sequence is convergent.

$x_n = (-1)^n$ not convergent $|x_n| \leq 1 \ \forall n \in \mathbb{N}$

T 9) Any convergent sequence is bounded.

Theorem $|x_n| \leq \max\{|x|+1, |x_1|, \ldots, |x_{N_1-1}|\} = M$

F 10) If $x_n y_n \to a$ and $x_n \to x$, then y_n is convergent.

$x = 0$ it is possible to have $y_n = (-1)^n$.

$a = 0$ $\quad x_n = \frac{1}{n}$