Lecture 3

- Sup and inf on $\mathbb{R}$
- Completness of $\mathbb{R}$

We want to show that

$$A = \{ p \in \mathbb{Q} \mid p > 0 , p^2 < 2 \}$$

does not have a supremum in $\mathbb{Q}$.

Let $B = \{ p \in \mathbb{Q} \mid p > 0 , p^2 > 2 \}$.

If $x, y > 0$ $\quad x < y \iff x^2 < y^2$,

because $\quad y - x > 0 \iff (y-x)\underbrace{(y+x)}_{>0} > 0$

This implies that $M = 3$ is an upper bound for A: $\quad M^2 = 9 > 2 > p^2$, $\forall p \in A$.

Of course there are infinitely many upper bounds for A. But there is no smallest upper bound in $\mathbb{Q}$.

Let $a = \sup A$ and suppose $a \in \mathbb{Q}$

Either $a^2 < 2$, then $a \in A$

$a^2 = 2$, then $a = \sqrt{2} \in \mathbb{Q}$ (false)

$a^2 > 2$, then $a \in B$.

If $a \in A$, then

$$a' = a + \underbrace{\frac{2-a^2}{a+2}}_{>0} \in A \ , \ a' > a$$

$\Rightarrow$ a cannot be $\sup A$ $\Rightarrow$ contradiction (since it does not dominate $a' \in A$).

If $a \in B$, then

$$a' = a + \underbrace{\frac{2-a^2}{a+2}}_{<0} \in B, \ a' < a$$

$\Rightarrow (a')^2 > 2 > p^2 \Rightarrow a' > p$

$\Rightarrow$ a' is an upper bound of A yet $a' < a$! contradiction.

The Real numbers are the set with the properties:

$(\mathbb{R}, +, \cdot)$ is a commutative field

$(\mathbb{R}, \leq)$ is totally ordered and compatible with $+, \cdot$

$\mathbb{R}$ is <u>complete</u>, meaning

$\forall A \subseteq \mathbb{R}$, A bounded above

$\Rightarrow \sup A \in \mathbb{R}$.

We derive the Archimedean property:

(i) $\forall x > 0 \quad \exists n \in \mathbb{N} \quad n > x$.

(ii) $\forall x, y \quad \exists p \in \mathbb{Q}$

$x < p < y$.

<u>Remark</u> (ii) is a consequence of (i).

Let $A = \{ k \in \mathbb{N} \mid k \le x \}$

If there is no $n \in \mathbb{N}$, $n > x$,

then A is bounded above by $x \in \mathbb{R}$.

~~Let~~ $a = \sup A$. $\Rightarrow$ $a \geq k \quad \forall k \in \mathbb{N}$

$\Rightarrow \quad a \geq k+1 \quad \forall k \in \mathbb{N}$

$\Rightarrow \quad a-1 \geq k \quad \forall k \in \mathbb{N}$

$\Rightarrow \quad a-1 < a$ and is an upper bound of A. <u>Contradiction</u>.

(ii) $\quad y > x \Rightarrow y - x > 0 \Rightarrow (y-x)^{-1} > 0$

(prove) and let $n > 0$ such that

$$n > \frac{1}{y-x}$$ or equivalently $n(y-x) > 1$

$\Rightarrow \quad ny - nx > 1$. This implies that in (nx, ny) there is at least one integer m

$\Rightarrow \quad p = \frac{m}{n}$ is in (x, y)

The property shows that $\mathbb{Q}$ is dense in $\mathbb{R}$.

A situation like the one described above by

$$A = (0, \sqrt{2}) \cap \mathbb{Q}$$

$$B = (\sqrt{2}, \infty) \cap \mathbb{Q}$$

cannot appear because the missing value $\sqrt{2}$ has been added to the extended set $\mathbb{R}$.
