3.4, 3.6, 3.5 MTH433

3.4 (4) odd $\to 2$ even $\to 0$

Two limits $\Rightarrow$ divergent.

(10) Prove this statement

$\forall (N, \varepsilon)$ $\exists n_\varepsilon > N$ $|x_{n_\varepsilon} - s| < \varepsilon$.

proof $S_n \downarrow s$ by construction.

$\exists n_1$ $s \le S_{n_1} < s + \varepsilon$

(def. of inf). because $S_n \downarrow$, this will be true for any S_n with $n \geq n_1$.

Define $N_1 = \max\{n_1, N\}$

$S_{N_1} = \sup\{x_k \mid k \geq N_1\}$. then

$\exists k_\varepsilon$ $s - \varepsilon \le S_{N_1} - \varepsilon < x_{k_\varepsilon} \le S_{N_1}$

So. $s - \varepsilon < x_{k_\varepsilon} < s + \varepsilon$

and $k_\varepsilon \geq N_1 > N$.

From here on, pick. $(1, 1) \mapsto n_1$ $|x_{n_1} - s| < 1$,

$(n_1, \frac{1}{2}) \mapsto n_2$ $|x_{n_2} - s| < \frac{1}{2}$.

etc $(n_2, \frac{1}{3}) \mapsto n_3$ $|x_{n_3} - s| < \frac{1}{3}$.

12

$\exists$ subsequence $x_{k_n} \to \infty$.

$$\forall (N, M) \ \exists \ n_M > N \quad x_{n_M} > M$$

3.4 proof

$$\exists (N, M) \ \forall (n > N). \quad (x_n \leq M)$$

the other $x_1, \ldots, x_N$ are bdd (finite) bdd. false.

Let $(1,1)$ $\exists$ $n_1 > 1$ $x_{n_1} > 1$.

$(n_1, 2)$ $\exists$ $n_2 > n_1$ $x_{n_2} > 2$

$\vdots$

(n_{k-1}, k) $\exists$ $n_k > n_{k-1}$ $x_{n_k} > k$.

---

done.

3.4 (14) $\boxed{\forall (N, \varepsilon) \quad \exists\, n_\varepsilon > N \qquad x_{n_\varepsilon} > s - \varepsilon}$

(proof) Negation.

$$\exists (N, \varepsilon) \quad \forall (n > N) \quad (x_n \leq s - \varepsilon)$$

this implies that $\sup\{x_n \mid n > N\} \leq s - \varepsilon$.

$\Rightarrow$ the supremum must be among $x_1, \dots, x_N$

i.e. it is a maximum, contradiction.

Then we start. $(1, 1) \longrightarrow n_1 \qquad x_{n_1} > s - 1$

$(n_1, \frac{1}{2}) \longrightarrow n_2 \qquad x_{n_2} > s - \frac{1}{2}$

---

(19) Use def with inf sup.

$$S_n = \sup\{x_k + y_k \mid k \geq n\}$$

$$\limsup_n (x_n + y_n) = \inf_n S_n \quad \text{but}$$

$$S_n \leq \sup\{x_k \mid k \geq n\} + \sup\{y_k \mid k \geq n\}$$

and $\inf_n S_n = \lim_{n \to \infty} S_n$ pass to the limit

$$\Rightarrow \limsup_n (x_n + y_n) \leq \limsup_n x_n + \limsup_n y_n$$

3.6

(3) $x_n > 0$ $\quad \lim x_n = 0 \iff \lim \frac{1}{x_n} = +\infty.$

$\Longrightarrow$

Let $M > 0$ take $N_M^{\frac{1}{x}} := N_{\frac{1}{M}}^{x}$

in other words $\forall n \geq N_{\frac{1}{M}}^{x}$ we have $0 < x_n < \frac{1}{M} = \varepsilon$

$\Rightarrow \quad \frac{1}{x_n} > M$ for the same n.

$\Longleftarrow$ Let $\varepsilon > 0$. take $N_\varepsilon^{x} := N_{\frac{1}{\varepsilon}}^{\frac{1}{x}}$,

Same proof.

3.6 8 (a) $\to \infty$ divergent

(b) $\to 0$

(c) $\to \infty$ div.

(d) divergent more than two limit pts.

10 $\forall M > 0 \quad \exists N_M \quad \forall n \geq N_M \quad a_n \geq M$.

WTS

We know. $\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad \forall n \geq N_\varepsilon$

$\frac{a_n}{n} > L - \varepsilon$ Take $\varepsilon = \frac{L}{2} > 0$

then $a_n > n\frac{L}{2} \quad \forall n \geq N_{L/2}$.

$\Rightarrow \lim_{n\to\infty} a_n = +\infty$. (comparison).

---

6 $|x_n| \to \infty$ and. $\lim_{n\to\infty} |x_n y_n| \in \mathbb{R}$.

(given). We can assume all are positive.

Let $y^+ = \limsup_n |y_n| \Rightarrow$ [If $y^+ > 0$, then

$x_{n_k} y_{n_k} \longrightarrow +\infty$] (where $|y_{n_k}| \to y^+$).

$\Longrightarrow y^+ = 0 \Longrightarrow y^- = 0$ ■

3.5 10 $x_n - x_{n-1} = -\frac{1}{2}(x_{n-1} - x_{n-2})$

$\Rightarrow$ (x_n) contractive with $C = \frac{1}{2}$.

$x_n - x_{n-1} = C(x_{n-1} - x_{n-2})$

$x_{n-1} - x_{n-2} = C(x_{n-2} - x_{n-3})$.

$\vdots$

$x_3 - x_2 = C(x_2 - x_1)$

$x_n - x_{n-1} = C^{n-2}(x_2 - x_1)$.

$x_3 - x_2 = C(x_2 - x_1)$

$x_n = (x_2 - x_1)\left[C^{n-2} + C^{n-3} + \cdots + 1\right] + x_1$.

$x_n \to (x_2 - x_1)\dfrac{1 - C^{n-1}}{1 - C} + x_1$.

$x_n \to (x_2 - x_1)\dfrac{1}{\frac{3}{2}} + x_1 = \frac{2}{3}x_2 + \frac{1}{3}x_1$

See back

3.5 (-10)

$$2x_n - x_{n-1} - x_{n-2} = 0.$$

$$2\lambda^2 - \lambda - 1 = 0. \qquad \lambda = \frac{1 \pm \sqrt{1+8}}{\cancel{2}4} = \frac{1 \pm 3}{4} = 1, -\frac{1}{2}.$$

$$c_1 + \left(-\frac{1}{2}\right)^n c_2 = x_n.$$

$$x_1 = c_1 - \frac{1}{2} c_2 \qquad x_2 = c_1 + \frac{1}{4} c_2$$

$$\frac{1}{2} x_1 + x_2 = \frac{3}{2} c_1 \Rightarrow c_1 = \frac{x_1 + 2x_2}{3}.$$

$$\ell = \frac{x_1 + 2x_2}{3}.$$

3.5 12 $x_1 > 0 \quad x_{n+1} = (2+x_n)^{-1}, \ n \geq 1,$

$$x_{n+1} - x_n = \frac{1}{2+x_n} - \frac{1}{2+x_{n-1}} = \frac{x_{n-1} - x_n}{(2+x_n)(2+x_{n-1})}$$

$$\Rightarrow \quad |\text{—}| \leq |x_{n-1} - x_n| \frac{1}{4} \qquad C = 1/4$$

$$x = \frac{1}{2+x} \Rightarrow 2x + x^2 = 1$$

$$x^2 + 2x - 1 = 0 . \qquad \frac{-1 \pm \sqrt{2}}{} = -1 \pm \sqrt{2}$$

$x = \sqrt{2} - 1$ only valid solutions
