Lecture 2

Order relations
Ordered fields

A relation $\mathcal{R}$ on G is a subset of $G \times G$. That is, we say

$(x,y) \in \mathcal{R} \iff x \,\mathcal{R}\, y$

" x in relation $\mathcal{R}$ to y ".

This is very general. All functions

$f : A \to B$, $A, B \subseteq G$ are relations on G! To see that we write $y = f(x)$ if $x \,\mathcal{R}\, y$.

Then all we need is

$\forall x \in A \quad \exists y \in B \quad x \,\mathcal{R}\, y$

$\forall x \in A \quad x \,\mathcal{R}\, y_1$ and $x \,\mathcal{R}\, y_2 \Rightarrow y_1 = y_2$

Exercise Write what f one-to-one & onto mean with this notation.

<u>Definition</u> $\mathcal{R}$ is an <u>order relation</u> on G if

$\forall x \quad x \mathcal{R} x$

$\forall x, y \quad x \mathcal{R} y$ and $y \mathcal{R} x \Rightarrow x = y$

(reflexivity)

$\forall x, y, z \quad x \mathcal{R} y$ and $y \mathcal{R} z \Rightarrow x \mathcal{R} z$

(transitivity)

<u>Hint</u> put $\leq$ instead of $\mathcal{R}$ to make these axioms obvious.

Let's simply write $\leq$ instead of $\mathcal{R}$ from now on.

What does $<$ as opposed to $\leq$ mean?

<u>Def</u> $x < y$ if $x \leq y$ and $x \neq y$.

We may have an ordered set yet not any two elements are comparable.

$(\mathbb{R} \times \mathbb{R}, \leq)$ $(p_1, q_1) \leq (p_2, q_2)$

when both $p_1 \leq p_2$ and $q_1 \leq q_2$

satisfies the order axioms yet

$(2,1) \leq (1,2)$

$(2,1) \geq (1,2)$

are both false! So the two elements are not comparable.

Def An ordered set where any two elements are comparable is said totally ordered.

This is equivalent to: For any x, y

(O) Only one of the three

$x < y$, $x = y$, $x > y$

is true.

We need to add an order relation to our commutative field.

A commutative field $(G, +, \cdot)$ is an ordered field if there exists an order relation $\leq$ on G such that

(O1) $\forall x, y$ exactly one is true

$$x < y, \quad x = y, \quad x > y$$

(O2) $$x > y \Rightarrow x + z > y + z$$

(O3) $$x, y > 0 \Rightarrow xy > 0.$$

Notice that we demand O1, O2, O3 be satisfied. One way to define $x > y$ is to say $x - y$ is positive. This is not wrong, but we need to say what positive means (i.e. $x > 0$) without making a circular argument. If we can just have (O1)-(O3) we are consistent and our construction is correct.

With all the axioms of a commutative field and the order axioms, we might think that $\mathbb{R}$ is well defined.,

Example $(\mathbb{Q}, +, \cdot, \leq)$ is an ordered field, yet $\mathbb{Q} \subsetneq \mathbb{R}$.

It is not enough to prescribe algebraic properties of $\mathbb{R}$ and a compatible order.

The next lecture gives the last property needed to define $\mathbb{R}$: completeness.

Definition $(G, \leq)$ is an ordered set.

- $A \subseteq G$ is bounded above if $\exists\, M \in G$ such that $\forall x \in A \quad x \leq M$.

  M is said an upper bound of A.

$A \subseteq G$ is bounded below if

$\exists\, m \in G \quad \forall x \in A \quad m \leq x.$

$\underline{m}$ is said a lower bound of A.

<u>Remark</u> there are many lower/upper bounds of a set A, in general.

$A = [-1, 3) \subseteq G = \mathbb{R}$

$M = 3$, $M = 4$, $M = 1{,}000{,}000$, $M = 3.001$

are all upper bounds. Note $3 \notin A$

$m = -1$, $m = -5$, $m = -1.0001$

are lower bounds of A. $m = -1 \in A$

There is something special about 3 and -1

3 = lowest upper bound of A (lub)

-1 = greatest lower bound of A (glb)

In modern terminology lub = supremum

glb = infimum

A set may not have an upper bound:

$\mathbb{N} = \{0, 1, 2, \dots\}$ does not have an upper bound.

$\mathbb{Z}$ does not have either upper nor lower bounds.

lub = <u>Supremum</u> of A $\stackrel{not}{=}$ sup A (notation)
glb = <u>infimum</u> of A $\stackrel{not}{=}$ inf A

Formal definition

$a = \sup A$ is defined by

$$\begin{cases} \forall x \in A & x \leq a \\ \forall a' & (\forall x \quad x \leq a') \Rightarrow (a \leq a') \end{cases}$$

The first says that a is an upper bound
The second says that if a' is another upper bound, $a \leq a'$.

A similar definition holds for the infimum.

<u>Exercise</u> $\sup(-A) = -\inf A$

where $-A = \{-x \mid x \in A\}$.

<u>The completness axiom in a set $(G, \leq)$</u>

Any subset with an upper bound has a smallest upper bound in G

To understand the value of this axiom; we look at $(\mathbb{Q}, +, \cdot, \leq)$ with the usual addition, multiplication and ~~inequality~~ order relation

let

$$A = \{p \in \mathbb{Q}^+ \mid p^2 < 2\}$$

this set does not have a supremum in $\mathbb{Q}$ because we know it is equal to $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ the set of irrationals.