Hwk 3.7

(Series)

$\boxed{3.7}$ (36)

$$\sum_{n\geq 0} \frac{1}{(\alpha+n)(\alpha+n+1)} = \sum_{n=0}^{\infty} \left[\frac{1}{\alpha+n} - \frac{1}{\alpha+n+1} \right]$$

$$S_n = \left[\frac{1}{\alpha} - \frac{1}{\alpha+1} \right] + \left[\frac{1}{\alpha+1} - \frac{1}{\alpha+2} \right] + \dots + \left[\frac{1}{\alpha+n} - \frac{1}{\alpha+n+1} \right]$$

$$= \frac{1}{\alpha} - \frac{1}{\alpha+n+1} . \qquad \lim_{n\to\infty} S_n = \frac{1}{\alpha} - \lim_{n\to\infty} \frac{1}{\alpha+n+1}$$

$$= \frac{1}{\alpha} \checkmark$$

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(4) $S_n^x + S_n^y = S_n^{x+y} \Rightarrow$ from prop of sequences that

$$\underbrace{\lim_{n\to\infty} S_n^x + \lim_{n\to\infty} S_n^y}_{LHS} = s^x + s^y$$

$$\underset{RHS}{=} \lim_{n\to\infty} S_n^{x+y} = s^{x+y} .$$

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(10) This is simply the thm on alt. series convergence: $\sum \underbrace{(-1)^n b_n}_{a_n}$ $b_n \downarrow 0$ converges.

10 → continued:

If we want a direct proof:

Cauchy's criterion says: Say $m \geq n > N$

then $|a_{n+1} + \dots + a_m| \leq |a_{n+1}| = b_n$

Since $(|a_n|) \to 0$

we have

$\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad \forall n \geq N_\varepsilon \quad |a_n| < \varepsilon.$

then

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad \forall \begin{matrix} n \geq N_\varepsilon \\ m \geq N_\varepsilon \end{matrix} \qquad |a_{n+1} + \dots + a_m| < \varepsilon.$$

⇒ the partial sums are Cauchy ⇒

the series converges.

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11 yes. $a_n \to 0$ if $\sum a_n$ converges ⇒ $|a_n| \leq M$.

$$\Rightarrow \sum a_n^2 \leq \sum M a_n = M \sum a_n < \infty$$

$\Rightarrow \sum a_n^2$ convergent.

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Cauchy Condensation test: $a_n \downarrow 0$.

$\sum a_n$ converges $\Leftrightarrow$ $\sum 2^n a_{2^n}$ conv.

(a) $\sum \frac{1}{n \ln n}$ $n \mapsto 2^n$ obtain

$$\sum \frac{1}{\cancel{2^n} \ln 2^n} \cdot \cancel{2^n} = \sum \frac{1}{n \ln 2} = \frac{1}{\ln 2} \sum \frac{1}{n}$$

divergent b/c $\sum \frac{1}{n^p} < \infty \Leftrightarrow p > 1$.

(b) $\sum \frac{1}{n \ln n \ln \ln n}$ $n \to 2^n$

$$\sum \frac{2^n}{2^n \cdot [n \ln 2][\ln n + \ln \ln 2]} = \frac{1}{\ln 2} \sum \frac{1}{n(\ln n + c)}$$

$c = \ln \ln 2$ notice that we can apply the test again!

$$\sum \frac{2^n}{2^n (n \ln 2 + c)} = \sum \frac{1}{n \ln 2 + c} = \sum c_n$$

since $\lim \frac{c_n}{1/n} = \frac{1}{\ln 2} > 0$ the series DIVERGES

Section 3.7

17 (c) $$\sum \frac{1}{n(\ln n)(\ln\ln n)(\ln\ln\ln n)},$$

(equivalent)

$$\sim \sum \frac{\cancel{2^n}}{\cancel{2^n}(\ln 2)(n)(\ln\ln 2+\ln n)\ln[\ln\ln 2+\ln n]}$$

$$\frac{1}{\ln 2}\sum \frac{1}{n(\ln n+c)\ln(\ln n+c)}.$$

$c=\ln\ln 2<0 \Rightarrow \ln n+c<\ln n$

so the original series

$$> \frac{1}{\ln 2}\sum \frac{1}{n\ln n\ln\ln n}$$

but this is 17(b) $\Rightarrow$ divergent