Sets

Sets are fundamental mathematical concepts. We need to talk about some known objects to do the simplest reasoning.

A set is given by its elements.

$A = \{1, 2, a, b, 13, \square\}$ is a set. Its elements are characters, in this case. Sets are denoted by capital letters A, B, C and their elements by lower-case letters a, b, c, x, y, z.
Most of the time we shall have sets of numbers

$$A = \{1, 2, 3\} \quad B = \{0, 1, 2, 3, 5\}$$

$$2 \in A, \quad 3 \in A, \quad 3 \in B, \quad 5 \in B$$

Where " $\in$ " denotes the fact that

an element <u>belongs to</u> a set.

$2 \in A$ means

" 2 belongs to the set A "

$0 \notin A$ means

" zero does not belong to the set A ".

<u>Inclusion</u>

$A \subseteq B$

" A included in B " or

" A a subset of B "

means

$$\boxed{\text{If } x \in A, \text{ then } x \in B}$$

$A = B$ if and only if

$A \subseteq B$ and $B \subseteq A$

<u>This is called double inclusion.</u>

Let $A = \{ x \mid x(x-1)(x+1) = 0 \}$

$B = \{ -1, 0, 1 \}$.

If $x \in A$ then

$$x(x-1)(x+1) = 0$$

So at least one of the factors must be equal to zero.

In case $x = 0$, then $0 \in B$ $\checkmark$

$x - 1 = 0$, then $1 \in B$ $\checkmark$

$x + 1 = 0$, then $-1 \in B$ $\checkmark$

We have shown that $A \subseteq B$.

Now If $x \in B$ then

in case $x = -1$, the factor $x+1 = 0$

in case $x = 0$, — " — $x = 0$

in case $x = +1$, — " — $x - 1 = 0$

So in any case $x(x-1)(x+1) = 0$

which says that $x \in \cancel{B} A$.

We conclude that $B \subseteq A$.

This has proven the equality $A = B$.

The <u>empty set</u> and the <u>universal set</u>

$\emptyset$ $\qquad\qquad$ U

Sometimes we want to know the largest possible set we can talk about. This will contain all elements we can encounter in the problem or theory we currently discuss.

In the examples before we could have assumed that U = set of real numbers.

ϕ the empty set is the set with no elements.

We could say "$x \in \phi$ is always not true."

U is the set containing all elements.

We could say "$x \in U$ is always true".

The complement

$\emptyset \subseteq A \subseteq U$ is always true.

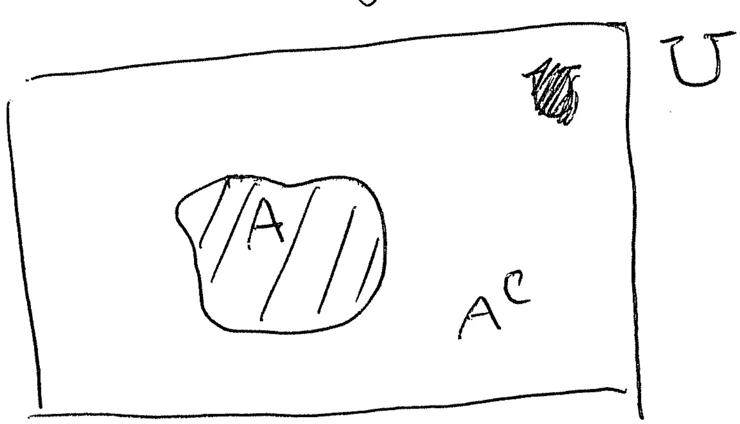
The set A^c = "Complement of A"
"A complement"

also denoted A' sometimes

is

$$A^c = \{ x \mid x \notin A \}$$

A Venn diagram



A = shaded area
A^c = the outside of A
U = the full rectangle

$U = \{0, 1, \ldots, 9\}$

1. $A = \{0, 1, 2, 3\}$

   $A^c = \{4, 5, 6, 7, 8, 9\}$

2. $A = \{$ even between 0 and 9 $\}$

   $A^c = \{$ odd — " — $\}$

3. $A = \emptyset \qquad A^c = U$

4. $(A^c)^c = A$

5. $A \subseteq B \iff B^c \subseteq A^c$

How to describe a set?

Two ways:

(i) Enumerate its elements

$A = \{2, 3, 4, 5\}$

(ii) By giving a defining property

$A = \{ x \mid x \text{ integer}, 2 \le x \le 5 \}$

(ii) can be written

$A = \{ x \mid P(x) \text{ true} \}$

Where $P(x)$ is a property of x.

$A = \{ x \mid P(x) \text{ true} \}$

$A^c = \{ x \mid P(x) \text{ false} \}$.

$B = \{ \text{even numbers} \}$

$B^c = \{ \text{odd numbers} \}$

Operations with sets

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

This is not "exclusive or".

$\{0,1,2\} \cup \{0,2,5,6\} = \{0,1,2,5,6\}$

Either $x \in A$, or $x \in B$ One must be true, or both.

$A \cup A^c = U$

$A \cup \phi = A$ for any set A.

$U \cup A = U$

<u>Note</u> $A = \{1,2,3\} = \{1,2,2,1,3\}$

repetitions don't matter
every element is considered only once. So even if we would think $B = \{2,3,4\}$

$A \cup B = \{1,2,3,2,3,4\} = \{1,2,3,4\}$.

Notice that $A \subseteq A \cup B$ for any pair A, B.

<u>Associativity</u>

$(A \cup B) \cup C = A \cup (B \cup C)$

Because of this property we can write
$A \cup B \cup C$ because we are allowed to compute from either end.

Intersection

$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

only the common elements belong to the intersection.

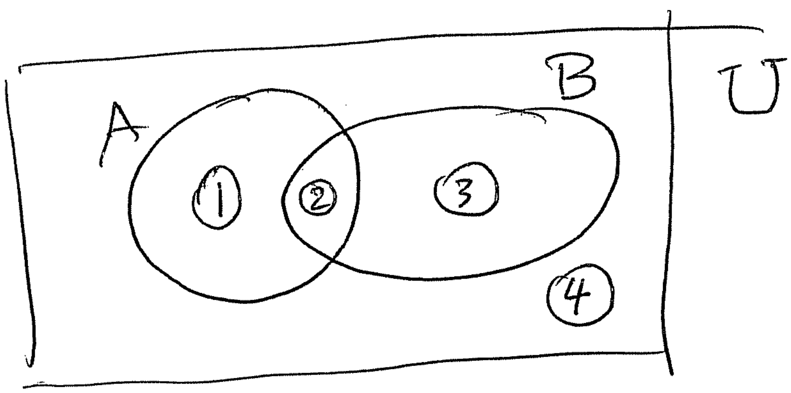


Figure 1

Two sets A and B :

$A \cap B$ is region 2

$A \cup B$ is region 1, 2, 3

Region 4 is $(A \cup B)^c$

$A = \{1, 2, 3\}$ $\quad B = \{3, 4, 2, 0\}$

$A \cap B = \{2, 3\}$ region 2

Notice more sets:

$A \cap B^c = \{x \mid x \in A \text{ and } x \notin B\}$

$= \text{region } 1$

$A^c \cap B = \text{region } 3$

$A^c \cap B^c = \text{region } 4$

Since 4 is also $(A \cup B)^c$

we have shown (based on Venn diagrams)

that

$$(A \cup B)^c = A^c \cap B^c$$

It is also true that

$$(A \cap B)^c = A^c \cup B^c$$

De Morgan Laws

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

EXAMPLE 8 Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$. ◁

EXAMPLE 9 Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. ◁

Set Identities

Table 1 lists the most important set identities. We will prove several of these identities here, using three different methods. These methods are presented to illustrate that there are often many different approaches to the solution of a problem. The proofs of the remaining identities will be left as exercises. The reader should note the similarity between these set identities and the logical equivalences discussed in Section 1.2. In fact, the set identities given can be proved directly from the corresponding logical equivalences. Furthermore, both are special cases of identities that hold for Boolean algebra (discussed in Chapter 11).

One way to show that two sets are equal is to show that each is a subset of the other. Recall that to show that one set is a subset of a second set, we can show that if an element belongs to the first set, then it must also belong to the second set. We generally use a direct proof to do this. We illustrate this type of proof by establishing the second of De Morgan's laws.