Practice problems for Midterm # 1.MATH 224

1. Let $p(x) = cx^2$ be the probability mass function (pdf) of a random variable X with range of values $x = 0, 1, 2$. a) Calculate the value of c

b) Calculate the mean value μ and the standard deviation σ of X.

- do the same problem with $x = -1, 0, 1$.

Solution. a) c must be such that $\sum p(x) = 1$. Then $c = 1/5$ (work out the details).

b)

$$\mu = \sum xp(x) = (0)(\frac{0^2}{5}) + (1)(\frac{1^2}{5}) + (2)(\frac{2^2}{5}) = \frac{9}{5} = 1.8$$

$$\mu^2 + \sigma^2 = \sum x^2 p(x) = (0^2)(\frac{0^2}{5}) + (1^2)(\frac{1^2}{5}) + (2^2)(\frac{2^2}{5}) = \frac{17}{5} = 3.4$$

$$\sigma = \sqrt{\frac{17}{5} - \left(\frac{9}{5}\right)^2} = \frac{2}{5} = 0.4\,.$$

2. An urn contains eight green marbles and twelve blue marbles. We draw two marbles from the urn without replacement. We define the events

$$A = \{\text{the first marble is blue}\}$$
$$B = \{\text{the second marble is blue}\}$$
$$C = \{\text{both are blue}\}$$
$$D = \{\text{at least one is blue}\}\,.$$

Calculate the probability of the four events. All marbles are identical except for color and equally likely to be selected at each draw. *Hint: use conditional probability.*

Solution. $P(A) = \frac{12}{20}$. The probability $P(B)$ should be equal to $P(a)$. We can calculate it in the following way. Notice that A' is the event that the first marble is green. Then

$$P(C) = P(C) = P(A \cap B) = P(B \mid A)P(A) = \left(\frac{11}{19}\right)\left(\frac{12}{20}\right) = \frac{132}{380} = .3473$$

and

$$P(B \cap A') = P(B \mid A')P(A') = \left(\frac{12}{19}\right)\left(\frac{8}{20}\right) = \frac{96}{380}\,.$$

This gives

$$P(B) = P(B \cap A) + P(B \cap A') = P(B \mid A)P(A) + P(B \mid A')P(A') = \frac{228}{380} = .6\,.$$

Finally

$$P(D) = 1 - P(B' \cap A') = \frac{324}{380} = .8526\,.$$

3. A cell phone depends on a certain type of batteries which have a probability of $p = 75\%$ to charge properly. How many batteries are needed to ensure a 99% chance of being able to use the cell phone on a long trip? We assume the batteries are manufactured independently of each other. *Hint: write down the events A_i that* *a battery fails for each battery $i = 1, 2, \ldots, n$ and then express the event that you cannot use the phone in terms of these events.*

Solutions. $P(A_i) = 1 - p = .25$. We are interested in

$$P(\text{at lease one is ok}) = 1 - P(\text{all fail})$$
$$= 1 - P(A_1 \cap A_2 \cap \ldots \cap A_n) = 1 - (1-p)^n \geq .99$$

which implies

$$.01 \geq (1-p)^n \quad \Rightarrow \quad n \geq \frac{\ln(.01)}{\ln(1-p)} = 3.32$$

(we mention that the inequality changed sign when we divided by $\ln(1-p) < 0$). The nearest larger integer is $n = 4$.

4. Suppose we are offered to buy a portfolio made of six stocks from a set of twenty stocks, seven of which are high risk (H), five are moderate risk (M) and eight are low risk (L).

a) How many such portfolios are available ? *Hint: the risk category is irrelevant here.*

b) A portfolio is said balanced if it contains exactly two stocks of each risk category. How many balanced portfolios are available ? *Hint: use combinations.*

Solution.

a) There are 6 objects to choose out of 20 objects, that is

$${}_{20}C_6 = \frac{(20)(19)(18)(17)(16)(15)}{(1)(2)(3)(4)(5)(6)} = 38760\,.$$

b) There are ${}_7C_2 = 21$ ways to choose the (H) stocks. There are ${}_5C_2 = 10$ ways to choose (M) stocks and there are ${}_8C_2 = 28$ ways to choose (L) stocks. By the counting principle there are $({}_7C_2)({}_5C_2)({}_8C_2) = 5880$ ways to choose a balanced portfolio.

5. Let A and B be events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cap B) = \frac{1}{6}$. Find

a) $P(A' \cup B')$

b) $P(A' \cap B)$

c) $P(A' \cup B)$

d) $P(A' \cap B')$

and write in words what the events from a) - d) mean. Example: $A \cup B'$ means that *either A occurs or B does not occur.* **Hint:** Draw a diagram.

Solution. a)

$$P(A' \cup B') = 1 - P(A \cap B) = \frac{5}{6}$$

"either A does not occur or B does not occur" =the negate of "both A and B occur"

b)

$$P(A' \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = \frac{1}{12}$$

"A does not occur and B occurs"

c)

$$P(A' \cup B) = 1 - P(A \cap B') = 1 - P(A \setminus B) = 1 - (P(A) - P(A \cap B)) = \frac{5}{6}$$

"either A does not occur or B occurs"

d)

$$P(A' \cap B') = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = \frac{7}{12}$$

"A does not occur and B does not occur"= "neither A nor B occurs".

The "wording" part is worth one point and the calculation of the probability is four points for each question.

Grading: a), b), c) are worth 6 points each, 2 points are free. d) is worth 5 points extra credit, listed separately.

6.

a) How many distinct six-letter words can one generate with the letters of the word "CANADA"?

b) A typical password for a computer system has three letters and three digits, in that order. How many passwords are available ? (there are 26 letters and 10 digits).

c) My friend's family has seven members. In how many ways can they be seated in a row?

d) Same setting as as c). How about being seated at a round table?

Solution.

a)There are 6 letters in the word CANADA, but only 4 are distinct. Henceforth we can form ${}_6P_6 = 6!$ words. However, there are repetitions: the letters 'A' can be permuted in ${}_3P_3 = 3!$ ways and the word is the same for any particular choice of three positions of the letters 'A' in the word. The other letters appear only once, so they don't repeat. Formally we can say they repeat ${}_1P_1 = 1!$ times. This gives the multinomial formula for the number of words N

$$N = \frac{6!}{3!1!1!1!} = 120\,.$$

b) The multiplication principle:

$$\#\ \text{passwords} = 26^3 \cdot 10^3 = 17,576,000\,.$$

c) Seating them in a row means choosing a particular permutation of the seven people taken seven at a time, that is ${}_7P_7 = 7!$.

d) At a round table, the relative position of the party matters but the seating is considered identical if we can obtain the same configuration by simply rotating the table. Let's establish one seat to be 'Seat # 1' and order the other seats, from 1 through 7, clockwise. This makes the seating identical to seating in a row. Consequently, there are 7! configurations. To eliminate the repetitions, we see that we could have obtained the same configurations by choosing any of the other seats as the 'Seat # 1'. We conclude there are $(7!) \div (7) = 6!$ seating possibilities at the round table.

7. Four white socks and eight olive socks are kept in a drawer. We can only pick one sock at a time. We draw two socks randomly and without replacement.

a) Find the conditional probability that the second sock be white if the first was white.

b) Find the probability that we pick the same color.

Note: Show your work. A nice setup will be appreciated.

Solution. Let X_1 be the color of the first sock we draw and X_2 the color of the second sock we draw. Then, let's abbreviate the color white with W and orange with O.

a)

$$P(X_2 = W \mid X_1 = W) = \frac{4-1}{8+(4-1)} = \frac{3}{11} = .2727 = 27.27\% \,.$$

b)

$$\begin{gathered} P(\text{same color}) = P(\text{either } \{\text{both white}\} \text{ or } \{\text{both olive}\}) = \\ P(X_1 = W \,,\, X_2 = W) + P(X_1 = O \,,\, X_2 = O) = \\ P(X_2 = W \mid X_1 = W)P(X_1 = W) + P(X_2 = O \mid X_1 = O)P(X_1 = O) = \\ \frac{3}{11} \cdot \frac{4}{12} + \frac{7}{11} \cdot \frac{8}{12} = \frac{17}{33} = .5151 = 51.51\% \,. \end{gathered}$$

8. An engine has a built-in redundant system. The engine needs one microchip to work properly. The safety system consists of n microchips connected in parallel fashion, such that the engine will work unless all microchips fail at the same time. The probability of failure of one microchip is 28%. Determine the minimum n needed to ensure a 99.9% probability of proper operation of the engine.

Solution. Denote by A_i the event that the ith microchip fails. Then

$$\begin{aligned} P(\{\text{the engine works}\}) &= 1 - P(\{\text{all microchips fail}\}) \\ &= 1 - P(A_1 \cap A_2 \cap A_3 \dots A_n) \,. \end{aligned}$$

The events A_i, for all $i = 1, 2, 3, \dots, n$ are INDEPENDENT so that

$$P(A_1 \cap A_2 \cap A_3 \dots A_n) = P(A_1)P(A_2)P(A_3) \dots P(A_n) = p^n \,.$$

We want that

$$1 - p^n \geq \frac{99.9}{100} = .999 \Rightarrow .001 \geq p^n$$

so that

$$n \geq \frac{\ln(0.001)}{\ln(0.28)} = 5.42 \Rightarrow n \geq 6 \,.$$