Homework 1

August 26, 2015

De Morgan Law

$$(B \cup C)^c = B^c \cap C^c$$

Proof:
$x \in (B \cup C)^c \Leftrightarrow x \notin B \cup C \neg(x \in B \cup C) \Leftrightarrow \neg(x \in B \vee x \in C) \Leftrightarrow \neg(x \in B) \wedge \neg(x \in C) \Leftrightarrow (x \notin B) \wedge (x \notin C) \Leftrightarrow (x \in B^c) \wedge (x \in C^c) \Leftrightarrow x \in B^c \cap C^c$

Ex 4.

$A \setminus (B \cup C) = A \cap (B \cup C)^c$
$= A \cap (B^c \cap C^c)$ [De Morgan]
$= (A \cap B^c) \cup (A \cap C^c)$ [Distributivity]
$= (A \setminus B) \cup (A \setminus C)$ [Definition of set difference]

Image of sets in relation to set operations

(i) $f(E \cup F) = f(E) \cup f(F)$ Always true

(ii) $f(E \cap F) \subseteq f(E) \cap f(F)$ The other inclusion is false by Ex 12. Find another simple example. *It is true when f injective.*

Preimage of sets in relation to set operations

(i) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$
(ii) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$
True even when f is neither injective, nor surjective. *Here f^{-1} denotes the preimage and does not assume the existence of a proper inverse function.*

Divisibility by prime numbers

$u|v$ means u divides v
p, q are prime
$p|a$ and $q|a$ then $pq|a$

17. We prove that $m = 6$. $n^3 - n = (n-1)n(n+1)$ the product of three consecutive numbers.
Take $m \geq 7$ and $n = m - 2$ then
$(m-3)(m-2)(m-1) = Multiple(m) - 6$ so it must be that $m|6$, which is impossible.
Prove that 2 and 3 divide $(n-1)n(n+1)$ for any n. Then use the result above. Hint: case by case modulo 2, respectively modulo 3.

Section 1.3. Number of functions between two finite sets

$|S| = n$ and $|T| = m$

All functions

m^n

Injective functions

$\binom{m}{n} n! = \frac{m!}{(m-n)!}$ if $n \leq m$
[Arrangements or permutations of m objects taken n at a time]

$n!$ if $n = m$
[Permutations of n objects]

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0 if $n > m$
[Pigeonhole principle]

Surjective functions

0 if $n < m$

$S(n, m)$ if $n \geq m$ where
$S(n, m) = m^n - \sum_{l=1}^{m-1} \binom{m}{l} S(n, l)$

Explanation: We subtract out of the total number of functions the number of functions with image having l elements, $1 \leq l \leq m - 1$.

3. Injective: Equals the number of permutations of 3 objects taken 2 at a time, also known as arrangements, equal to $\binom{3}{2}\, 2! = 3 \cdot 2 = 6$
Surjective: There are 2^3 function in total from T to S. Not all are surjective. Those that are not must have image exactly one element of S (there must be at least one). These are only two, constant functions. So the answer is $2^3 - 2 = 6$.
6. $f(n) = n + 1$

9. Suppose S, T are denumerable and disjoint. We can reduce the problem to this case or the simpler case when S is denumerable and T is finite. Let $s : \mathbb{N} \to S$, $t : \mathbb{N} \to T$ be the bijections showing the sets are denumerable. Then define
$f(n) = s(\frac{n}{2})$ when n is even
$f(n) = t(\frac{n+1}{2})$ when n is odd.
This function is bijective. Prove.

12. Show that there exists a bijection between the subsets of $\mathbb{N}_{n+1}$ containing $n+1$ and the subsets of $\mathbb{N}_n$.

13. A finite subset of $\mathbb{N}$ has a maximal element, say m. Then the subset is contained in $A_m = \mathcal{P}(\mathbb{N}_m)$.

Now $A_m \subseteq A_{m+1}$ and all are finite sets. Denote $B_m = A_{m+1} \setminus A_m$, taking $A_0 = \emptyset$. Then we have $\mathcal{F}(\mathbb{N}) \subseteq \cup_{m=1}^{\infty} B_m$. The countable union of disjoint finite sets is countable.

Remark. Since $\{n\}$ is a subset of $\mathbb{N}$ for sure $\aleph_0 \leq |\mathbb{N}| \leq |\mathcal{F}(\mathbb{N})| \lneq |\mathbb{P}(\mathbb{N})| = 2^{\aleph_0}$. The last inequality is strict due to Cantor's Theorem (Thm 1.3.13), see next.

Bijection between subsets and indicator functions

For any $A \subset S$ define a function from S to $\{0, 1\}$ (binary) called *the indicator function of A*)
$\chi_A(s) = 1$ if $s \in A$ and $\chi_A(s) = 0$ if $x \notin A$.
Let χ be the set of all such functions. Then $|\chi| = 2^n$ if $|S| = n$.
The function that takes A into χ_A is a bijection between $\mathbb{P}(S)$ and χ.
We have $|\mathcal{P}(S)| = 2^{|S|}$ for any set S, including infinite. Cantor's theorem says that $|S| \lneq 2^{|S|}$.

Cardinality of the real numbers **c**, the Continuum

$|\mathbb{N}| \lneq |2^{|\mathbb{N}|}| = \mathbf{c} = |\mathbb{R}|$
The set of all sequences of 0 and 1 is not countable and has the same cardinal number as the real numbers. This corresponds to the binary form of a real number in $(0, 1)$, which is in bijection with $\mathbb{R}$ - see Ex 1.1.16.