Relations on sets

1. Cross product of sets

For two sets A and B, the set of *ordered pairs* is denoted

$$A \times B = \{(a,b) \,|\, a \in A \,, b \in B\} \,.$$

The pairs (a,b) and (b,a) are not equal unless $a = b$. This is a natural way to juxtapose elements from two sets while keeping track of order.

If $A = B = \mathbb{R}$, then $A \times B$ is the plane $\{(x,y)| -\infty < x < \infty, -\infty < y < \infty\}$. For $A = \mathbb{R}$ and $B = [0,\infty)$ we obtain the upper half plane above the x - axis. When A and B are intervals, we obtain a rectangle.

Multiplication principle. When A, B are finite then $|A \times B| = |A||B|$, where we denoted $|A|$ the number of elements of A.

2. Relations

We are interested in defining *relations between elements of two sets.* Many times the "two sets" turn out to be the same, but there is no difficulty in considering the general case of distinct sets. In that case we say the relation $\mathcal{R}$ is on the set A, with the understanding that $\mathbb{R} \subseteq A \times A$.

$\boxed{\textbf{Definition. } \text{A relation on } A \times B \text{ is simply a } \textit{subset } \mathcal{R} \subseteq A \times B.}$

Notation. When $(a,b) \in \mathcal{R}$ we can use the notation

$a\mathcal{R}b$ and read *a is in the relation $\mathcal{R}$ with b*

A relation is
(i) **reflexive**, if $\forall a \in A$, then $a\mathcal{R}a$
(ii) **symmetric**, if $\forall a, b \in A$, if $a\mathcal{R}b$, then $b\dashv$.
(ii)' **antisymmetric**, if $\forall a, b \in A$, if $a\mathcal{R}b$ and $b\mathcal{R}a$, then $a = b$.
(a relation cannot have property (ii)' and (ii) at the same time, unless it is the equality)
(iii) **transitive**, if $\forall a, b, c \in A$, if $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a\mathcal{R}c$.

Definition. A relation with properties (i), (ii) and (iii) is said an **equivalence relation**.
A relation with properties (i), (ii)', (iii) is said an **order relation**.

If $\mathcal{R}$ is an equivalence relation on A, then for any $x \in A$ we define
the equivalence class of x and denote $\hat{x} = \{y \in A | x\mathcal{R}y\}$, the set of all elements in relation to x.

- $\hat{x} \cap \hat{y} \neq \emptyset \quad \Leftrightarrow \quad \hat{x} = \hat{y} \quad \Leftrightarrow \quad x\mathcal{R}y$
- $\cup_{x \in A}\, \hat{x} = A$

These two properties say that the sets $\hat{x}$, $x \in A$ form a partition of A.

3. Exercises

P 1.

- Determine the complement of $[0,1] \times [-1,1]$.
- Describe the first, second quadrants of the plane as cross products.
- If $V_{a,b} = \{(x,y) | x \leq a, y \leq b\}$, describe the set difference
$V_{1,4} \setminus V_{0,-1}$
- Describe the intersection of two rectangles.

P 2.

Determine which of the properties $(i), (ii), (ii)', (iii)$ are satisfied by the relations on the set of all people:
- a is taller than b
- a and b were born the same day
- a and b have a common grandparent.

P 3.

Let $A = \mathbb{R}$ and answer the same question form Problem 2 about $x\mathcal{R}y$ iff (if and only if):
1) $x + y = 0$
2) $x - y \in \mathbb{Q}$
3) $x = 2y$
4) $xy \geq 0$

P 4.

Let A be the set of all buildings owned by the university. Making reasonable assumptions, give examples of relations (number of floors, purpose, new/old, campus) and argue briefly if they are relations of equivalence or of order. If they are of equivalence, what are the equivalence classes?

P 5.

Let $A = \{1,2,3\}^2$ (ordered pairs in A) and say $(a,b) \equiv (c,d)$ if $a = d$ and $b = c$. Enumerate the classes of equivalence of this relation. Put the pairs in an array and argue that the classes are represented by the elements above the diagonal.

P 6.

Set $A = B = \mathbb{Z}$ and $\mathcal{R}$ one of the following relations:
$=, \equiv \pmod{m}$, where m is a positive integer. Show these are equivalence relations.
$\leq, \geq$ Show these are order relations.

P 7.

On $A = B = \mathbb{R} \times \mathbb{R} \setminus \{0\}$ define
$(x,y)\mathbb{R}(x',y')$ if and only if $xy' = x'y$. Show that this is an equivalence relation.

P 8.

On $\mathbb{R}$ define the relation

1) $a\mathcal{R}b$ if $a - b \in C = [0, \infty)$.

2) The set C can be generalized. Let A be the (x, y) plane. Define $C = \{(x, y) | x \geq 0\,, y = mx\,, |m| \leq t\}$, where $t \geq 0$ is a fixed non-negative number. This is an infinite cone with vertex at the origin. Show that the relation from part 1) is an order relation.

P 9.

Show that whenever C satisfies

$(0, 0) \in C$ then it is reflexive; $C = -C$ the relation is symmetric; $C + C \subseteq C$, then it it transitive. Give exact definitions for the sets $-C$, $C + C$ etc.

P 10.

Show that $a|b$ (a divides b) on $\mathbb{Z}_+$ is an order relation. Show that not all pairs can be ordered.

P 11.

Let $\mathcal{R}$ be a relation of equivalence on a set A. Then the classes of equivalence form a partition of A.

P 12.

Let $A = \{1, 2, 3, \ldots, 10\}$ and let B_1, B_2, B_3 and B_4 form a partition of A. Define $x\mathcal{R}y$ if and only if x and y belong to the same subset B_i of the partition. Show that $\mathcal{R}$ is an equivalence relation and $\hat{x}$ (the class of equivalence of x) is exactly B_i if $x \in B_i$, $i = 1, 2, 3, 4$. *Note that this construction can be generalized to any set and any partition. In fact, any relation of equivalence is given by a partition and the reciprocal is also true.*

P 13.

Lexicographic order. Words w of the same length L in the dictionary are ordered as follows:

$w \geq w'$ if

1) either for $1 \leq k \leq L$, the first $k - 1$ characters of w and w' coincide and the k-th character of w precedes the k - th character of w' in the alphabet,

2) or all characters are the same, then $w = w'$.

Show that the lexicographic order is an order relation.

P 14.

- Determine the pairs of integers $(0, 0) \leq (x, y) \leq (2, 1)$ in lexicographical order.
- order the two strings $(1, 1, 2)$ and $(1, 2, 1)$
- order the two strings $(1, 0, 1, 0, 1)$ and $(0, 1, 1, 1, 0)$
- explain how to order strings (or words) of possibly different length (think again of a dictionary). Try the example:

zoo, zero, zoom, zoology, zoological