Solutions Chapter 4

Section 4.1

4.1 6 Let $\varepsilon > 0$. then pick $\delta = \varepsilon / K$

we immediately have

$\forall \varepsilon > 0 \quad \exists \delta = \varepsilon / K \quad 0 < |x - c| < \delta$

then $|f(x) - L| < K|x - c| = K\delta = \varepsilon$.

This case is covered by the more general notion of Lipschitz bound / continuity

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7 With sequences:

I $\quad x_n \to c$ then $x_n \cdot x_n \cdot x_n \to c \cdot c \cdot c = c^3$.

from basic operations with conv. sequences.

II $\underline{\varepsilon - \delta}$: Fix $x_0 = c$.

$|x^3 - c^3| = |x - c| \cdot |x^2 + xc + c^2|$

Let $|x - c| < \delta \leq 1$. Then

$|x^3 - c^3| \leq |x - c| \cdot [|x|^2 + |x||c| + |c|^2]$.

$\longrightarrow$

7 II then $|x| \leq |x-c| + |c| \leq |c|+1$.

So $|x|^2 + |x||c| + |c|^2 \leq [|c|+1]^2 + |c| \cdot [|c|+1] + |c|^2 \leq 3(1+|c|)^2 = K$.

Then

$\forall \varepsilon > 0 \quad \exists\, \delta = \min\{1, \frac{\varepsilon}{K}\}$

$\forall x \quad 0 < |x-c| < \delta$ then

$$|x^3 - c^3| < \varepsilon.$$

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12 (a) I am sorry but the text is simply absurd. The limit does not exist in $\mathbb{R}$ but is of course ∞ !!!

Limit exists $= +\infty$; not finite.

(a) Let $M > 0$. $\exists\, \delta > 0$ s.t.

$\frac{1}{|x|^2} > M$ as soon as $\delta = \frac{1}{\sqrt{M}}$

(def of $\to \infty$)

12 (b)

$$\lim_{x \to 0} \frac{1}{\sqrt{x}} = +\infty. \quad \text{here}.$$

$\forall M > 0 \quad \exists\ \delta = \frac{1}{M^2}$ s.t.

$|x| < \delta \implies \frac{1}{\sqrt{x}} > M.$

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12. (c) This truly DNE because it has more than one limit point,

Let $x_n = +\frac{1}{n}$ then.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\left(\frac{1}{n} + 1\right) = 1.$$

Let $x_n' = -\frac{1}{n}$.

$$\lim_{n\to\infty} f(x_n') = \lim_{n\to\infty}\left(-\frac{1}{n} - 1\right) = -1.$$

There is no unique limit L as sequences $\to 0 \implies$ qed

12(d). $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$.

Take. $\frac{1}{x_n^2} = 2n\pi$ and $\frac{1}{(x_n')^2} = 2n\pi + \frac{\pi}{2}$

then $x_n = \frac{1}{\sqrt{2n\pi}}$ $x_n' = \frac{1}{\sqrt{2n\pi + \frac{\pi}{2}}}$

these are different seq. $\to 0$.

and

$\lim_{n\to\infty} f(x_n) = 0$

$\Rightarrow$ no limit exists.

$\lim_{n\to\infty} f'(x_n') = 1$.

4.1 (15) Any $x_0 \in \mathbb{R}$, $\exists\, x_n \in \mathbb{Q}$ (5)

$x_n \to x_0$

and $\exists\, x_n' \in \mathbb{R} \setminus \mathbb{Q}$, $x_n' \to x_0$

(find the ref - in ~~text~~).

The part is the conseq. of the density of rationals in $\mathbb{R}$, itself a consequence of the Archimedean property.

A proof (simplified) is the following (for x_n')

$x_0 \in \mathbb{Q}$. then $x_n' = x_0 + \dfrac{\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$

and $x_n' \to x_0$.

$x_0 \in \mathbb{R} \setminus \mathbb{Q}$. $x_n' = x_0 + \frac{1}{n} \in \mathbb{R} \setminus \mathbb{Q}$

and $x_n' \to x_0$

(I) $f(x)$ has limit at $x_0 = 0$ because $L = 0 = f(0)$

$|f(x) - f(0)| \leq |x - 0| = |x|$

So K from Ex (6) is $\underline{K = 1}$

The inequality is TRUE for both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$

15 (ii) if $x_0 = c \neq 0$ then.

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Let $x_n \in \mathbb{Q}$. $x_n \to c$.

Let $x_n' \in \mathbb{R} \setminus \mathbb{Q}$ $x_n' \to c$.

We have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c$

$\lim_{n\to\infty} f(x_n') = \lim_{n\to\infty} 0 = 0$.

So $c \neq 0$. Lim DNE.

Section 4.2

4.2 (2) Using sequences the only thing to show is $z_n \to z \Rightarrow \sqrt{z_n} \to \sqrt{z}$ when $z > 0$ (Ex 15) but also $\boxed{\text{Thm } 3.2.10 \text{ p } 68}$

(a) $\to \sqrt{\lim \frac{2x_n+1}{x_n+3}} = \sqrt{\frac{2\cdot 2+1}{2+3}} = \sqrt{\frac{5}{5}} = 1.$

$x_n \to 2 \Rightarrow \begin{array}{l} 2x_n+1 \to 5 \\ \text{and} \\ x_n+3 \to 5 \neq 0 \end{array} \Rightarrow \lim_{n\to\infty} \underbrace{\frac{2x_n+1}{x_n+3}}_{=z_n} = 1$

$\Rightarrow \lim_{n\to\infty} \sqrt{z_n} = \sqrt{1} = 1.$

(b) Must simplify $\frac{x^2-4}{x-2} = x+2$; $x \neq 2$.

and then $x_n \to 2 \Rightarrow x_n + 2 \to 4$.

(c) Do the algebra. $x \neq 0$ we can write $f(x) = \frac{x^2+2x}{x} = x+2 \Rightarrow$ as before.

(d) $\frac{\sqrt{x}-1}{\sqrt{x}^2-1^2} = \frac{1}{\sqrt{x}+1}$ use similar sequence of reasoning to (a).

$\boxed{4.2}$ (4)

$\lim_{x\to 0} \cos\frac{1}{x}$ dne.

pick x_n s.t. $\frac{1}{x_n} = 2n\pi + \frac{\pi}{2}$ (goes to 0)

$= 2n\pi$ (goes to 1).

$\lim_{x\to 0} x\cos\frac{1}{x} = 0$ because $0 \le \left|x\cos\frac{1}{x}\right| \le |x|$

so by the squeeze thm. we obtain $0 = L$.

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(9) (a) write $h = f+g$ then

$\lim_{x\to x_0}[\underbrace{h(x)-f(x)}_{=g(x)}] =$ exists. (Thm 4.2.4)

so $\lim_{x\to x_0} g(x)$ exists $= L_{f+g} - L_f$.

(b) Not necessarily, unless $L_f \neq 0$.

if $f(x) = x$ and $g(x) = \cos\frac{1}{x}$ Ex 4

Show $\lim fg$ exist at $x_0 = 0$ yet $\lim g(x)$ dne.

But if $L_f \neq 0$ then we can use Thm 4.2.4 (b).

4.2 12 We have

$\forall x, y \in \mathbb{R} \quad f(x+y) = f(x) + f(y)$.

Let $x = x_n = \frac{1}{n} = y_n = y$; then.

$f(\frac{2}{n}) = f(\frac{1}{n}) + f(\frac{1}{n})$ Let $n \to \infty \implies$

$L = 2L \implies L = 0$.

Set $x = y = 0 \implies f(0+0) = f(0) + f(0)$

$\implies f(0) = L = 0$.

Take x_0 and $x_n \to x_0$. Then set $x_n - x_0 = y_n$.

AND $y_n \to 0$

$f(x_n) = f(x_0 + y_n) = f(x_0) + f(y_n)$.

$\implies 0 \le | f(x_n) - f(x_0) | = | f(y_n) |$ but

$\lim_{n\to\infty} | f(y_n) | = |L| = 0 \implies$ (squeeze theorem) $\lim_{n\to\infty} f(x_n) = f(x_0)$.

The truth is $f(x) = Cx$, C constant

This is a well know problem, will re-appear in Chapter 5.

Section 4.3

$\boxed{4.3}$ 5 (a) $\displaystyle\lim_{x\to 1^+} \frac{x}{x-1} = +\infty$.

(I) Let $M>0$ we want $\delta>0$ s.t.

$1 < x < 1+\delta$ then $\dfrac{x}{x-1} > M$.

$$\left.\begin{array}{l} x-1>0 \\ x \quad >0 \end{array}\right\} \Rightarrow \frac{x}{x-1} > M \iff x > M(x-1)$$

$\iff \quad M > (M-1)\,x$ but M is large so $\underline{M>1}$

$$\iff x < \frac{M}{M-1} = \frac{M-1+1}{M-1} = 1+\frac{1}{M-1}.$$

$\forall M>1 \quad \exists\, \delta = \dfrac{1}{M-1} \quad 1<x<1+\delta \Rightarrow \dfrac{x}{x-1} > M$.

(II) $\displaystyle\lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} f(1+h)$ here. $\dfrac{1+h}{h} = 1+\dfrac{1}{h}$.

and $\displaystyle\lim_{h\to 0^+} \frac{1}{h} = +\infty$ one of the basic limits.

4.3 5 (b) d.n.e. $\lim_{x\to 1^+} f(x) = +\infty$

$\lim_{x\to 1^-} f(x) = -\infty$.

5(c) $+\infty$

5(d) $+\infty$ why?

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{x} + \frac{2}{\sqrt{x}}$$

$$= \infty + 0 = \infty.$$

5(e). dne. 0^+, 0^- different.

5(f) $= 0$ because $\frac{\sqrt{x+1}}{x} = \sqrt{\frac{x+1}{x^2}} = \sqrt{\frac{1}{x} + \frac{1}{x^2}}$

$$\lim_{x\to\infty} f(x) = \sqrt{\lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{1}{x^2}} = \sqrt{0+0} = 0.$$

5(g) $\lim_{x\to\infty} \frac{\sqrt{x}-5}{\sqrt{x}+3} = 1$. Divide by $\sqrt{x}$.

5(h). Divide by x. $\frac{\frac{1}{\sqrt{x}} - 1}{\frac{1}{\sqrt{x}} + 1} \longmapsto \frac{0-1}{0+1} \longmapsto \boxed{-1}$