problem 95. Ch 2

Why is $\sum_{k=1}^{n} \frac{1}{k}$, $n \geq 2$

NOT equal to an integer ?

The reason is that we shall show

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{p}{q} \quad \text{where} \quad \begin{array}{l} p \text{ odd} \\ q \text{ even} \end{array}$$

when $n \geq 2$

$\Rightarrow$ cannot be simplified to an integer.

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proof $n = 2 \Rightarrow \frac{3}{2}$ ✓

$n = 3$ $\frac{3}{2} + \frac{1}{3} = \frac{11}{6}$ true ✓

Suppose $S_n = \frac{p}{2^{\ell} m}$, p, m odd, $\ell \geq 1$

Case I $n+1$ even $\longrightarrow$

Case II $n+1$ odd

Case I $\quad n+1 = \text{even} = 2^{\alpha} r$, r odd. $\quad \alpha \geq 1$

Then $\dfrac{p}{2^{l} m} + \dfrac{1}{2^{\alpha} r} = \dfrac{2^{\alpha} pr + 2^{l} m}{2^{l+\alpha} m r}$

if $\alpha \geq l$ we simplify

$$\frac{2^{\alpha - l} pr + m}{2^{\alpha} m r} = \frac{\text{odd}}{\text{even}} \quad \text{because } \alpha \geq 1.$$

if $l > \alpha$ we simplify

$$\frac{pr + 2^{l-\alpha} m}{2^{l} m r} = \frac{\text{odd}}{\text{even}}, \quad \text{since } l \geq 1,$$

Case II $\quad n+1 = \text{odd} \Rightarrow$

$$\frac{p}{2^{l} m} + \frac{1}{n+1} = \frac{p(n+1) + 2^{l} m}{2^{l} m (n+1)} = \frac{\text{odd}}{\text{even}}.$$

since $l \geq 1$

In all cases the fraction cannot be simplified

$\Rightarrow S_n \notin \mathbb{Z}$.