SETS AND LOGIC

1. Sets

Sets are collections of elements. Usually they are denoted by capital letters (from the beginning of the alphabet): A, B, C. There are special notations for some sets:

$\emptyset$ the *empty set*, the set without any element.
$\mathbb{N}$ the set of positive integers (natural numbers)
$\mathbb{Z}$ the set of integers
$\mathbb{Q}$ the set of rational numbers
$\mathbb{R}$ the set of real numbers
$\mathbb{C}$ the set of complex numbers.

Sets can be given by
- *listing all elements* $A = \{1, 3, 5\}$
- *stating a property distinguishing its elements* $A = \{x \,|\, x \text{ odd positive less or equal to } 5\}$

$x \in A$ means that the element x belongs to the set A

$x \notin A$ means that the element x does not belong to A

Note that elements count only once in a set: $\{1, 2, 3\} = \{1, 2, 2, 3\}$ and their order *does not matter*: $\{1, 2, 3\} = \{2, 1, 3\}$. If we wish to distinguish two occurrences of the digit 2, we could create the set $\{2\}$ and have the new set $\{1, 2, \{2\}, 3\}$. This also shows that sets may be elements of larger sets.

1.1. Relations between sets.

We say that A is a *subset* of B (A included in B) and write $A \subseteq B$ when all elements of A also belong to B. The relation $A \subset B$ means that A is a *strict* subset of B (A strictly included in B), meaning that $A \subseteq B$ and $A \neq B$. The order of inclusion may be reversed. Then $B \supseteq A$ means that B includes A (the same as $A \subseteq B$), and $B \supset A$ means that B *strictly* includes A (the same as $A \subset B$).

To prove $A = B$, we have to show the *double inclusion* $A \subseteq B$ and $B \subseteq A$.

2. Operations with sets

2.1. Union.

$A \cup B$ is a new set of elements x such that
$x \in A \cup B$ if and only if $x \in A$ or $x \in B$
We read "A union with B". The key word is *or*.

2.2. Intersection.

$A \cap B$ is a new set of elements x such that
$x \in A \cap B$ if and only if $x \in A$ and $x \in B$
We read "A intersection with B". The key word is *and*.

2.3. Complement.

$A' = A^c$ is a new set of elements x such that
$x \in A'$ if and only if $x \notin A$; this is the same as *not* $(x \in A)$.
We read "A complement". The key word is *not*.

2.4. **Set difference.** $A \setminus B$ is a new set of elements x such that
$x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$.
We read "A minus B" or A set-minus B. The key word is *minus*. We see that $A \setminus B = A \cap B'$.

2.5. **The universal set.** Sometimes it is useful to have a *universal set* U containing all elements under consideration. Then $A' = U \setminus A$ for any set A. Note that $\emptyset' = U$, $U' = \emptyset$ and $U = A \cup A'$.

3. Propositional calculus

Logical statements are also known as *propositions*. A proposition can only be true (has value T or the number 1) or false (has value F or the number 0). There is a close relation between mathematical logic (propositional calculus) and set theory. Let's denote a the statement $x \in A$, b the statement $x \in B$, and so on. Then $x \notin A$ corresponds to *not* A, denoted by $\neg A$. The truth value of a and $\neg a$ are opposite: $\neg a$ is true when a is false and $\neg a$ is false when a is true. The *disjunction* of a and b, denoted by $a \vee b$ is true when *at least one* of a <u>or</u> b is true; the *conjunction* of a and b, denoted $a \wedge b$ is true only when *both* a <u>and</u> b are true - see the tables in [3] and [2].

The correspondence between sets and propositions is given by

$$A = \{x \,|\, a \text{ is true}\}.$$

In this way

$$A' = \{x \,|\, \neg a \text{ is true}\} \qquad A \cup B = \{x \,|\, a \vee b \text{ is true}\}.$$

$$A \cap B = \{x \,|\, a \wedge b \text{ is true}\} \qquad A \setminus B = \{x \,|\, a \wedge \neg b \text{ is true}\}.$$

We then have the table:

$x \in A$	a	
$x \notin A$	$\neg a$	not a
$x \in A \cup B$	$a \vee b$	a or b
$x \in A \cap B$	$a \wedge b$	a and b
$x \in A \setminus B$	$a \wedge \neg b$	a minus b

To summarize:

$\cup$	or	$\vee$
$\cap$	and	$\wedge$
$'$	not	$\neg$

3.1. **Logical equivalence.** Two statements (propositions) with the same truth values are *logically equivalent*. A statement which is always true is a *tautology* and one that is always false is a *contradiction*.

Example: $a \vee a \equiv a$, $a \wedge a \equiv a$ are logically equivalent and $a \vee \neg a$ is a tautology.

a	$a \vee a$	$a \wedge a$	$\neg a$	$a \vee \neg a$
1	1	1	0	1
0	0	0	1	1

3.2. **De Morgan's laws.** The relations

$$(B \cup C)' = B' \cap C', \qquad (B \cap C)' = B' \cup C'$$

between sets can be formulated in terms of propositional calculus

$$\neg(b \vee c) \equiv \neg b \wedge \neg c, \qquad \neg(b \wedge c) \equiv \neg b \vee \neg c$$

and verified with a truth value table. Both are referred as *de Morgan's laws.*

Proof.

b	c	$b \vee c$	$\neg(b \vee c)$	$\neg b$	$\neg c$	$\neg b \wedge \neg c$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Note that the fourth and seventh columns have the same truth values irrespective of the truth values of b, c. This concludes the proof.

3.3. **Set identities.** All set identities can be proven in this fashion. Verify the distributivity of the intersection with respect to union: For any sets A, B, C, we have the equality $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. First, note that for x to belong to the first set, the statement $a \wedge (b \vee c)$ must be true and for x to belong to the second set, the statement $(a \wedge b) \vee (a \wedge c)$ must be true. The two sets are equal if and only if the two propositions are logically equivalent. Construct the truth value table and verify.

Once we derive a sufficient number of set formulas, most others need not be proven with truth tables, but by simple application of basic rules of logic. **Venn diagrams** [4] are non-rigorous but useful tools to visualize such identities.

3.4. **The implication.** We say that a implies b and write $a \to b$ if b is true whenever a is true. In other words, the truth value of $a \to b$ is false only when a is true and b is false. Note that this implies that a false statement implies any other statement. Verify that

$$a \to b \equiv \neg a \vee b\,.$$

We say that a is *equivalent* to b and write $a \leftrightarrow b$ if a is true only when b is true. The relation is symmetric, meaning that if $a \leftrightarrow b$ then also $b \leftrightarrow a$. We have

$$a \leftrightarrow b \equiv (a \to b) \wedge (b \to a)\,.$$

This means that both b if a and a if b must be true in order to have a true statement. In common language, the double implication is expressed as *a if and only if b*. Also, the expression *if and only if* is abbreviated by *iff.*

3.5. **Composite statements.** Composite statements may be formulated by combining the propositions with the operators $\neg$, $\vee$, $\wedge$ and parentheses. An example is $\neg a \vee \neg(b \wedge \neg c) \vee d$. It is convened that $\neg$ has precedence over other operations. Thus $\neg p \vee q$ means $(\neg p) \vee q$ and not $\neg(p \vee q)$.

3.6. **Negation of a composite statement.** To negate a composite logical proposition, we replace each proposition p by $\neg p$, $\vee$ by $\wedge$ and $\wedge$ by $\vee$, while keeping the precedence order between parentheses. To negate the example form above we have

$$\neg[\neg\, a \vee \neg\,(b \wedge \neg\, c) \vee d] \equiv a \wedge (b \wedge \neg\, c) \wedge \neg\, d\,.$$

This rule is a consequence of the *double negation* (i.e. $\neg\neg p \equiv p$) and the de Morgan rules.

3.7. **Quantifiers.** We shall use mainly two quantifiers, *the existential quantifier*, denoted by $\exists$, and *the universal quantifier*, denoted by $\forall$. When we say "for any even integer n there exists an integer k such that $n = 2k$" we write

$$(\forall n)(\exists k)p(n,k)\,,$$

where $p(n,k)$ is the statement $n = 2k$ (as we see, it depends on n and k).

3.8. **Negation of statements containing quantifiers.** To negate a statement containing quantifiers, we change $\exists$ into $\forall$ and $\forall$ into $\exists$ and negate the statement according to the negation rules. The negation of the example above is

$$\neg\left[(\forall n)(\exists k)p(n,k)\right] = (\exists n)(\forall k)\ \neg\, p(n,k)\,.$$

In other words, "there exists n even such that, for all integers k, we have $n \neq 2k$". Equivalently, "there exists n even that cannot be written as the double of another integer". Notice that if we replace the set of even numbers with the set of multiples of three, then the negation is no longer false.

Universal statements are easy to disprove (prove that they are false) by finding a counterexample (example when they are false). For instance, "all prime numbers are odd" is false, since $n = 2$ is prime and even.

4. The Cartesian product

The Cartesian product (from René Descartes) of two sets A and B is denoted $A \times B$ and contains all *ordered pairs* (a,b) of elements $a \in A$ and $b \in B$. In other words, we consider all possible pairs when one element is in the first set, and the other is in the second set; the pair is ordered because $(1,2) \neq (2,1)$. Notice that $\{1,2\} = \{2,1\}$ as a set. Example:

$$\{1,2\} \times \{-1,0,1\} = \{(1,-1),(1,0),(1,1),(2,-1),(2,0),(2,1)\}$$

$$[1,2] \times \{-1,0,1\} = \{(x,y)\,|\,1 \le x \le 2\,, y = -1,0,1\}$$

$$[1,2] \times [-1,1] = \{(x,y)\,|\,1 \le x \le 2\,, -1 \le y \le 1\}$$

In the $x - y$ plane the first set represents six points, the second set represents three parallel segments at three levels $y = -1,0,1$ and the third represents a closed rectangle.

4.1. **Relations.** A *relation* between elements of A and elements of B is a subset $\mathcal{R}$ of $A \times B$. There are many kinds of relations, for instance *equivalence relations*, *order relations*, satisfying different sets of axioms. We are interested in a special kind of relation, called *function*.

4.2. **Functions.** A function f from A to B is a procedure that associates to *every* $a \in A$ a *unique* element $b \in B$. We write $b = f(a)$. It is possible that $f(a') = f(a'')$ for two different elements a' and a'' of A. However it is not accepted that $f(a) = b'$ and $f(a) = b''$ for $b' \neq b''$. The element $b \in B$ is the *image* of $a \in A$. Summarizing, a function has two constraints: all elements have an image, and that image is unique. We may regard f as a relation on $A \times B$ with the property that

(i) for all $a \in A$ there exists $b \in B$ such that $(a, b) \in f$,

(ii) if $(a, b') \in f$ and $(a, b'') \in f$ then $b' = b''$. In that case $(a, b) \in f$ is written $b = f(a)$ for convenience.

When A and B are subsets of the real numbers, we may plot A on the x-axis and B on the y- axis. A simple criterion to have a proper function is the *vertical line test*: Any vertical line should intersect the set f (seen as a subset of $A \times B$) at most once.

Formally, a function is a *triple* (A, B, f) where A is called the *domain* of the function (also denoted by D_f), B is the *co-domain* (also denoted by C_f) of the function and f is the procedure associating elements from A to elements from B. Two functions are equal if and only if all three elements are equal. Let $A = \{-1, 0, 1\}$ and $B = \{0, 1\}$, $f(x) = x^2$ and $g(-1) = 1$, $g(0) = 0$, $g(1) = 1$. Then $(A, B, f) = (A, B, g)$ while $(A, B, f) \neq (A, A, f)$.

- The set $\{(a, f(a)) \in A \times B \,|\, a \in A\}$ is called *the graph* of f.

- If $A' \subseteq A$, the set $f(A') = \{b \in B \,|\, \exists a \in A' \text{ such that } b = f(a)\}$ is called *the image* of A'.

- If $B' \subseteq B$, the set $f^{-1}(B') = \{a \in A \,|\, f(a) \in B'\}$ is called *the pre-image* of B'.

- When $A' = A$, $R_f = f(A)$ is the *range* of f.

- Sometimes a function is regarded as a procedure, or a formula; by abuse of notation we sometimes identify the function with the formula $y = f(x)$. This may make sense only because we are making assumptions on the correct domain and co-domain of the function, consistent with the context. In that case, the domain is assumed to be the largest set where the formula makes sense, and the co-domain is the full line $\mathbb{R} = (-\infty, \infty)$, unless more information is specified. Typically $f(x) = \sqrt{x - 3}$ will have domain $[3, \infty)$ and $g(x) = 1/(x - 1)$ will have domain $\mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, \infty)$.

4.3. **Composition.** Let (A, B, f) and (B, C, g) be two functions. Then the *composition* of f and g, denoted by $g \circ f$ is $(A, C, g(f(a)) = c)$. The composition is possible only if the co-domain of f equals the domain of g. The procedure associating $a \in A$ to $c \in C$ is: Transform a into $b = f(a)$ and the b into $g(b)$. In short form, $g \circ f(a) = g(f(a))$.

- Notice that $f \circ g$ may not make sense, and even if it does, in general we have $f \circ g \neq g \circ f$. Give examples of all these cases.

- For any set A, we define I_A, the *identity function of* A by the triple $(A, A, I_A(a) = a)$. In other words, $I_A(a) = a$ does not make any change on $a \in A$.

4.4. **Injective, surjective and bijective functions.** Let (A, B, f) be the function where A is the set of inhabitants in a city and B is the set of ten digit numbers, while $b = f(a)$ is the phone number of person a. There are many problems with this definition. First problem: If there were people without a phone, or one person would have more than one phone number at their residence, then it would not even be a function. Second problem: Supposing that everybody has only phone at home, in many cases more than one person is associated to the same phone number. If we change A with the set of residences in the city, then the function becomes one-to-one, meanwhile the original function was not one-to-one. Third problem: The image of the function f does not cover all ten digit numbers. Once we restrict B to just numbers starting with the city's area code, the function becomes what we call onto.

We say that the function is

- An injective (also called *one-to-one*) function has different images for different elements. That is, if $f(a) = f(a')$, then $a = a'$.

- A surjective function (also called *onto*) function takes all values in its co-domain. That is, $f(A) = B$.

- A function that is both injective and surjective is said *bijective*.

4.5. **The inverse function.** Consider the function (A, B, f) where A is the set of words in language 1 and B is the set of words in language 2, while $b = f(a)$ is the translation of word a from language 1 into word b from language 2 (a dictionary). Ideally this should be a bijection. Then the function translating from language 2 into language a (reverse dictionary) is called the inverse function to f.

A bijective function f always has an *inverse* function denoted by f^{-1}. More precisely, (B, A, f^{-1}) has domain equal to the co-domain of f, co-domain equal to the domain of f and $f^{-1}(b)$ is the unique element $a \in A$ such that $f(a) = b$. The existence of such a is guaranteed by surjection and the uniqueness by injection.

- Informally, finding the inverse of a function f amounts to solving $y = f(x)$ for a known y and an unknown x. For each y, the answer will be $f^{-1}(y) = x$. An example is $(\mathbb{R}, \mathbb{R}, f(x) = x + 2)$ with the inverse $(\mathbb{R}, \mathbb{R}, f^{-1}(y) = y - 2)$ because the solution to $y = x + 2$ is $x = y - 2$. Another example is $([0, \infty), [0, \infty), f(x) = x^2)$ with inverse $([0, \infty), [0, \infty), f(y) = \sqrt{y})$.

- The inverse of the inverse is the function itself, or $(f^{-1})^{-1} = f$.

- The function f and its inverse f^{-1} satisfy $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$. These are the *cancellation identities*. As an example, $\sqrt{a^2} = a$ and $(\sqrt{b})^2 = b$ when $A = B = [0, \infty)$.

References

[1] Robert G. Bartle, Donald R. Sherbert *Introduction to Real Analysis,* John Wiley and Sons 2000, Third Edition (Our textbook) Chapter 1 and Appendix 1.

[2] *Discrete Mathematics and Its Applications*, Fourth Edition by Kenneth H. Rosen, AT& T Laboratories `http://www.mhhe.com/math/advmath/rosen/` (used for MTH309)

[3] Truth tables, Wikipedia. `http://en.wikipedia.org/wiki/Truth_table`

[4] Interactive Real Analysis `http://www.mathcs.org/analysis/reals/logic/proofs/distlaw.html`