Relations on sets

1. Solutions

P 1. *Set $A = B = \mathbb{Z}$ and $\mathcal{R}$ one of the following relations:*
$=$, $\equiv$ *(mod m), where m is a positive integer. Show these are equivalence relations.*
$\leq$, $\geq$ *Show these are order relations.*

Solution. $=$

1) $a = a$; $a = b$ then $b = a$; $a = b$ and $b = c$ then $a = c$. These prove (i), (ii), (iii). The "equality" relation $=$ is the template for all equivalence relations.

2) $m|a - a = 0$ implies $a \equiv a$ (i)

$m|a - b$ then $m|b - a$ implies reflexivity (ii)

$m|a - b$ and $m|b - c$ then $m|(a - b) + (b - c) = a - c$ proves transitivity (iii)

3) $a \leq a$ proves (i);

$a \leq b$ and $b \leq a$ implies $a = b$ proves (ii)';

$a \leq b$ and $b \leq c$ then $a \leq c$ proves (iii).

The "less or equal" relation $\leq$ is the template for order relations.

There is no difference to prove $\geq$. Notice that $a \leq b \Leftrightarrow (-a) \geq (-b)$ so any property proven for one transfers to the other with a change of sign.

P 2. *On $A = B = \mathbb{R} \times \mathbb{R} \setminus \{0\}$ define*

$(x, y)\mathbb{R}(x', y')$ if and only if $xy' = x'y$. Show that this is an equivalence relation.

Solution. This relation says that $x'/y' = x/y$. Notice that we can divide by y, y' which are not zero.

$xy = yx$ gives reflexivity.

$xy' = x'y$ implies $x'y = xy'$ which gives symmetry.

$xy' = x'y$ and $x'y'' = x''y'$ gives by multiplication $xy'x'y'' = x'yx''y'$. If $x' = 0$, then x and $x'' = 0$ and then $xy'' = 0 = x''y$. If not, then we may simplify by $x'y'$ and obtain $xy'' = x''y$, which proves transitivity in all cases.

P 3. *On $\mathbb{R}$ define the relation*

1) $a\mathcal{R}b$ if $a-b \in C = [0,\infty)$.

2) The set C can be generalized. Let A be the (x,y) plane. Define $C = \{(x,y)|x \geq 0\,, y = mx\,, |m| \leq t\}$, where $t \geq 0$ is a fixed non-negative number. This is an infinite cone with vertex at the origin. Show that the relation from part 1) is an order relation.

Solution. 1) The regular order relation $\geq$ is equivalent on the real line with $a \geq b$ if and only if $a-b \geq 0$ or equivalently $a-b \in C = [0,\infty)$. The role of this set C can be taken by other sets, like in part 2).

$a-a = 0 \in C$ (reflexivity);

$a-b \in C$ and $b-a \in C$ should imply that $a=b$. This is guaranteed by the fact that $-C = (-\infty, 0]$ and $-C \cap C = \{0\}$ (antisymmetry).

$a-b \in C$ and $b-c \in C$ should imply that $a-c \in C$. This is true because the sum of two elements of C is in C, a fact we write as $C + C \subseteq C$.

2) The properties from 1) will be satisfied in this new setting. Here $a = (x,y)$ is a pair of numbers, also a point in the plane.

$a-a = (0,0) \in C$ (reflexivity)

For $a = (x,y)$ and $b = (x',y')$, $a-b \in C$ and $b-a \in C$ implies that the pair $(x-x', y-y') \in C$ and $(x'-x, y'-y) \in C$. The only opposites in C are exactly $(0,0)$.

Proof: By definition, $x'-x \geq 0$ and at the same time $x-x' \geq 0$ because elements in C have non-negative first component. So $x = x'$. Then $y-y' = m(x-x') = 0$ for some m, $|m| \leq t$. We conclude that $(x,y) = (x',y')$.

Let $a = (x,y)$, $b = (x',y')$ and $c = (x'',y'')$. If $(x-x', y-y') \in C$ and $(x'-x'', y'-y'') \in C$ we want to show that $(x-x'', y-y'') \in C$. This is true if the sum of two elements in C is in C.

Since $x-x' \geq 0$ and $x'-x'' \geq 0$ for sure $x-x'' = x-x'+x'-x'' \geq 0$.

If $x-x' = 0$ and $x'-x'' = 0$ then any m (take $m=0$) will prove the statement. We assume that $x-x'' > 0$. We have to show that

$$|\frac{y-y''}{x-x''}| \leq t\,.$$

We know that $m = \frac{y-y'}{x-x'}$ and $m' = \frac{y'-y''}{x'-x''}$ with $|m| \leq t$, $|m'| \leq t$. Then

$$|\frac{y-y''}{x-x''}| = \frac{|y-y''|}{x-x''} \leq \frac{|m|(x-x') + |m'|(x'-x'')}{x-x''} \leq \frac{t(x-x') + t(x'-x'')}{x-x''} \leq t$$

which proves that $(x-x'', y-y'') \in C$.

P 4. *Show that $a|b$ (a divides b) on $\mathbb{Z}_+$ is an order relation. Show that not all pairs can be ordered.*

Solution. $a|a$ (a divides a) is always true, implies reflexivity (i).

$a|b$ and $b|a$ implies that $a = b$. If you really want to prove that, notice that $a|b$ implies $b = ak$, $k \in \mathbb{Z}$ and $k \geq 1$. It cannot be negative because both a and b are positive, and it cannot be zero because then b would be zero. So $b \geq a$. We combine the other inequality to obtain $a = b$. We proved the antisymmetry (ii)'.

$a|b$ and $b|c$ then $a|c$. We have $b = ak$ and $c = bk'$ then $c = a(kk')$ so $a|c$. This proves the transitivity property (iii).

P 5. - *Determine the pairs of integers* $(0, 0) \leq (x, y) \leq (2, 1)$ *in lexicographical order.*

- *order the two strings* $(1, 1, 2)$ *and* $(1, 2, 1)$

- *order the two strings* $(1, 0, 1, 0, 1)$ *and* $(0, 1, 1, 1, 0)$

- *explain how to order strings (or words) of possibly different length (think again of a dictionary). Try the example:*

zoo, zero, zoom, zoology, zoological

Solution.

• (x, y) must be between $(0, 0)$ and $(2, 1)$.

If $x = 0$ then $(0, 0) \leq (0, y)$ if $0 \leq y$. On the other hand, $(0, y) \leq (2, 1)$ for any y. $A = \{(0, y)|0 \leq y\}$

If $x = 1$ then y can be any integer. $B = \{(1, y)|y \in \mathbb{Z}\}$.

If $x = 2$ then $y \leq 1$. $C = \{(2, y)|y \leq 1\}$.

Answer: $A \cup B \cup C$.

• $(1, 1, 2) \leq (1, 2, 1)$ because the first components are equal, but the second component is larger in the second string.

• $(1, 0, 1, 0, 1) \geq (0, 1, 1, 1, 0)$ They are different starting with the first component, and the first component $1 > 0$.

•. The rule is the same except that if two words of different lengths coincide for the full length of the shorter word, the shorter word is greater than the longer word in lexicographical order, e.g. *search* > *searching*, *like* > *likelihood*, *equal* > *equalitarian*.

An alternative (more formal) way to deal with words of different lengths is by extending the alphabet by one character (say ∞), not present in the alphabet, that will be set to precede all letters in the alphabet. A finite set of words can be ordered by adding the infinity character at the end of shorter words, until we obtain only words of the same length. Then we use the lexicographic order directly on the transformed set of words.

Then

zero > zoo > zoology > zoological > zoom