Midterm # 1.

March 1st 2002
Math 224, Section E
Introduction to Probability and Statistics

Do all problems. Show your work.

Name

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1. Let A, B and C be three *mutually independent* events such that

$$P(A) = \frac{1}{3}, \qquad P(B) = \frac{1}{2}, \qquad P(C) = \frac{2}{3}.$$

Find the probability that *exactly* one of the two events occurs.

Solution. We have to calculate

$$P(A' \cap B' \cap C) + P(A \cap B' \cap C') + P(A' \cap B' \cap C')$$

since the event we are looking at is the union of the events from above and they are mutually exclusive (cannot occur in the same time; as sets, they are disjoint).

We calculate $P(A') = 1 - P(A) = \frac{2}{3}$, $P(B') = 1 - P(B) = \frac{1}{2}$, $P(C') = 1 - P(C) = \frac{1}{3}$. By independence we have

$$P(A' \cap B' \cap C) = P(A')P(B')P(C) = (1 - P(A))(1 - P(B))P(C) = \frac{4}{18}$$

and, calculating all three probabilities and summing up we get the answer: $\frac{7}{18}$.

2. An urn contains 7 white and 5 red marbles. Two marbles are drawn at random without replacement. Use conditional probability to find the probability that they have the same color.

Solution. Let W_1 be the event that the first marble is white, W_2 the event that the second marble is white; the same for R_1 and R_2.

$$\{\text{same color}\} = (W_1 \cap W_2) \cup (R_1 \cap R_2).$$

The events that both marbles be white and the event that both marbles be red are mutually exclusive. Then

$$\begin{aligned} P(\{\text{same color}\}) &= P(W_1 \cap W_2) + P(R_1 \cap R_2) \\ &= P(W_2|W_1)P(W_1) + P(R_2|R_1)P(R_1) \end{aligned}$$

$$= \left(\frac{6}{11}\right)\left(\frac{7}{12}\right) + \left(\frac{4}{11}\right)\left(\frac{5}{12}\right) = \frac{62}{132}\,.$$

3. Two companies share the market for a certain electronic component such that company I has 40% of the market and company II has 60% of the market. It is known that 1% of products of company I are defective and 3% of products of company II are defective. We sample one product from the market and see that it is defective. What is the probability that it was manufactured by company II ?

Solutions. The companies represent a partition of the market. Let B_1 be the event that a product is manufactured by company I and B_2 by company II. Let D be the event that the item be defective. We know $P(B_1) = .40$, $P(B_2) = .60$ and $P(D|B_1) = .01$ and also $P(D|B_2) = .03$. We have to find $P(B_2|D)$.

$$P(B_2|D) = \frac{P(D|B_2)P(B_2)}{P(D|B_1)P(B_1) + P(D|B_2)P(B_2)}$$

$$= \frac{(0.03)(0.60)}{(0.03)(0.60) + (0.01)(0.40)} = \frac{9}{11} \simeq 82\%\,.$$

4. The probability density function of the profit X for a certain item is $f(x) = \frac{1}{3}x^2$ when $-1 \le x \le 2$, measures in dollars (the negative values correspond to losses).

a) Calculate the mean μ and the standard deviation σ of the

b) What is the median? Calculate the ninetieth percentile $\pi_{.90}$. How do you define $\pi_{.90}$ in words?

Solution.

a)

$$\mu = \int_{-1}^{2} x\left(\frac{1}{3}x^2\right)dx = \int_{-1}^{2}\left(\frac{1}{3}x^3\right)dx = \frac{1}{12}x^4\Big|_{-1}^{2} = \frac{15}{12} = 1.25$$

$$\int_{-1}^{2} x^2\left(\frac{1}{3}x^2\right)dx = \int_{-1}^{2}\left(\frac{1}{3}x^4\right)dx = \frac{1}{15}x^5\Big|_{-1}^{2} = \frac{11}{5} = 2.2$$

Then $\sigma^2 = 2.2 - (1.25)^2 = 0.6375$ and $\sigma = 0.798$.

b) Let p be a probability, $1 \le p \le 1$. We want to find $\pi_p = z$.

$$\int_{-1}^{z}\left(\frac{1}{3}x^2\right)dx = \frac{1}{9}x^3\Big|_{-1}^{z} = \frac{z^3 - (-1)^3}{9} = \frac{z^3+1}{9}\,.$$

We derive

$$fracz^3 + 19 = p \quad \Rightarrow \quad z = \sqrt[3]{9p-1}$$

we plug in $p = .5$ and obtain $\pi_{.5} = m = 1.632$ and $p = .90$ to obtain $\pi_{.9} = 1.921$.

5. How many different car plates can one make with the letters in the English alphabet (there are 26 letters) and the ten digits 0, 1, 2, ..., 8, 9 if

a) there is no restriction other than the plate must have only six symbols (repetitions allowed)

b) like a), but no repetitions allowed

c) the plate must have exactly three letters followed by three digits (repetitions allowed)

d) the plate must have exactly two letters followed by four digits without repetition

e) the digit 0 is present exactly three times (other symbols can repeat) and there are only six symbols.

Solutions.

a) $(26 + 10)^6 = 36^6 = 2,176,782,336$. Enough for all cars?

b) ${}_{36}P_6 = 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 = 1,402,410,240$. About half.

c) $26^3 \cdot 10^3 = 17,576,000$. Enough for Florida?

d) ${}_{26}P_2 \cdot_{10} P_4 = 3,276,000$.

e)

$$\binom{6}{3} \cdot 35^3 = 857,500\,.$$

It is enough to decide the positions occupied by the zeros; once this is done, the remaining $36 - 1$ symbols can be chosen without restriction.

6. Two events A and B have probabilities $P(A) = .25$ and $P(B) = .5$. Their union has probability $P(A \cup B) = .60$.

a) Find $P(A \cap B)$

b) Find $P(A|B)$.

c) Find $P(A|B^c)$.

d) Are the events independent?

Solutions.

a)

$$P(A \cap B) = .25 + .50 - .60 = .15$$

b)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.50} = .30$$

c) $P(A \cap B) + P(A \cap B') = P(A)$ then $P(A \cap B') = .25 - .15 = .10$. Also $P(B') = 1 - P(B)$. Then

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{0.10}{0.50} = .20$$

d)

$$P(A \cap B) = 0.10 \neq 0.125 = P(A)P(B)$$

The events **are not** independent.