The probability distribution of a random variable $Y$ can be given in three ways:

1. **Probability mass function (pmf)** $p(y)$, with $y \in \mathcal{R}(Y)$ (the range of $Y$). The only restrictions are
   $$p(y) \geq 0 \quad \text{and} \quad \sum_{y} p(y) = 1$$

2. **Cumulative distribution function (cdf)** $F_Y(y) = P(Y \leq y)$. For discrete r.v.
   $$F_Y(y) = \sum_{y' \leq y} p(y')$$

**Example 1.**

$\mathcal{R}(Y) \subseteq \{0, 1, 2, 3, \ldots\}$ (integer values) then

$$p(y) = F_Y(y) - F_Y(y - 1)$$

$$P(a < Y \leq b) = F_Y(b) - F_Y(a)$$

(pay attention to $\leq$ and $<$)

$$F_Y(y) \uparrow \lim_{y \to -\infty} F_Y(y) = 0 \quad \lim_{y \to +\infty} F_Y(y) = 1$$

When $A \leq y \leq B$, meaning that the range $\mathcal{R}(Y) \subseteq [A, B]$ we have

$$y < A \Rightarrow F_Y(y) = 0 \quad \text{and} \quad y \geq B \Rightarrow F_Y(y) = 1$$

3. **Moment generating function (mgf)** $M_Y(t) = E[e^{tY}]$ (a function of $t$)

$$M_Y(t) = \sum_{y} e^{ty}p(y)$$

Definition: the $k$-th moment of $Y$ is $M_k = E[Y^k]$. Special cases: $k = 0$ then $M_0 = 1$; $k = 1$ then $M_1 = \mu$; $k = 2$ then $M_2 = \sigma^2 + \mu^2$ notice that if we know $M_1$ and $M_2$ we know $\mu, \sigma^2$

Since $e^{tY} = \sum_{k=0}^{\infty} \frac{Y^k}{k!} t^k$ then

$$M_Y(t) = \sum_{k=0}^{\infty} \frac{M_k}{k!} t^k, \quad M_Y^0(0) = M_k$$

The $k$-th order derivative at $t = 0$ equals the $k$-th moment.
Properties of the mgf.

- **Uniqueness** two r.v. $Y_1$ and $Y_2$ have the same MGF $\Leftrightarrow$ they have the same distribution.
- **Summation of independent r.v.**
  
  If $Y_1$ and $Y_2$ are independent, then $M_{Y_1+Y_2}(t) = M_{Y_1}(t)M_{Y_2}(t)$

- $M'_Y(0) = E[Y]$ $M''_Y(0) = E[Y^2] = \sigma^2 + \mu^2$

- Define $R(t) = \ln M(t)$ - the cumulant of $Y$.
  
  $R'(0) = \mu$ $R''(0) = \sigma^2$

Example 2.

Binomial distribution $Y \sim Bin(n, p)$ ($q = 1 - p$).

$$M_{Bin(n, p)}(t) = \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y q^{n-y} = \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y q^{n-y} = (pe^t + q)^n$$

$n = 1 \Rightarrow M_{Bernoulli(p)}(t) = pe^t + q$

Example 3.

Geometric distribution $Y \sim Geo(p)$ $p(y) = pq^{y-1}, y = 1, 2, 3, \ldots$

$$M_{Geo(p)}(t) = \sum_{y=1}^{\infty} e^{ty} pq^{y-1} = pe^t \left( \sum_{y=1}^{\infty} (qe^t)^{y-1} \right) = \frac{pe^t}{1 - qe^t}$$

Example 4.

Let $Y$ be the number of successes in $n$ independent Bernoulli($p$) trials $Y_1, Y_2, \ldots, Y_n$ (known as “coin flips”). Because $Y_i$ is zero or one, we have $Y = \sum_{i=1}^{n} Y_i$. Then

$$M_Y(t) = (pe^t + q)(pe^t + q)\ldots(pe^t + q) = (pe^t + q)^n = M_{Bin(n, p)}(t)$$

from Example 2. By uniqueness we have

$$\sum_{i=1}^{n} Y_i \sim Bin(n, p).$$

(we already knew that)
Example 5.
We know that if $Y_1, Y_2, \ldots, Y_r$ are $r$ independent identically distributed geometric r.v. ($Y_i \sim Geo(p)$). Since

$$Y = \sum_{i=1}^{r} Y_i \sim NB(r, p)$$

We have a formula for the MGF of $NB(r, p)$, based on Example 3

$$M_{NB(r,p)}(t) = \left( \frac{pe^t}{1-qe^t} \right)^r$$

Example 6.
For $Y \sim NB(r, p)$, $R(t) = r(\ln p + t - \ln(1 - qe^t))$.

$$R'(t) = r(1 + \frac{qe^t}{1 - qe^t}) \quad \mu = R'(0) = r \left( 1 + \frac{q}{1 - q} \right) = \frac{r}{p}$$

$$R''(t) = r \left( 1 + \frac{qe^t}{1 - qe^t} \right)' = \frac{r q e^t}{(1 - q e^t)^2} \quad \sigma^2 = R''(0) = \frac{rq}{p^2}.$$  

Example 7.
An easy way to calculate the distribution of $Y = Y_1 + Y_2$ where $Y_i, i = 1, 2$ are the outcomes of rolling two independent fair die - $Y_i \sim Unif(1, 2, 3, 4, 5, 6)$.

$$M_Y = \frac{1}{36}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})^2 = \frac{1}{36}(e^{2t} + 2e^{3t} + 3e^{4t} + 4e^{5t} + 5e^{6t} + 5e^{7t} + 4e^{8t} + 3e^{9t} + 2e^{10t} + 2e^{11t} + e^{12t})$$

$$= \sum_{y=2}^{12} e^{yt}p(y)$$

with