Hydrodynamic Limit for a Fleming-Viot Type System

Ilie Grigorescu [a,*], Min Kang [b]

[a] *Department of Mathematics, University of Miami, 1365 Memorial Drive, Ungar Building, room 525, Coral Gables, FL 33124-4250*

[b] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695*

Abstract

We consider a system of N Brownian particles evolving independently in a domain D. As soon as one particle reaches the boundary it is killed and one of the other particles is chosen uniformly and splits into two independent particles resuming a new cycle of independent Brownian motion until the next boundary hit. We prove the hydrodynamic limit for the joint law of the empirical measure process and the average number of visits to the boundary as N approaches infinity.

Key words: Fleming-Viot, hydrodynamic limit, catalytic branching, absorbing Brownian motion
1991 MSC: 60K35, 60J50, 35K15

1 Introduction

In [2], Burdzy, Hołyst, Ingerman and March propose a variant of the Fleming-Viot model in which the branching mechanism is triggered by the event that a random walk reaches the boundary of an open set from the Euclidean space. Later on (in [3]), the same authors propose a continuous time Brownian model, which motivates our work.

Let $d \in \mathbb{Z}_+$ and D be a bounded open subset of $\mathbb{R}^d$ with piecewise smooth boundary of class C^2 satisfying the exterior cone condition. We fix a positive integer N and consider the Nd-dimensional process with values in D^N defined

* Corresponding author.
Email addresses: igrigore@math.miami.edu (Ilie Grigorescu), kang@math.ncsu.edu (Min Kang).

iteratively as follows. The bold notation will be used to designate vectors. Let $\{\mathbf{w}_i(\cdot)\}$, with $1 \leq i \leq N$ be N independent Brownian motions on $\mathbb{R}^d$ with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, starting at (possibly random) points $x_i \in D$, respectively. As soon as one Brownian particle is killed upon reaching the boundary ∂D, one of the remaining $N-1$ particles chosen with equal probability gives birth to a new independent Brownian particle at the same location. The total number of particles is preserved, and the new system of N particles, with starting points inside D, perform again independent Brownian motions until one of them hits the boundary, when the branching procedure is repeated. The consistency of the construction is discussed in [3]. The particles can never reach the boundary more than one at a time and the number of boundary hits in any bounded time interval is finite, almost surely. If $\mathbf{x}(0) = (x_1, \ldots, x_N) \in D^N$ is the initial configuration, then for a fixed N we shall denote by $P^N_{\mathbf{x}}$ or simply P^N the law of the process. In general, we shall consider that all processes $\{\mathbf{x}^N(\cdot)\}$, for all N, are constructed on the same probability space $(\Omega, \mathcal{F}, P)$ with the same filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

While the construction from above underscores the analogy with the Fleming-Viot evolution, it is equivalent to a dynamics on $\mathbf{D}([0,\infty), D^N)$, the Skorohod space, where the Brownian particles, once they have reached the boundary ∂D, jump with uniform probability to one of the locations of the remaining $N-1$ particles. For each time $t > 0$ and each path $x_i(\cdot)$, we shall denote by $A^N_i(t)$ the total number of visits to the boundary ∂D of $x_i(s-)$, when $s \in [0,t]$. With probability one, this number is equal to the number of jumps of the particle $x_i(\cdot)$ up to time t. This fact is a consequence of the continuity of the Brownian paths, and the continuity of the distribution of the hitting times to the boundary of independent Brownian motions, which prevents the possibility that two particles be on the boundary at the same time and forces that any jump be nontrivial.

In comparison to the Fleming-Viot branching system, where Brownian particles die and choose uniformly the location where they are reborn among the positions of the remaining particles at *independent exponential times*, the present model is self-pacing the redistribution of particles with a clock counting the hitting times to the boundary. One could regard this as a form of *catalytic branching* with the boundary acting as a catalyst. In both models there is conservation of mass, however in the present model the correlation between update times and the location of particles makes impossible the speed up of the branching process leading to a deterministic limit as opposed to a superprocess.

Let D be the space of admissible genetical configurations (the allelic profile) of a certain population. The Fleming-Viot and the present models are descriptions of the slow diffusion of the profiles for a fixed population size N. The boundary ∂D represents a collection of 'extinction' or non-viable profiles. It is reasonable although an idealization to consider that an individual with a viable profile is added to the population at the moment when another one becomes non-viable. Our result proves the deterministic nature of the sample mean profile and the average number of 'extinctions' for a large but fixed population. In that sense, one can note the convergence to an equilibrium configuration given by the normalized first eigenfunction of the Dirichlet Laplacian (Corollary 1).

The main objective of the paper is to prove the hydrodynamic limit for the branching Brownian particles confined to the domain D (Theorem 1). The construction of the process is based on Theorem 1.1 from [3]. All the other results are independent, with the exception of Corollary 1, which is not needed in the proof of Theorem 1, the main result.

One has to differentiate between the original result from [3] and Theorem 1. A preliminary benefit of this proof is that we can drop the requirement that particles start at deterministic location. The law of large numbers at the level of the path space is a result about the joint law of the process, as opposed to the one valid for the one-dimensional marginals. The question of convergence of the time-dependent empirical distributions is more natural in the context of the study of measure-valued processes (in this case a branching process). It is the full trajectory of the particle profile which becomes deterministic in the scaling limit satisfying equation (19), and not only its distribution at a given time. In order to evaluate this, one has to prove the law of large numbers for the average number of visits to the boundary, also at the level of the path space, which is a completely new result. In addition, the limits allow very strong absolute continuity estimates, for example showing that the average number of particles located in a certain subset of the domain remains roughly proportional to the volume, *uniformly* in time - the contents of (44), (47), (49) and (50). These estimates can be extended, in the end, all the way to the boundary ∂D as a consequence of (20).

It is worth mentioning that the method used can be generalized to diffusions under natural regularity conditions. Finally, this approach leads to an exact derivation of the asymptotic law of the tagged particle, together with a proof of the propagation of chaos presented in [4].

Plan of the proof. The interaction between particles consists in the redistribution mechanism activated as soon as they reach the boundary. The average number of visits to the boundary (16) and the empirical measure (15) vary at the same rate and on the same scale N^{-1} and proving tightness for one implies tightness for the other as seen in (91). The first step is to obtain a hydrodynamic limit (Lemma 1) for a transformation of the empirical measure (21) which puts negligible mass in a neighborhood of the boundary, which is done in Section 3. The technical difficulty here is to prove that a measure-valued weak solution to the heat equation with Dirichlet boundary conditions is a function (absolute continuity). This result gives us control over the average number of particles in any set $D_0 \subset\subset D$ (Corollary 2) through Propositions 3, 4 and 5. Section 4 proves Theorem 2, which establishes control over the number of particles visiting (or at least situated near) the boundary. Section 5 proves the tightness of the average number of visits to the boundary in Theorem 3, which is based on a very careful accounting of the activity near ∂D. Violating the tightness estimate is equivalent to the occurrence of either one of two very unlikely events. One is the accumulation of a large number of particles $[\epsilon N]$ in a layer of thickness $r << \epsilon$ neighboring the boundary ∂D, which was already taken care of in Section 4. The other is the migration of a massive number of particles $O(\epsilon N)$ across a macroscopically thick region in a short amount of time $\eta << \epsilon$. The last section completes the proof of Theorem 1.

2 The Results.

Let $f \in C(\overline{D}^N)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ be a point in $\overline{D}^N$ and let i, j be two indices between 1 and N. We shall denote by $f^{ij}(\mathbf{x})$ the $N-1$ variable function depending on $\mathbf{x}$ with the exception of the component x_i which is replaced by x_j, that is

$$f^{ij}(\mathbf{x}) = f(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_N) \tag{1}$$

and by Δ_N the Nd dimensional Laplacian. The family of independent $\mathbb{R}^d$ Brownian motions $\mathbf{w}_i(\cdot)$, $1 \le i \le d$ is adapted to $\mathcal{F} = \{\mathcal{F}_t\}_{t \ge 0}$. The point processes $\{A_i^N(\cdot)\}_{1 \le i \le N}$ counting the number of boundary hits for each particle $1 \le i \le N$ are adapted to the filtration $\mathcal{F}$, finite and converge to infinity almost surely, as shown in [3]. The construction of the process implies the following proposition.

Proposition 1 *For any function $f \in C(\overline{D}^N)$, with f smooth up to the boundary, we write*

$$\mathbf{A}_f(t) = \sum_{i=1}^{N} \frac{1}{N-1} \int_0^t \sum_{j \neq i} \Big(f^{ij}(\mathbf{x}(s-)) - f(\mathbf{x}(s-)) \Big) dA_i^N(s) \,. \tag{2}$$

Then,

$$f(\mathbf{x}(t)) - f(\mathbf{x}(0)) - \int_0^t \frac{1}{2} \Delta_N f(\mathbf{x}(s)) ds \; - \mathbf{A}_f(t) = \mathcal{M}_f^{N,B}(t) + \mathcal{M}_f^{N,J}(t) \tag{3}$$

where

$$\mathcal{M}_f^{N,B}(t) = \sum_{i=1}^{N} \int_0^t \nabla_{x_i} f(\mathbf{x}(s)) \cdot d\mathbf{w}_i(s) \tag{4}$$

is the Brownian martingale and $\mathcal{M}_f^{N,J}(t)$ *is the jump martingale for which*

$$(\mathcal{M}_f^{N,J}(t))^2 - \frac{1}{N-1}\sum_{i=1}^{N}\int_0^t \sum_{j\neq i}\Big(f(\mathbf{x}^{ij}(s-)) - f(\mathbf{x}(s-))\Big)^2 dA_i^N(s) \tag{5}$$

is a martingale. All martingales are P*-martingales with respect to the filtration* $\mathcal{F}$*.*

Remark 1: Since the support of the counting measures $\{dA_i^N(t)\}_{t\geq 0}$ is the set of hitting times of the boundary, the function $f(\mathbf{x}(s-))$ in (2) has the i^{th} component situated on ∂D.

Remark 2: By construction $f^{ij}(\mathbf{x}(s)) - f(\mathbf{x}(s-)) = f^{ij}(\mathbf{x}(s-)) - f(\mathbf{x}(s-))$ on the support of $dA_i^N(t)$, which makes the integrand $\mathcal{F}_{s-}$-measurable.

Proof. The continuous part of the semi-martingale (3) is obtained by applying the Itô formula on the time intervals between jumps. The pure jump martingale is equal to

$$\sum_{\tau\in J(\omega)\cap[0,t]} f(\mathbf{x}(\tau)) - f(\mathbf{x}(\tau-)) - \mathbf{A}_f(t) \tag{6}$$

where $J(\omega)$ is the discrete set of random jump times and has the quadratic variation from (5). More precisely, for a given deterministic time $t > 0$, let $\ell(t)$ be the number of boundary hits $\{\tau_l\}_{0\leq l\leq \ell(t)}$ in the time interval $[0,t]$. The probability distributions of the visits to the boundary are continuous, hence with probability one t is not a jump time. For simplification, write $t = \tau_{\ell(t)+1}$. Then, almost surely

$$\begin{gathered} f(\mathbf{x}(t)) - f(\mathbf{x}(0)) = \\ = \textstyle\sum_{l=0}^{\ell(t)}\Big(f(\mathbf{x}(\tau_{l+1}-)) - f(\mathbf{x}(\tau_l))\Big) + \sum_{l=1}^{\ell(t)}\Big(f(\mathbf{x}(\tau_l)) - f(\mathbf{x}(\tau_l-))\Big). \end{gathered} \tag{7}$$

Again with probability one,

$$f(\mathbf{x}(\tau_{l+1}-)) - f(\mathbf{x}(\tau_l)) = \int_{\tau_l}^{\tau_{l+1}-}\frac{1}{2}\Delta_N f(\mathbf{x}(s))ds + \int_{\tau_l}^{\tau_{l+1}-}\nabla_{x_i} f(\mathbf{x}(s))\cdot d\mathbf{w}_i(s) \tag{8}$$

and

$$\begin{aligned} \mathcal{M}_f^{N,J}(t) = \sum_{l=1}^{\ell(t)}\Bigg\{ & f(\mathbf{x}(\tau_l)) - f(\mathbf{x}(\tau_l-)) \\ & - \sum_{i=1}^{N}\mathbf{1}_{\partial D}(x_i(\tau_l-))\Big[\frac{1}{N-1}\sum_{j\neq i}\Big(f^{ij}(\mathbf{x}(\tau_l-)) - f(\mathbf{x}(\tau_l-))\Big)\Big]\Bigg\} \end{aligned} \tag{9}$$

equal to (6). The pure jump martingale (9) is such that

$$(\mathcal{M}_f^{N,J}(t))^2 - \sum_{l=1}^{\ell(t)} \left\{ \sum_{i=1}^{N} \mathbf{1}_{\partial D}(x_i(\tau_l-)) \left[\frac{1}{N-1} \sum_{j\neq i} \Big(f^{ij}(\mathbf{x}(\tau_l-)) - f(\mathbf{x}(\tau_l-)) \Big)^2 \right] \right\} \tag{10}$$

is a martingale, thus establishing (5). □

Let $\frac{1}{2}\Delta_N$ be the Nd dimensional half Laplacian on $L^2(D^N)$ with domain

$$\mathcal{D} = \left\{ f \in C^2(\overline{D}^N) \;:\; \forall\, i\,, (N-1)^{-1} \sum_{j\neq i} f^{ij}(\mathbf{x}) = f(\mathbf{x}) \,\text{whenever}\, x_i \in \partial D \right\}. \tag{11}$$

We notice that $\mathcal{D}$ contains the functions $f \in C^2(\overline{D}^N)$ vanishing on the boundary as well as on any diagonal $x_{j'} = x_{j''}$ (where $j' \neq j''$ between 1 and N), a subset of functions dense in $L^2(D^N)$. Constants are included in $\mathcal{D}$ and the maximum principle is valid in D^N. This allows us to regard $(\frac{1}{2}\Delta_N, \mathcal{D})$ as a Markov pregenerator on $L^2(D^N)$. Then, the measure P^N solves the martingale problem $(\frac{1}{2}\Delta_N, \mathcal{D})$ starting at $\mathbf{x}_0 = \mathbf{x}(0)$.

Let $\phi \in C^2(\overline{D})$ and for a given index i let $f(\mathbf{x}) = \phi(x_i)$. Formula (3)-(2) reduces to

$$\begin{aligned}\phi(x_i(t)) = \phi(x_i(0)) + \int_0^t \frac{1}{2}\Delta_d \phi(x_i(s))ds \, + \qquad\qquad & (12)\\ + \int_0^t \Big(\frac{1}{N-1} \sum_{j\neq i} \phi(x_j(s)) - \phi(x_i(s-)) \Big) dA_i^N(s) + & \\ + \int_0^t \nabla\phi(x_i(s)) \cdot d\mathbf{w}_i(s) + \mathcal{M}_\phi^{N,J}(t)\,. & \end{aligned}$$

In a similar fashion, if $f(\mathbf{x}) = N^{-1}\sum_{i=1}^N \phi(x_i)$, formula (3) reads

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N \phi(x_i(t)) = \frac{1}{N}\sum_{i=1}^N \phi(x_i(0)) + \int_0^t \frac{1}{N}\sum_{i=1}^N \frac{1}{2}\Delta_d\phi(x_i(s))ds \qquad & (13)\\ + \frac{1}{N}\sum_{i=1}^N \int_0^t \Big(\frac{1}{N-1}\sum_{j\neq i} \phi(x_j(s)) - \phi(x_i(s-)) \Big) dA_i^N(s) & \\ + \frac{1}{N}\sum_{i=1}^N \int_0^t \nabla\phi(x_i(s)) \cdot d\mathbf{w}_i(s) + \mathcal{M}^{N,J}_{\langle\phi,\mu^N\rangle}(t)\,. & \end{aligned}$$

For $r > 0$ sufficiently small we define the set

$$D_r = \left\{ x \in D \;:\; d(x, \partial D) > r \right\}. \tag{14}$$

Definition 1 *Let r_D be the inner radius of the domain D, defined as the supremum of all $r > 0$ with the properties that D_r has the same number of connected components as D and ∂D_r is of the same regularity class as ∂D, in our case, C^2. For $r \in (0, r_D/2)$, we define the function $\gamma_r \in C^2(\overline{D})$ as a smooth version of $\mathbf{1}_{D_r^c}$ with the properties (i) $0 \leq \gamma_r(x) \leq 1$ if $x \in \overline{D}$, (ii) $\gamma_r(x) = 1$ if $x \in D_r^c$, (iii) $\gamma_r(x) = 0$ if $x \in D_{2r}$ and (iv) $\|\Delta\gamma_r(x)\|_\infty \leq c(D)r^{-2}$ for a constant $c(D)$ determined by the domain D and independent of $r > 0$.*

Definition 2 *For any $N \in \mathbb{Z}_+$ we define the empirical distribution process*

$$\mu^N(t, dx) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)} \tag{15}$$

and the average number of jumps

$$A^N(t) = \frac{1}{N-1} \sum_{i=1}^{N} A_i^N(t)\,. \tag{16}$$

In general, when $\alpha(\cdot, dx)$ is an element of $\mathbf{D}([0,\infty), \mathcal{M}(D))$ and ϕ is a bounded continuous function on D we shall write $\int_D \phi(x)\alpha(\cdot, dx) = \langle \phi, \alpha(\cdot, dx)\rangle$.

Remark: The pre-factor $(N-1)^{-1}$ is only technical in order to simplify the formula (28). Asymptotically as $N \to \infty$, $A^N(t)$ as defined in (16) is the same as the actual average of the boundary hits by all particles.

Definition 3 *The family of empirical distributions $\{\mu^N(dx)\}_{N>0}$ on the set D is said asymptotically nondegenerate at the boundary if, for any $\epsilon > 0$*

$$\lim_{r\to 0} \limsup_{N\to\infty} P\Big(\int_D \gamma_r(x)\mu^N(dx) > \epsilon \Big) = 0\,. \tag{17}$$

Remark: In case $\mu^N(0, dx)$ converges weakly to a probability measure concentrated on D the condition (17) is automatically fulfilled. Since $D \subseteq \overline{D}$ any family of measures on D is precompact yet we want to prevent the mass from running away to the boundary.

Let $p_{abs}(t, x, y)$ be the absorbing Brownian kernel on the set D and, for a finite measure $\mu(dx) \in \mathcal{M}(D)$, we denote by $u(t, y) = \int_D p_{abs}(t, x, y)\mu(dx)$ the solution in the sense of distributions to the heat equation with Dirichlet boundary conditions

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2}\Delta_x u(t, x) \qquad u(t, x)\Big|_{x\in\partial D} = 0 \qquad u(0, x) = \mu(dx)\,. \tag{18}$$

We also define $z(t) = \int_D u(t,x)dx > 0$ the probability of survival up to time $t > 0$ of a Brownian particle killed on the boundary ∂D and starting with distribution $\mu(dx)$. The solution to the heat equation with Dirichlet boundary conditions conditional on survival up to time t is $v(t,x) = z(t)^{-1}u(t,x)$ and $\mu(t,dx) = v(t,x)dx$ is the weak solution of

$$\frac{\partial}{\partial t}v(t,x) = \frac{1}{2}\Delta_x v(t,x) - \frac{z'(t)}{z(t)}v(t,x) \qquad v(t,x)\Big|_{x\in\partial D} = 0 \qquad v(0,x) = \mu(dx)\,. \tag{19}$$

We can state the main result.

Theorem 1 *If $\mu^N(0,dx)$ converges in probability in weak sense to a deterministic initial density profile $\mu(dx) = \mu(0,dx)$ such that $\mu(D) = 1$, then, for any $T > 0$, the joint distribution of $(A^N(\cdot), \mu^N(\cdot,dx)) \in \mathbf{D}([0,T], \mathbb{R}_+ \times \mathcal{M}(D))$ is tight in the Skorohod topology and the set of limit points is a delta function concentrated on the unique continuous trajectory $(-\ln z(\cdot), \mu(\cdot,dx))$ as defined in (19) and, for any $\phi \in C^2(\overline{D})$ and any $\epsilon > 0$*

$$\lim_{N\to\infty} P\Big(\sup_{t\in[0,T]}\Big|\frac{1}{N}\sum_{i=1}^{N}\phi(x_i(t)) - \int_D \phi(x)\mu(t,dx)\Big| > \epsilon\Big) = 0\,. \tag{20}$$

Let $M^N(dx)$ be the unique stationary distribution of the process $\{\mathbf{x}(\cdot)\}$ (the measure exists according to [3]) and $\Phi_1(x)$ be the first eigenfunction of the Laplacian with Dirichlet boundary conditions normalized such that it integrates to one over D. It is known that $\Phi_1(x) > 0$ in D and under general regularity conditions for D (smooth ∂D) is continuous on $\overline{D}$ and vanishes at the boundary. This allows us to regard $\Phi_1(x)$ as a probability density function over the domain D.

Corollary 1 *Assume the process $\{\mathbf{x}(\cdot)\}$ is in equilibrium at time $t = 0$. Then, the family of empirical measure processes $\{\mu^N(\cdot,dx)\}_{N\in\mathbb{Z}_+}$ is tight in the Skorohod space $\mathbf{D}([0,T], \mathcal{M}(D))$ and the unique limit point is the delta function concentrated on the constant measure $\Phi_1(x)dx$.*

Proof. The proof has two parts. The first part requires to show that the empirical measures

$$\mu^N(0,dx) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(0)}$$

when $x_i(0)_{1\le i\le N}$ have joint distribution $M^N(dx)$ are tight as measures on the open set D. The proof of this fact is a consequence of Theorem 1.4 from [3]. However, the only fact we need is the asymptotic non-degeneracy at the boundary, and not the limit proper.

The second part is an immediate consequence of Theorem 1. Since the process is in equilibrium, any weak limit of the empirical measure process must be constant in time and satisfy (19). Unless $v(x)$ is identically zero, which is impossible since the empirical measures have mass one, the factor $z'(t)/z(t)$ is constant, which implies that the limit is an eigenfunction. On the other hand, we know that the first eigenfunction $\Phi_1(x)$ is positive on D. If another eigenfunction were nonnegative, the inner product with Φ_1 will show it must be zero almost surely. Since the solution is a probability measure (the total mass is one), the proof is complete. $\square$

3 General estimates.

In the following, $\nu(t, dx)$ denotes the finite measure $u(t, x)\, dx$ defined in (18). Let

$$\nu^N(t, dx) = exp(-A^N(t))\mu^N(t, dx) \tag{21}$$

be a transformation of the empirical measure process. For D_0 an open set such that $\overline{D_0} \subset D$ we define the restriction of $\nu^N(t, dx)$ to D_0 by $\nu^N_{D_0}(t, dx)$.

Proposition 2 *Let ν^N be as in (21). Assume that the deterministic measure $\mu(0, dx)$ with $\mu(0, D) = 1$ is the weak limit in probability of $\mu^N(0, dx)$. Let $T > 0$. (a) For any $\phi \in C^2(\overline{D})$ vanishing at the boundary the family of processes $\{\langle \phi, \nu^N(t, dx)\rangle\}_{N>0}$ is tight in the Skorohod topology on $\mathbf{D}([0, T], \mathbb{R})$ and any limit point belongs to $C([0, T], \mathbb{R})$. (b) The law of $\nu^N_{D_0}(t, dx)$ converges weakly to the delta function on $C([0, T], \mathcal{M}(D_0))$ concentrated on the unique deterministic solution of (18) with initial value $\nu(0, dx) = \mu(0, dx)$.*

Remark: The processes $\nu^N(t, dx)$ are tight in $\mathbf{D}([0, T], \mathcal{M}(D))$ but the proof of this fact will be completed in Section 6, Theorem 1. The conditions for weak tightness are fulfilled as long as we concentrate on test functions vanishing on the boundary. If we examine (13) we see that for arbitrary functions ϕ the integrands of the jump terms $A^N_i(t)$ contain a boundary term which would not reduce in the differential formula for the derived process $\nu^N(t, dx)$. Essentially Proposition 2 is the hydrodynamic limit of the transformed processes $\nu^N(\cdot, dx)$ seen as measure-valued processes on open sets $D_0 \subset\subset D$. However, the uniform estimates in the current section can be obtained from the present result at no further cost and will be used in the proof of Theorem 1.

Proof. Step 1: part (a) and tightness for (b).

Let $(X, \|\cdot\|)$ be a Polish space. The conditions for tightness of a family of processes $\{y^N(\cdot)\}_{N>0}$ with values in X seen as measures on the Skorohod space

$\mathbf{D}([0,T],X)$ which ensure that any limit point belongs to $C([0,T],X)$ are

$$\text{(i) there exists an } M>0 \text{ such that } \quad \limsup_{N\to\infty} P\Big(\|y^N(0)\|>M\Big)=0 \quad \text{and} \tag{22}$$

$$\text{(ii) for any } \epsilon>0 \quad \lim_{\delta\to 0}\limsup_{N\to\infty} P\Big(\sup_{\substack{s,\,t\,\in\,[0,T]\\ |t-s|\,<\,\delta}} \|y^N(t)-y^N(s)\|>\epsilon\Big)=0\,. \tag{23}$$

To prove the tightness of $\nu^N_{D_0}(t,dx)$ in weak sense we have to verify (i) and (ii) for any test function $\phi \in C^2(\overline{D_0})$ (which include the smooth bounded functions on D_0). Tightness for the processes $\nu^N_{D_0}(t,dx)$ is implied by the proof of tightness for $\langle \phi, \nu^N(t,dx)\rangle$ by considering functions $\phi \in C^2(\overline{D})$ which vanish on the boundary ∂D restricted to D_0. In order to verify (i) it is sufficient to see that the test functions are bounded and the total mass of the empirical measures is one.

Condition (ii) (23) will be shown to be fulfilled as a consequence of 1) the time integral on the right side of (28) satisfies (23) due to the uniform boundedness of $\Delta\phi$, 2) the martingale term has a quadratic variation of order N^{-1} and 3) the pure jump part is of order N^{-1} as well.

Let $\mathbf{X}(t)=(X_1(t),\ldots,X_m(t))$ be an m-dimensional semi-martingale and F a smooth function on $\mathbb{R}^m$. Denote

$$\tilde{\Delta}X(t)=\sum_{0\le s\le t}\Big(X(s)-X(s-)\Big) \tag{24}$$

and $\langle (X_k)^c,(X_l)^c\rangle(s)$ the cross variation of the continuous martingale parts of $X_k(t)$ and $X_l(t)$. Then, we can write

$$F(\mathbf{X}(t))-F(\mathbf{X}(0))=\sum_{l=1}^{m}\int_0^t \partial_l F(\mathbf{X}(s-))dX_l(s) \tag{25}$$

$$+\frac{1}{2}\sum_{k,l=1}^{m}\int_0^t \partial_{k\,l}F(\mathbf{X}(s-))d\langle (X_k)^c,(X_l)^c\rangle(s) \tag{26}$$

$$+\sum_{0\le s\le t}\Big[F(\mathbf{X}(s))-F(\mathbf{X}(s-))-\sum_{k=1}^{m}\partial_k F(\mathbf{X}(s-))\tilde{\Delta}X_k(s)\Big]\,. \tag{27}$$

For any fixed $N>0$ and any $\phi\in C^2(\overline{D})$ vanishing on the boundary ∂D we can apply Itô's formula for semimartingales (25)-(27)(see [5], Chapter I, Section 4) in the two-dimensional case $m=2$ to the pair of bounded semimartingales $(X_1(t),X_2(t))=(A^N(t),\int_D \phi(x)\mu^N(t,dx))$ and the function

$F(X_1, X_2) = exp(-X_1)X_2$. We obtain that $\nu^N(t, dx) = F(X_1(t), X_2(t))$ satisfies for any $t > 0$

$$\int_D \phi(x)\nu^N(t, dx) - \int_D \phi(x)\nu^N(0, dx) = \tag{28}$$

$$\int_0^t \int_D \frac{1}{2}\langle \Delta_d \phi(x)\nu^N(u, dx)\rangle du + \tag{29}$$

$$\int_0^t \exp\Big(- \frac{1}{N-1}\sum_{i=1}^N A_i^N(u)\Big)\frac{1}{N}\sum_{i=1}^N \nabla\phi(x_i(u)) \cdot d\mathbf{w}_i(u) + \mathcal{E}^N(t) \tag{30}$$

with the error term such that $E[\sup_{0\le t\le T} |\mathcal{E}^N(t)|^2]$ is of order N^{-1}. In order to see this, keeping in mind the integral formula (13) expressing $X_2(t)$, the error term obtained by applying (25)-(27) to the special case (28) will be divided in two parts. Let $\mathcal{E}_1(t)$ be the error term issued from the right hand side of (25) and $\mathcal{E}_2(t)$ be the error term equal to (27). On the right hand side of (25), the $dA^N(t)$ term cancels out, this being the feature motivating the transformation (21). The du (29) term and the Brownian martingale term (30) are not part of $\mathcal{E}^N(t)$. Consequently $\mathcal{E}_1(t)$ is equal to the integral against the jump martingale

$$\mathcal{E}_1(t) = \int_0^t \exp(-A^N(s-))d\mathcal{M}^{N,J}_{\langle\phi,\mu^N\rangle}(s)\,. \tag{31}$$

Doob's maximal inequality and the computation of the quadratic variation for a pure jump process provide a bound uniform in time

$$E\Big[\sup_{0\le t\le T} |\mathcal{E}_1^N(t)|^2\Big] \le N^{-1}C_\phi E\left[\int_0^T \exp(-2A^N(s-))dA^N(s)\right]. \tag{32}$$

This estimate is based on (5) applied to (13), using the fact that the absolute value of the integrand is bounded by a multiple of $N^{-1}\|\phi\| \exp\{-A^N(s)\}$, where the constant C_ϕ depends on the dimension d and the supremum norm of ϕ. It is obviously sufficient to provide a bound for the case when the integrand is $\exp(-A^N(s))$. We remind that the average number of jumps (16) is divided by $N-1$ (and not by N) for convenience. Then (32) is bounded above by

$$\begin{aligned} N^{-1}C_\phi E\Big[\int_0^T \exp(-A^N(u-))dA^N(u)\Big] &\\ \le N^{-1}(N-1)^{-1}C_\phi E\Big[\sum_{l=0}^{(N-1)A^N(t-)} \exp(-\frac{l}{N-1})\Big] &\\ \le N^{-1}C_\phi\Big((N-1)(1-e^{-\frac{1}{N-1}})\Big)^{-1} = O(N^{-1})\,. &\end{aligned}$$

We move on to investigate the error $\mathcal{E}_2^N(t)$ equal to the pure jump term (27). Assume without loss of generality that the jump consists of particle k situated at time $\tau-$ on the boundary going to the location of particle j. If J denotes the set of jump times, the pure jump term (27) is the sum over all $\tau \in J$ of

$$\Big[e^{-(A^N(\tau-)+\frac{1}{N-1})}\Big(\langle\mu^N(\tau-,dx),\phi(x)\rangle \;+\frac{1}{N}\phi(x_j(\tau-))\Big)-$$
$$e^{-A^N(\tau-)}\langle\mu^N(\tau-,dx),\phi(x)\rangle\;\Big]$$
$$-\Big[\Big(-e^{-A^N(\tau-)}\frac{1}{N-1}\Big)\langle\mu^N(\tau-,dx),\phi(x)\rangle+\frac{1}{N}\phi(x_j(\tau-))e^{-A^N(\tau-)}\Big)\Big]=$$
$$e^{-A^N(\tau-)}\Big[(e^{-\frac{1}{N-1}}-1+\frac{1}{N-1})\langle\mu^N(\tau-,dx),\phi(x)\rangle$$
$$+\frac{1}{N}(e^{-\frac{1}{N-1}}-1)\phi(x_j(\tau-))\Big]\;.$$

The absolute value of each jump has upper bound $C_1\|\phi\|(N-1)^{-2}e^{-A^N(\tau-)}$ where C_1 depends only on the exponential function. Finally, the sum of the jump terms is bounded above by

$$\frac{C_1\|\phi\|}{(N-1)^2}\sum_{\tau\in J\cap[0,t]}e^{-A^N(\tau-)}\le\frac{C_1\|\phi\|}{(N-1)^2}\sum_{l\ge 0}e^{-\frac{l}{N-1}}=O(\frac{1}{N})\,,$$

uniformly in time, which provides a stronger bound that the needed maximal inequality.

In order to complete the proof of (ii) (23) for $\langle\phi,\nu^N(\cdot,dx)\rangle$, we see that for any $\epsilon>0$,

$$\limsup_{N\to\infty}P\Big(\sup_{t\in[0,T]}\Big\{\frac{1}{N}\sum_{i=1}^N\int_0^t\exp(-A^N(u))\nabla\phi(x_i(u))\cdot d\mathbf{w}_i(u)\Big\}>\epsilon\Big)\tag{33}$$

$$\le\limsup_{N\to\infty}N^{-1}\epsilon^{-2}E\left[\int_0^T\frac{1}{N}\sum_{i=1}^N\|\nabla\phi(x_i(u))\|^2du\right]=0$$

by the martingale maximal inequality. Finally, the du term in (28) is bounded uniformly in N by $|t-s|(\|\Delta\phi\|/2)$.

Step 2: the weak heat equation. We have shown that the joint distribution of the processes

$$Q^N_{D_0}=P\circ\nu^N_{D_0}(\cdot,dx)^{-1}=P\circ\left[\exp\Big(-\frac{1}{N-1}\sum_{i=1}^N A_i^N(\cdot)\Big)\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i(\cdot)}\Big)\right]^{-1}\tag{34}$$

is tight on $\mathbf{D}([0,T],\mathcal{M}(D_0))$ and any limiting measure Q_{D_0} is concentrated on the set $C([0,T],\mathcal{M}(D_0))$. For $\psi\in C^2(\overline{D})$ vanishing on the boundary ∂D, the functional

$$C([0,T],\mathcal{M}(D_0))\ni m(\cdot,dx)\to\Psi(m(t,dx))=\tag{35}$$
$$=\sup_{t\in[0,T]}\Big|\int_{D_0}\psi(x)m(t,dx)-\int_{D_0}\psi(x)m(0,dx)-\int_0^t\int_{D_0}\frac{1}{2}\Delta_d\psi(x)m(s,dx)ds\Big|$$

is continuous and bounded. Assume that Q_{D_0} is a limit point of the tight family of measures $\{Q^N_{D_0}\}$ and let $N' \to \infty$ be a subsequence converging to Q_{D_0}. It follows that, for any $\epsilon > 0$,

$$Q_{D_0}\Big(\nu \,:\, \Psi(\nu(t,dx)) > \epsilon\Big) \leq \liminf_{N'\to\infty} P\Big(\Psi(\nu^{N'}_{D_0}(t,dx)) > \epsilon\Big) = 0 \tag{36}$$

due to the presence of the vanishing martingale term. This shows that Q_{D_0} is concentrated on the set of measures $\nu_{D_0}(\cdot, dx)$ indexed by time which satisfy

$$\int_{D_0} \psi(x)\nu_{D_0}(t,dx) - \int_{D_0} \psi(x)\nu_{D_0}(0,dx) = \int_0^t \int_{D_0} \frac{1}{2}\Delta\psi(x)\nu_{D_0}(s,dx)ds \tag{37}$$

for any $\psi \in C^2(\overline{D})$ vanishing on the boundary ∂D.

Step 3: properties of the weak solution. We shall follow the proof of Proposition 3.4 from [6]. The weak equation (37) can be extended to smooth functions $\psi(t,x)$ vanishing on the boundary ∂D. The new form of the equation is

$$\begin{aligned} &\int_{D_0} \psi(t,x)\nu_{D_0}(t,dx) - \int_{D_0} \psi(0,x)\nu_{D_0}(0,dx) \\ &= \int_0^t \int_{D_0} \Big(\frac{\partial}{\partial s}\psi(s,x) + \frac{1}{2}\Delta\psi(s,x)\Big)\nu_{D_0}(s,dx)ds\,. \end{aligned} \tag{38}$$

Let $g \in L^1(D)$. The restriction of g to D_0 is in $L^1(D_0)$. We define

$$\tilde{g}(s,x) = \int_D g(y)p_{abs}(t+h-s,x,y)dy \tag{39}$$

for arbitrary $h > 0$ and $t \in [0,T]$. Then $\tilde{g}(s,x)$ is smooth and vanishes on the boundary ∂D. We apply (38) to $\psi(s,x) = \tilde{g}(s,x)$ and obtain

$$\int_{D_0} \tilde{g}(t,x)\nu_{D_0}(t,dx) = \int_{D_0} \tilde{g}(0,x)\nu_{D_0}(0,dx)\,.$$

We derive

$$\begin{aligned} \Big|\int_{D_0} \tilde{g}(t,x)\nu_{D_0}(t,dx)\Big| &= \Big|\int_{D_0} \tilde{g}(0,x)\nu_{D_0}(0,dx)\Big| \\ \leq \int_{D_0} |\tilde{g}|(0,x)\nu_{D_0}(0,dx) &\leq C(t+h)\int_D |g(x)|dx \end{aligned}$$

by Fubini's theorem, where $C(t+h)$ is $\sup_{x,y\in D} p_{abs}(t+h,x,y)$. Furthermore, $C(t+h)$ is bounded above by a constant $C_0(t) > 0$ uniformly in h. Let G be an open set $G \subseteq D_0$. Fatou's lemma applied to $\Big|\int_{D_0} \tilde{g}(t,x)\nu_{D_0}(t,dx)\Big|$ as $h \to 0$ shows that, for $g(x) = \mathbf{1}_G(x)$

$$\nu_{D_0}(t,G) \leq \liminf_{h\to 0} \int_{D_0} \tilde{g}(t,x)\nu_{D_0}(t,dx) \leq C_0(t)\int_D |g(x)|dx = C_0(t)|G|\,. \tag{40}$$

For any time $t > 0$, we have shown that 1) $\nu_{D_0}(t, dx)$ is absolutely continuous with respect to the Lebesgue measure having a density $u_{D_0}(t, x)$ and 2) $u_{D_0}(t, x)$ is uniformly bounded in x and D_0 by the constant $C_0(t)$.

Step 4: identification of the solution. At this stage we know the solution $\nu_{D_0}(\cdot, dx)$ only as a limit point of the tight measures $\{\nu^N(\cdot, dx)|_{D_0}\}_{N\in\mathbb{Z}_+}$. Since the measures depending on N are consistent, the limit points are consistent as well. The solutions $\nu_{D_0}(t, dx) = u_{D_0}(t, x)dx$ are consistent in the sense that, if $D_0' \subseteq D_0''$, then $\nu_{D_0'}(t, dx) = \nu_{D_0''}(t, dx)$ on D_0'. We have shown that, for any $t \in [0, T]$, there exists a function $u(t, x)$ defined on the open set D such that $u(t, x)|_{D_0} = u_{D_0}(t, x)$. Moreover, for $\psi \in C^2(\overline{D})$ vanishing on the boundary ∂D, the function u satisfies (37)

$$\int_{D_0} \psi(x)u(t'', x)dx - \int_{D_0} \psi(x)\nu(t', dx) = \int_{t'}^{t''} \int_{D_0} \frac{1}{2}\Delta\psi(x)u(s, x)dxds \quad (41)$$

for any $0 \le t' \le t'' \le T$. To make sure that $u(t, x)$ is the weak solution to (18), we must prove that, for any $\psi \in C^2(\overline{D})$ vanishing on the boundary ∂D

$$\int_{D} \psi(x)u(t'', x)dx - \int_{D} \psi(x)\nu(t', dx) = \int_{t'}^{t''} \int_{D} \frac{1}{2}\Delta\psi(x)u(s, x)dxds \quad (42)$$

Let $n \in \mathbb{Z}_+$ and D_0^n be an increasing sequence of open subsets of D such that $\overline{D_0^n} \subset D$ and $\cup_{n\ge1} D_0^n = D$. Because u has a uniform bound $C_0(t)$ for any $t > 0$ established in (40), we can pass to the limit in (41) as $D_0^n \to D$ which implies that u is the solution to the heat equation with zero boundary conditions on D.

The left hand side of equation (37) has a limit as $D_0^n \to D$ by dominated convergence. It is clear that the right hand side term of equation (37) has a limit as well. We need to prove that the limit is equal to the integral over the full domain D. We cannot use dominated convergence directly for times t approaching zero. However, the integral over D_0^n is known to have a unique limit over *any* choice of sequences D_0^n. Consequently we can pass to the limit in (37) as $D_0^n \to D$ and obtain (42). Standard PDE results for linear parabolic equations imply that the solution is unique in case $\nu(0, dx)$ is deterministic. The weak solutions will be in fact strong solutions if $\nu(0, dx) = \rho_0(x)dx$. □

Lemma 1 *For any function $\phi \in C_c^2(D)$, any $\epsilon > 0$ and for any $N \in \mathbb{Z}_+$ there exists an event $S^N_{unif}(\phi)$ such that*

$$\begin{gathered} S^N_{unif}(\phi) = \left\{ \sup_{t\in[0,T]} \left| \int_D \phi(x)\nu^N(t, dx) - \int_D \phi(x)\nu(t, dx) \right| > \epsilon \right\} \quad \text{and} \\ \limsup_{N\to\infty} P\Big(S^N_{unif}(\phi)\Big) = 0\,. \end{gathered} \quad (43)$$

Proof. We fix the function $\phi(x)$. The limit $\nu(t,dx) = u(t,x)dx$ is the solution to the heat equation with zero boundary conditions from (18), a continuous function of time when applied to ϕ. Since we are interested in establishing (43) in the case when ϕ has compact support in D, we shall identify the measures $\nu^N(t,dx)$ with their restrictions to an open set $D_0 \subset\subset D$ including the support of ϕ, as in the proof of Proposition 2. The functional $\Phi(g(\cdot)) = \sup_{t\in[0,T]} |g(t) - g_0(t)|$ is continuous on the Skorohod space $\mathbf{D}([0,T],\mathbb{R})$ when $g_0(\cdot)$ is continuous (see [1]). Let $g_0(t) = \langle \phi, \nu(t,dx)\rangle$. Since $\alpha(\cdot,dx) \to \langle \phi, \alpha(\cdot,dx)\rangle$ is continuous as a functional on $\mathbf{D}([0,T],\mathcal{M}(D_0))$ we derive that $\alpha(\cdot,dx) \to \Phi(\langle \phi, \alpha(\cdot,dx)\rangle)$ is continuous. We recall the law $Q^N_{D_0}$ from (34). Then

$$\limsup_{N\to\infty} Q^N_{D_0}\Big(\{\alpha \,:\, \Phi(\langle \phi, \alpha(\cdot,dx)\rangle) \geq \epsilon\}\Big) \leq$$
$$Q_{D_0}\Big(\{\alpha \,:\, \Phi(\langle \phi, \alpha(\cdot,dx)\rangle) \geq \epsilon\}\Big) = 0\,,$$

concluding the proof. □

Proposition 3 *Under the conditions of Proposition 2, for a given time interval* $[0,T]$*, there exists a constant* $C(T) > 0$ *and for each* $N \in \mathbb{Z}_+$ *an event* S^N_A *such that*

$$S^N_A = \Big\{A^N(T) > C(T)\Big\} \quad \textit{and} \quad \limsup_{N\to\infty} P\Big(S^N_A\Big) = 0\,. \tag{44}$$

Proof. Let $\Phi_1(x)$ be the normalized first eigenfunction of the Laplacian with Dirichlet boundary conditions on D such that $\int_D \Phi_1(x)dx = 1$. Let $\mu(dx)$ a probability measure on D and $u(t,x)$ given in (18). By applying Green's formula to $u(t,\cdot)$ and $\Phi_1(x)$ on D we see that the function $\int_D u(t,x)\Phi_1(x)dx$ is nonincreasing as a function of time. We derive that

$$\inf_{t\in[0,T]} \int_D u(t,x)\Phi_1(x)dx = 2\nu_T > 0\,. \tag{45}$$

For a sufficiently small $r > 0$, the function $\Phi^r_1(x) = \Phi_1(x)(1-\gamma_r(x))$ has compact support and, by continuity,

$$\inf_{t\in[0,T]} \int_D u(t,x)\Phi^r_1(x)dx = \nu_T > 0\,. \tag{46}$$

We take $C(T) = \log(2\nu_T^{-1}\|\Phi^r_1\|_\infty)$. The series of inclusions

$$\Big\{A^N(T) > C(T)\Big\} \subseteq \Big\{\exists\, t \in [0,T] \,:\, \exp(A^N(t)) > e^{C(T)}\Big\} \subseteq$$

$$\left\{\exists\, t \in [0,T] \,:\, \exp(A^N(t)) > \frac{\int_D \Phi^r_1(x)\mu^N(t,dx)}{\int_D \Phi^r_1(x)\nu(t,dx) - \frac{\nu_T}{2}}\right\} \subseteq$$

$$\left\{\exists\, t\in[0,T]\,:\,\exp(-A^N(t))\int_D \Phi_1^r(x)\mu^N(t,dx)-\int_D \Phi_1^r(x)\nu(t,dx)<-\frac{\nu_T}{2}\right\}\subseteq$$
$$\left\{\sup_{t\in[0,T]}\left|\exp(-A^N(t))\int_D \Phi_1^r(x)\mu^N(t,dx)-\int_D \Phi_1^r(x)\nu(t,dx)\right|>\frac{\nu_T}{2}\right\}$$

imply (44) by using (43) applied to $\phi=\Phi_1^r$ a smooth function with compact support in D and taking $\epsilon=\nu_T/2$. □

Proposition 4 *Recall $\gamma_r(x)$ from Definition 1 and define $\gamma_r^c(x)=1-\gamma_r(x)\geq 0$, which is smooth on $\overline{D}$ and vanishes on the boundary. Let $r_D(\mu)>0$ be the largest radius r less than r_D such that $\mu(0,D_r)>0$. Under the conditions of Proposition 2, for a given time interval $[0,T]$ and for any $r\leq r_D(\mu)$ there exists a constant $C_r>0$ and for each $N\in\mathbb{Z}_+$ an event $S_L^N(r)$ such that*

$$\begin{aligned} S_L^N(r)=\left\{\inf\nolimits_{t\in[0,T]}\tfrac{1}{N}\textstyle\sum_{i=1}^N\gamma_r^c(x_i(t))\leq C_r\right\}\quad \textit{and}\\ \limsup\nolimits_{N\to\infty}P\Big(S_L^N(r)\Big)=0\,. \end{aligned}\tag{47}$$

Remark: Proposition 4 gives a uniform lower bound for the number of particles in D_r or, equivalently, a uniform upper bound for the number of particles in a vicinity of the boundary D_r^c.

Proof. The function $\gamma_r^c(x)=1-\gamma_r(x)\geq 0$ is smooth on $\overline{D}$ and vanishes on the boundary. For $r>0$ we define

$$\nu_T(r)=\inf_{t\in[0,T]}\int_D\gamma_r^c(x)\nu(t,dx)\tag{48}$$

and $C_r=\nu_T(r)/2$. We make the observation that because $\mu(0,dx)$ puts zero mass on the boundary ∂D we can always choose $r>0$ such that $\nu_T(r)>0$. We look at the following inclusions

$$\left\{\inf_{t\in[0,T]}\int_D\gamma_r^c(x)\mu^N(t,dx)<C_r\right\}\subseteq$$

$$\left\{\exists\, t\in[0,T]\,:\,\int_D\gamma_r^c(x)\mu^N(t,dx)<\frac{\nu_T(r)}{2}\right\}\subseteq$$
$$\left\{\exists\, t\in[0,T]\,:\,\int_D\gamma_r^c(x)\mu^N(t,dx)<\int_D\gamma_r^c(x)\nu(t,dx)-\frac{\nu_T(r)}{2}\right\}\subseteq$$
$$\left\{\exists\, t\in[0,T]\,:\,e^{-A^N(t)}\int_D\gamma_r^c(x)\mu^N(t,dx)<\int_D\gamma_r^c(x)\nu(t,dx)-\frac{\nu_T(r)}{2}\right\}\subseteq$$
$$\left\{\exists\, t\in[0,T]\,:\,e^{-A^N(t)}\int_D\gamma_r^c(x)\mu^N(t,dx)-\int_D\gamma_r^c(x)\nu(t,dx)<-\frac{\nu_T(r)}{2}\right\}\subseteq$$
$$\left\{\sup_{t\in[0,T]}\left|e^{-A^N(t)}\int_D\gamma_r^c(x)\mu^N(t,dx)-\int_D\gamma_r^c(x)\nu(t,dx)\right|>\frac{\nu_T(r)}{2}\right\}.$$

We can apply (43) with $\phi=\gamma_r^c$ and $\epsilon=\nu_T(r)/2$ and derive (47). □

Proposition 5 *For any $t_0 > 0$ there exists a constant C_{t_0} such that, for each $N \in \mathbb{Z}_+$ and any $\phi \in C^2(\overline{D})$ vanishing on the boundary, there exists an event $S_U^N(\phi)$ satisfying*

$$\begin{gathered} S_U^N(\phi) = \left\{ \sup_{t\in[t_0,T]} \left| \int_D \phi(x)\mu^N(t,dx)\right| > C_{t_0} \int_D |\phi(x)|dx \right\} \quad \textit{and} \\ \limsup_{N\to\infty} P\Big(S_U^N(\phi) \Big) = 0\,. \end{gathered} \tag{49}$$

Proof. Without loss of generality we assume $\phi \geq 0$. Let

$$\epsilon = \left(\sup_{t\in[t_0,T]} \sup_{x\in D} \{|u(t,x)|\} \right) \int_D \phi(x)dx$$

applied to equation (43). We write the following inclusions.

$$\begin{aligned} U^N &= \left\{ \sup_{t\in[t_0,T]} \left| \int_D \phi(x)\nu^N(t,dx) - \int_D \phi(x)\nu(t,dx) \right| > \epsilon \right\} \\ &\supseteq \left\{ \exists\ t \ : \ \int_D \phi(x)\nu^N(t,dx) - \int_D \phi(x)\nu(t,dx) > \epsilon \right\} \\ &= \left\{ \exists\ t \ : \ \int_D \phi(x)\mu^N(t,dx) > e^{A^N(t)} \Big(\int_D \phi(x)\nu(t,dx) + \epsilon \Big) \right\} \\ &\supseteq \left\{ \exists\ t \ : \ \int_D \phi(x)\mu^N(t,dx) > e^{A^N(T)} \Big(\int_D \phi(x)\nu(t,dx) + \epsilon \Big) \right\}. \end{aligned}$$

Recall (44). For

$$C_{t_0} = 2\, e^{C(T)} \left(\sup_{t\in[t_0,T]} \sup_{x\in D} \{|u(t,x)|\} \right)$$

the set from above includes the intersection of V^N and C^N where

$$V^N = \left\{ \exists\ t \ : \ \int_D \phi(x)\mu^N(t,dx) > C_{t_0} \int_D \phi(x)dx \right\},\ C^N = \left\{ A^N(T) \leq C(T) \right\}.$$

We obtain that $V^N \cap C^N \subseteq U^N$. Therefore

$$V^N = (V^N \cap C^N) \cup (V^N \cap (C^N)^c) \ \subseteq\ U^N \cup (C^N)^c$$

which concludes the proof. □

The number of particles in a subset $F \subseteq D$ at time $t \in [0,T]$ will be denoted by $N(F,t)$.

Corollary 2 *Let $F \subseteq \overline{F} \subset D$. Then, for any $t_0 \in (0,T)$ and for any $N \in \mathbb{Z}_+$ there exists an event $S_U^N(F)$ such that*

$$\begin{gathered} S_U^N(F) = \left\{ \sup_{t\in[t_0,T]} N(F,t) > 2C_{t_0} vol(\overline{F}) N \right\} \quad \textit{and} \\ \limsup_{N\to\infty} P\Big(S_U^N(F) \Big) = 0 \end{gathered} \tag{50}$$

where C_{t_0} is the same constant from Proposition 5.

Remark: The constant $2C_{t_0}$ in the Corollary can be reduced to be exactly C_{t_0}.

Proof. Since $F \subseteq \overline{F} \subset D$ we can approximate the indicator function of $\overline{F}$ from above with a decreasing sequence of $\{\phi_l(x)\}_{l\geq 1} \in C_0^\infty(D)$. We apply Proposition 5 to the sequence ϕ_l to obtain (50). □

Remark. We shall use Corollary 2 only for sets F with $vol(\partial F) = 0$ (with negligible boundary in the sense of Lebesgue measure).

Lemma 2 *Let C_r be the constant in equation (47) and $T_r \geq 0$ be the stopping time defined as*

$$T_r = \inf\{t > 0 \,:\, \langle \gamma_r^c(x), \mu^N(t,dx)\rangle \leq C_r\} \tag{51}$$

where $\gamma_r^c(x)$ is as in Proposition 4 and $T_r = \infty$ if the infimum is taken over the empty set. Then there exists a constant $C(r,T)$ independent of N such that

$$E\Big[A^N(T\wedge T_r)^2\Big] \leq C(r,T)\,. \tag{52}$$

Remark. In this paper we actually only need the bound on the first moment of $A^N(T\wedge T_r)$. However, the estimate is valid for any moment $p>1$ along the same lines as in the following proof.

Proof. We apply (12) for the function $\phi = \gamma_r^c$ in conjunction with the optional stopping theorem to obtain

$$\begin{aligned}
C_r\Big(A^N(T\wedge T_r) - A^N(0)\Big) &\leq \inf_{u\in[0,T\wedge T_r]}\Big\{\frac{1}{N}\sum_{j=1}^N \gamma_r^c(x_j(u))\Big\}\Big(A^N(T\wedge T_r) - A^N(0)\Big) \qquad (53)\\
&\leq \frac{1}{N}\sum_{i=1}^N \int_0^{T\wedge T_r}\Big(\frac{1}{N-1}\sum_{j\neq i}\gamma_r^c(x_j(u)) - \gamma_r^c(x_i(u-))\Big)dA_i^N(u)\\
&= \frac{1}{N}\sum_{i=1}^N \gamma_r^c(x_i(T\wedge T_r)) - \frac{1}{N}\sum_{i=1}^N \gamma_r^c(x_i(0)) - \int_0^{T\wedge T_r}\frac{1}{N}\sum_{i=1}^N \frac{1}{2}\Delta_d \gamma_r^c(x_i(u))du\\
&\quad -\frac{1}{N}\sum_{i=1}^N \int_0^{T\wedge T_r}\nabla\gamma_r^c(x_i(u))\cdot d\mathbf{w}_i(u) - \mathcal{M}^{N,J}_{\langle \gamma_r^c,\mu^N\rangle}(T\wedge T_r)\,.
\end{aligned}$$

The parameter r is fixed. We divide by the constant $C_r > 0$ and take the expected value to see that the first moment of $A^N(T\wedge T_r)$ is bounded independently of N. To estimate the second moment, we square both sides of the inequality, apply Schwarz's inequality on the right hand side and obtain

$$E\Big[A^N(T\wedge T_r)^2\Big] \leq C_1(r,T) + \frac{C_2(r,T)}{N}E\Big[A^N(T\wedge T_r)\Big]\,.$$

Denote then $U^2 = E\Big[A^N(T \wedge T_r)^2\Big]$ and apply Schwarz's inequality once more to the first moment of $A^N(T \wedge T_r)$ from the right hand side. Since $(U - \frac{C_2}{2N})^2 \leq C_1 + (\frac{C_2}{2N})^2$ we conclude that $U^2 \leq 2C_1 + (\frac{C_2}{N})^2 := C(r,T)$ from (52). □

Proposition 6 *If the initial configuration of the process is asymptotically nondegenerate at the boundary ∂D (in the sense of Definition 3), then for any $\epsilon > 0$,*

$$\lim_{r \to 0} \limsup_{h \to 0} \limsup_{N \to \infty} P\Big(\sup_{0 \leq h' \leq h} \int_D \gamma_r(x) \mu^N(h', dx) > \epsilon \Big) = 0 \tag{54}$$

and

$$\limsup_{h \to 0} \limsup_{N \to \infty} P\Big(A^N(h) > \epsilon \Big) = 0 \, . \tag{55}$$

Remark. This result takes care of the asymptotic behavior at the boundary for a short time interval $[0, h]$ and is needed only because we do not assume regularity of the initial profile $\mu(0, dx)$.

Proof. We recall T_r from Lemma 2, with $r \leq r_D(\mu)$, denoted by r_D in the following. We write (13) for the function $\phi = \gamma_r$. The $dA_i^N(t)$ terms are all negative due to the form of the function γ_r. This gives the bound (valid pathwise)

$$0 \leq \frac{1}{N} \sum_{i=1}^{N} \gamma_r(x_i(h')) \leq \frac{1}{N} \sum_{i=1}^{N} \gamma_r(x_i(0)) + \tag{56}$$

$$\int_0^{h'} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{2} \Delta_d \gamma_r(x_i(u)) du + \frac{1}{N} \sum_{i=1}^{N} \int_0^{h'} \nabla \gamma_r(x_i(u)) \cdot d\mathbf{w}_i(u) + \mathcal{M}^{N,J}_{\langle \gamma_r, \mu^N \rangle}(h') \, .$$

The case $T_r < h$ is a subset of the asymptotically negligible event $S_L^N(r)$ from (47). Assume $T_r \geq h$. Since the function γ_r is positive, the supremum over $h' \in [0, h]$ of the left hand side of the inequality will be bounded above by 1) the du term which is bounded in absolute value by $\frac{1}{2} h c(D) r^{-2}$, 2) the martingale terms, which are of order N^{-1} after calculating the quadratic variation and using the martingale maximal inequality and 3) the first term that will vanish as $r \to 0$ according to (17). This proves (54).

To prove (55),

$$P\Big(A^N(h) - A^N(0) > \epsilon\Big) \leq P\Big(A^N(h) - A^N(0) > \epsilon \, , \, T_{r_D} > T\Big) + P\Big(T_{r_D} \leq T\Big) \, .$$

Proposition 4 shows that the second probability converges to zero as $N \to \infty$. The first probability can be written as

$$P\Big(A^N(h \wedge T_{r_D}) - A^N(0) > \epsilon \, , \, T_{r_D} > T\Big) \leq P\Big(A^N(h \wedge T_{r_D}) - A^N(0) > \epsilon\Big) \, . \tag{57}$$

Then, similarly to Lemma 2 we can have the inequality

$$C_{r_D}\Big(A^N(h\wedge T_{r_D}) - A^N(0)\Big) \leq \tag{58}$$

$$\inf_{u\in[0,h\wedge T_{r_D}]}\left\{\frac{1}{N}\sum_{j=1}^N \gamma_r^c(x_j(u))\right\}\Big(A^N(h\wedge T_{r_D}) - A^N(0)\Big) \leq$$

$$\frac{1}{N}\sum_{i=1}^N\int_0^{h\wedge T_{r_D}}\Big(\frac{1}{N-1}\sum_{j\neq i}\gamma_r^c(x_j(u)) - \gamma_r^c(x_i(u-))\Big)dA_i^N(u)$$

$$= \frac{1}{N}\sum_{i=1}^N \gamma_r^c(x_i(h\wedge T_{r_D})) - \frac{1}{N}\sum_{i=1}^N\gamma_r^c(x_i(0)) - \int_0^{h\wedge T_{r_D}}\frac{1}{N}\sum_{i=1}^N\frac{1}{2}\Delta_d\gamma_r^c(x_i(u))du$$

$$-\frac{1}{N}\sum_{i=1}^N\int_0^{h\wedge T_{r_D}}\nabla\gamma_r^c(x_i(u))\cdot d\mathbf{w}_i(u) - \mathcal{M}^{N,J}_{\langle\gamma_r^c,\mu^N\rangle}(h\wedge T_{r_D})$$

hence $A^N(h\wedge T_{r_D})$ will have an upper bound given by the right-hand side of (58) times $C_{r_D}^{-1}$. Since $\gamma_r^c = 1-\gamma_r$, we can write for any h

$$\frac{1}{N}\sum_{i=1}^N\gamma_r^c(x_i(h)) - \frac{1}{N}\sum_{i=1}^N\gamma_r^c(x_i(0)) = \frac{1}{N}\sum_{i=1}^N\gamma_r(x_i(0)) - \frac{1}{N}\sum_{i=1}^N\gamma_r(x_i(h))\,.$$

In order to estimate the probability in (57) we notice that we have to estimate the probabilities of the union of the events that either of the four terms of the right hand side of the inequality (58) exceeds $\epsilon/4$. The first term is bounded by $\langle\gamma_r(x),\mu^N(0,dx)\rangle$ and approaches zero as $r\to 0$ after $N\to\infty$ by the definition of the existence of the initial profile concentrated on the open set D. The time integral is of order h uniformly in N. The Brownian martingale has a quadratic variation of order hN^{-1}. The jump martingale has quadratic variation with an integrand of order N^{-1} and the process $A^N(h\wedge T_{r_D})$ has a first moment uniformly bounded in N from Lemma 2. This concludes the proof. □

4 Bound for the number of particles at the boundary.

This section proves Theorem 2, which shows that the empirical measure is asymptotically non-degenerate at the boundary ∂D in the sense of Definition 3, uniformly in time.

For $M\in\{1,2,\ldots,N\}$ and $\tau\geq 0$ a stopping time, we denote

$$\xi(r,\tau,M) = \inf\{t>\tau\,:\,N(D_r^c,t)=M\}\wedge T\,, \tag{59}$$

which is well defined because the times when particles enter and exit D_r^c, either through diffusive motion or jump occur one at a time with probability one. If

τ_D is the exit time from D for a d-dimensional Brownian motion starting with the probability measure $\mu(dx)$ coinciding with the initial profile from (18), let

$$p(\zeta, r) = \inf_{x \in D_r^c} P_x(\tau_D \leq \zeta) = \inf_{x \in D_r^c} \left(1 - \int_D p_{abs}(\zeta, x, dy)\right) \tag{60}$$

with the property that $\lim_{r \to 0} p(\zeta, r) = 1$.

Let $t_0 > 0$ be as in (49) and (50) and recall the upper bound for the number of particles in a set $F \subseteq D$ from Corollary 2. Based on equation (50), write

$$c_1(t_0) = 4C_{t_0} \sup_{0 < r \leq r_D} r^{-d} \, vol(D_r \setminus D_{2r}) \tag{61}$$

for a constant depending on the geometry of D, where C_{t_0} is the uniform constant from (50). Also, if $u(t, x)$ is the solution of the heat equation with Dirichlet boundary conditions (18), write

$$c_2(t_0) = \frac{e^{C(T)}}{2} \left(1 + \sup_{t_0 \leq t \leq T} \int_D |\Delta u(t, x)| dx\right) , \tag{62}$$

justified by the following proposition, which is a refinement of Corollary 2. In the following, we shall write c_1 and c_2 instead of $c_1(t_0)$ and $c_2(t_0)$ for simplification (see the remark after Theorem 2 in relation to the dependence on t_0).

Proposition 7 *For any fixed $t_0 \in (0, T)$, any $r > 0$, $\zeta > 0$, and any stopping time $\tau \geq t_0$, there exists an event $S^N(\zeta, r)$ such that*

$$\begin{gathered} S^N(\zeta, r) = \left\{ \sup_{0 \leq \zeta' \leq \zeta} \left| \int_\tau^{\tau + \zeta'} \tfrac{1}{2N} \textstyle\sum_{i=1}^N \Delta \gamma_r^c(x_i(s)) ds \right| > c_2 \zeta \right\} \\ \lim_{N \to \infty} P(S^N(\zeta, r)) = 0 \, . \end{gathered} \tag{63}$$

Notice that the constant c_2 depends only on the initial density profile $\mu(dx)$ in (18), the time t_0, and not on N, r, ζ or τ.

Remark. The proposition justifies the lower bound $-c_2\zeta$ for (72).

Proof. A supremum norm estimate of the integrand is of order ζr^{-2}, which would interfere with our argument. However, we can obtain a bound depending only on ζ as follows.

$$\sup_{0 \leq \zeta' \leq \zeta} \left| \int_\tau^{\tau + \zeta'} \frac{1}{2N} \sum_{i=1}^N \Delta \gamma_r^c(x_i(s)) \, ds \right| \leq \zeta \sup_{t_0 \leq t \leq T} \left| \frac{1}{2N} \sum_{i=1}^N \Delta \gamma_r^c(x_i(t)) \right|$$

$$\leq \frac{\zeta}{2} \sup_{t_0 \leq t \leq T} \left| e^{A^N(t)} \left[\left(\frac{1}{N} \sum_{i=1}^N \Delta \gamma_r^c(x_i(t)) \right) e^{-A^N(t)} \right] \right|$$

$$\leq \frac{e^{C(T)}\zeta}{2} \sup_{t_0 \leq t \leq T} \left| \langle \int_D \Delta\gamma_r^c(x)\nu^N(t,dx)\rangle - \langle \int_D \Delta\gamma_r^c(x)\nu(t,dx)\rangle \right| \tag{64}$$

$$+\frac{e^{C(T)}\zeta}{2} \sup_{t_0 \leq t \leq T} \left| \langle \int_D \Delta\gamma_r^c(x)\nu(t,dx)\rangle \right| \tag{65}$$

where (64) holds except on the exceptional set (44) with probability approaching zero as $N \to \infty$. Take $\epsilon = 1$ in (43). Notice that $\nu(t,dx) = u(t,x)dx$ as in (18) and the density profile is smooth for $t \geq t_0 > 0$. In addition, apply Green's second formula to see that (65) is bounded above by a constant independent of r. $\square$

In the following, $\epsilon > 0$ is a small but fixed number. Recall the universal constant C_{r_D} defined in Proposition 4, specifically for the case $r = r_D$ (the thickness r of the boundary layer D_r^c). We would like to find a set of positive constants (a, ζ, r, k, k') satisfying the system

$$\begin{cases} k > 1\,, \qquad ka < \epsilon/2\,, \qquad k'/k < p(\zeta, r) \\ \qquad a + c_1 r + c_2\zeta < \epsilon/2 \\ k - C_{r_D}k' + a^{-1}(c_1 r + c_2\zeta) < 1 \end{cases} \tag{66}$$

with the condition that a and ζ depend only on ϵ and the solution exists for any $r < r(\epsilon)$, where $r(\epsilon)$ is a critical value depending on ϵ. The explanation for the necessity of choosing the parameters according to (66) will become apparent with Lemma 3.

We shall construct a solution of the system (66) as follows. Take $\alpha \in (0, 1 \wedge C_{r_D}^{-1})$ and pick a solution of the system of inequalities in the variables (λ_1, λ_2)

$$\begin{cases} \frac{1}{6}(1 - \alpha C_{r_D}) < \lambda_1 < \frac{1}{2}(1 - \alpha C_{r_D}) \\ \lambda_1 + c_2\lambda_2 < \frac{1}{4} \\ 0 < \lambda_2 < \frac{\alpha C_{r_D}}{10c_2}\lambda_1 \end{cases} \tag{67}$$

bounded away from the boundary of the domain. Let $a = \lambda_1\epsilon$ and $\zeta = \lambda_2\epsilon$ and notice that the numbers a and ζ constructed in this way depend only on ϵ. Set $v = 1 - C_{r_D}\alpha p(\zeta, r) > 0$ which implies that

$$\frac{1+v}{2} \in (v, 1)\,, \qquad k = \frac{1+v}{2v}\,, \qquad k' = k\alpha p(\zeta, r)\,. \tag{68}$$

Notice that $1 - C_{r_D}\alpha \leq v < 1$ and $v = 1$ if and only if $p(\zeta, r) = 0$. We notice that $k \leq (1 - \alpha C_{r_D})^{-1}$. Combining (67) and (60), we can see that the original system (66) has a solution as prescribed for any sufficiently small r.

The main ingredient of the proof is Lemma 3. For a given ϵ, we choose a which is roughly of the same magnitude but strictly smaller, in the sense that $ka < \epsilon/2$ where $k > 1$. If the number of particles in D_r^c at start is $[aN]$, then there will be at least $[kaN]$ particles in D_r^c before the number $[\epsilon N]$ is reached. Denote the time when $[kaN]$ is reached by ξ_k. However, Itô's formula (71) ensures that a large number of particles reaching the boundary (at least $C_{r_D}[k'aN]$) will return in the complement D_r, where $k' \approx p(\zeta, r)k$. The second inequality on line one of the system (66) together with the second line of the system ensure that the number of particles will never reach $[\epsilon N]$ in the time interval $[\tau, \xi_k + \zeta]$. The third line of (66) says that the number of particles will drop below the initial number $[aN]$ at time $\xi_k + \zeta$, the right endpoint of the time interval. In addition, it is essential to note that a and ζ depend on ϵ *only*, which proves that the process spends a macroscopic time ζ (independent of N) away from the state with $[\epsilon N]$ particles in the boundary layer, with the exception of an event with probability approaching zero as $N \to \infty$.

Lemma 3 *Let $\epsilon > 0$, $k > 1$, $k'/k < p(\zeta, r)$, $a < \epsilon$, and $\zeta > 0$ be defined as in (66). We fix $0 < t_0 << T$ as in (49) and (50). Let τ be a stopping time $\tau \in [t_0, T]$. Then*

$$\lim_{N\to\infty} P\Bigg(\Big\{N(D_r^c, \tau) \le [aN]\,, \xi_k + \zeta \le T\Big\} \bigcap \qquad\qquad (69)$$
$$\Big\{\xi_k + \zeta < \xi(r, \tau, [\epsilon N])\,, N(D_r^c, \xi_k + \zeta) \le [aN]\Big\}^c\Bigg) = 0$$

and

$$\lim_{N\to\infty} P\left(\Big\{N(D_r^c, \tau) \le [aN]\,, \xi_k + \zeta > T\Big\} \bigcap \Big\{\sup_{\tau \le t \le T} N(D_r^c, t) < [\epsilon N]\Big\}^c\right) = 0\,. \qquad (70)$$

Proof.1) The limit (69). Recall T_{r_D} from (51) for the inner set D_{r_D}. Without loss of generality we assume that $r << r_D$. Given $\gamma_r^c(x)$ the indicator function of D_r convoluted with a mollifier approximating the delta function (Definition 1), we write for $\zeta' \le \zeta$,

$$\frac{1}{N}\sum_{i=1}^{N} \gamma_r^c(x_i([\xi_k + \zeta'] \wedge T_{r_D})) - \frac{1}{N}\sum_{i=1}^{N} \gamma_r^c(x_i(\xi_k \wedge T_{r_D})) \qquad (71)$$

$$= \int_{\xi_k \wedge T_{r_D}}^{[\xi_k + \zeta'] \wedge T_{r_D}} \frac{1}{2N}\sum_{i=1}^{N} \Delta\gamma_r^c(x_i(s))\, ds \qquad (72)$$

$$+ \int_{\xi_k \wedge T_{r_D}}^{[\xi_k + \zeta'] \wedge T_{r_D}} \frac{1}{N}\sum_{i=1}^{N} \gamma_r^c(x_i(s))\, dA^N(s) \qquad (73)$$

$$+ \mathcal{M}^N_{\gamma_r^c}([\xi_k+\zeta'] \wedge T_{r_D}) - \mathcal{M}^N_{\gamma_r^c}(\xi_k \wedge T_{r_D}) \,. \tag{74}$$

Consider Proposition 5 and Corollary 2 applied to $D_r \setminus D_{2r}$, a set appearing in the approximation of the error separating the number of particles in D_r and D_r^c and their smooth approximation counterparts $\langle \gamma_r^c, \mu^N(\cdot, dx)\rangle$ and $\langle \gamma_r, \mu^N(\cdot, dx)\rangle$. The cumulative errors are of order r, with constant c_1 independent of both N and r, as defined in (61). We notice that the exceptional sets (49) and (50) depend on r but their probability will vanish as $N \to \infty$ before we pass to the limit in the other parameters. We shall give a lower bound to the term (72) based on Proposition 7, with constant c_2 independent of N and r. Next, (73) will be bounded below by using the universal constant C_{r_D} from Proposition 4, again outside the exceptional event $S_L^N(r_D)$ from (47). Finally, the martingale part (74) has quadratic variation of order $O(N^{-1})$ with the same argument as in Proposition 6. With the notation $N(B,t)$ for the number of particles in subset B at time t, we have

$$\frac{1}{N} N(D_r, \xi_k + \zeta') \geq \tag{75}$$

$$\geq \frac{1}{N} N(D_r, \xi_k) - c_1 r - c_2 \zeta + C_{r_D}(A^N(\xi_k+\zeta') - A^N(\xi_k)) - O(N^{-1/2})$$

which can be written as

$$\frac{1}{N} N(D_r^c, \xi_k + \zeta') \leq \tag{76}$$

$$\frac{1}{N} N(D_r^c, \xi_k) + c_1 r + c_2 \zeta - C_{r_D}(A^N(\xi_k+\zeta') - A^N(\xi_k)) + O(N^{-1/2})$$

outside of an exceptional event $U^N(r,\tau)$, defined as the union of the exceptional events allowing the lower bound described in the paragraph from above. For r sufficiently small as in (66), $\lim_{N\to\infty} P(U^N(r,\tau)) = 0$. This proves that, for any $\zeta' \leq \zeta$, the average number of particles $\frac{1}{N} N(D_r^c, t)$ for $t \in [\xi_k, \xi_k + \zeta']$ has an upper bound in probability

$$\limsup_{N\to\infty} P\Big(\frac{1}{N} \sup_{t\in[\xi_k, \xi_k+\zeta']} N(D_r^c, t) > a + c_1 r + c_2 \zeta' \Big) = 0 \tag{77}$$

as a consequence of (76). As soon as $\epsilon/2 > a + c_1 r + c_2\zeta$, since $2ka < \epsilon$ once again from (66), we have shown that $\xi(r,\tau,[\epsilon N]) > \xi_k + \zeta'$.

Proposition 8 implies that at time $\xi_k + \zeta$ the lower bound for the term in (73) is $C_{r_D}[k'aN]/N$, bringing (76) to the form

$$\frac{1}{N} N(D_r^c, \xi_k+\zeta) \leq \frac{1}{N} N(D_r^c, \xi_k) + c_1 r + c_2\zeta - C_{r_D}\frac{[k'aN]}{N} + O(N^{-1/2}) \,. \tag{78}$$

Let $N \to \infty$ to see that if line three of (66) is true, then (78) implies that at time $\xi_k + \zeta$ the average number of particles in D_r^c drops again below $[aN]$, without having ever reached $[\epsilon N]$ in $[\tau, \xi_k + \zeta]$.

2) The limit (70). The first case is when $\xi_k = T$. It is clear that for any $t \in [\tau, T]$, $N(D_r^c, t) \leq [kaN] \leq [(\epsilon/2)N]$. Suppose $\xi_k < T$ but $\xi_k + \zeta > T$. For the interval $t \in [\tau, \xi_k]$ case 1) applies, while for the interval $t \in [\xi_k, T]$ we apply the estimate (77).

□

Proposition 8 *If $k'/k < p(\zeta, r)$, and ξ_k is defined as in Lemma 3 and satisfies $\xi_k + \zeta \leq T$, then*

$$\lim_{N \to \infty} P\Big(\#\, \textit{jumps in } [\xi_k, \xi_k + \zeta] < [k'aN]\Big) = 0 \,. \tag{79}$$

Proof. Denote

$$p_j(\zeta, r) = P_{x_j(\xi_k)}(\tau_D \leq \zeta) \,. \tag{80}$$

Construct the Bernoulli random variables Z_j by setting $Z_j = 1$ if the particle of index j, $1 \leq j \leq N$ starting at time ξ_k hits the boundary ∂D before time $\xi_k + \zeta$ and $Z_j = 0$ otherwise. Then, if $J = \{j = 1, 2, \ldots, N \, : \, x_j(\xi_k) \in D_r^c\}$,

$$P\Big(\#\,\text{jumps in } [\xi_k \,, \xi_k + \zeta] < [k'aN]\Big) \leq P\Big(\sum_{j=1}^{N} Z_j < [k'aN]\Big)$$

$$\leq P\Big(\sum_{j \in J} Z_j < [k'aN]\Big)$$

$$\leq P\Big(\sum_{j \in J} (\langle p_j(\zeta, r)\rangle - Z_j) > \sum_{j \in J} \langle p_j(\zeta, r)\rangle - [k'aN]\Big) \,.$$

Since

$$\sum_{j \in J} \langle p_j(\zeta, r)\rangle - [k'aN] \geq |J|\Big(p(\zeta, r) - \frac{[k'aN]}{|J|}\Big) \geq [kaN]\Big(p(\zeta, r) - \frac{[k'aN]}{[kaN]}\Big) > 0 \,,$$

the particles are independent Brownian motions until they hit the boundary, and the actual number of boundary hits can only be larger than $\sum Z_j$, Chebyshev's inequality gives that the probability from (79) has an asymptotic upper bound as $N \to \infty$

$$\frac{\max_{j \in J} Var(Z_j)|J|}{|J|^2\Big(p(\zeta, r) - \frac{[k'aN]}{|J|}\Big)^2} \leq \frac{1}{4aN}\Big(p(\zeta, r) - \frac{k'}{k}\Big)^{-2} \sim O(\frac{1}{N}) \,.$$

□

Theorem 2 *Let $\epsilon > 0$. Then*

$$\lim_{r\to 0} \limsup_{N\to\infty} P\Big(\xi(r, 0, [\epsilon N]) \le T\Big) = 0\,. \tag{81}$$

Remark. All the estimates in this section are obtained for times starting after a positive t_0, in order to avoid the possible singularity of the initial profile $\mu(dx)$ and take advantage of the uniform estimates in Section 3. The only limit where t_0 is not fixed (namely $t_0 \to 0$) is the asymptotic bound for the event (83). In regards to (82), the limit is zero after letting $N \to \infty$ due to Proposition 6.

Proof. Let $t_0 \in (0, T)$ be a small positive number exactly as in Lemma 3. Then

$$\Big\{\xi(r, 0, [\epsilon N]) \le T\Big\} \subseteq U_1^N \cup U_2^N$$

where

$$U_1^N = \Big\{\xi(r, 0, [\epsilon N]) \le T\,,\ \xi(r, 0, [aN]) > t_0\Big\} \tag{82}$$

$$U_2^N = \Big\{\xi(r, 0, [\epsilon N]) \le T\,,\ \xi(r, 0, [aN]) \le t_0\Big\} \subseteq \Big\{\xi(r, 0, [aN]) \le t_0\Big\}\,. \tag{83}$$

Let $\tau \in [t_0, T]$ be a stopping time, a, ζ, r, k, k' be chosen as in Lemma 3. We shall call a *regular cycle* a random time interval $[\tau, (\xi(r, \tau, [aN]) + \zeta) \wedge T]$ with either one of the properties (p) or (p'), where

$$\text{(p)}\ \begin{cases} (i)\ N(D_r^c, \tau) \le [aN], \\ (ii)\ \xi(r, \tau, [aN]) + \zeta \le T\,, \\ (iii)\ \xi(r, \tau, [\epsilon N]) > \xi(r, \tau, [aN]) + \zeta \\ (iv)\ N(D_r^c, \xi(r, \tau, [aN]) + \zeta) \le [aN] \end{cases}$$

and

$$\text{(p')}\ \begin{cases} (i') = (i)\,, \\ (ii')\ \xi(r, \tau, [aN]) + \zeta > T\,, \\ (iii')\ \sup_{\tau \le t \le T} N(D_r^c, t) < [\epsilon N]\,. \end{cases}$$

Let $l = 0$, $\tilde\tau_0 = t_0$. Define $\tilde\tau_{l+1} = (\xi(r, \tilde\tau_l, [aN]) + \zeta) \wedge T$ if $[\tilde\tau_l, \tilde\tau_{l+1}]$ is a regular cycle. Proceed inductively until we reach a non-regular cycle. Then, since ζ defined in (66) does not depend on N and r, U_1^N from (82) belongs to

$$\Big\{\text{there are at most } [\frac{T}{\zeta}] + 1 \text{ regular cycles starting at } t_0\,,\ \xi(r, 0, [\epsilon N]) \le T\Big\}$$

$$\subseteq \bigcup_{l=0}^{[\frac{T}{\zeta}]+1} \Big\{ \text{there are exactly } l \text{ regular cycles before } T\,,\ \xi(r,0,[\epsilon N]) \le T \Big\}$$

$$\subseteq \bigcup_{l=0}^{[\frac{T}{\zeta}]} \Big\{ [\tilde{\tau}_l, (\xi(r,\tilde{\tau}_l,[aN]) + \zeta) \wedge T] \text{ is not a regular cycle} \Big\} .$$

At this point we apply Lemma 3 to $\tau = \tilde{\tau}_l$ to see that $\lim_{N\to\infty} P(U_1^N) = 0$. We have shown that

$$\limsup_{N\to\infty} P\Big(\xi(r,0,[\epsilon N]) \le T \Big) \le \limsup_{N\to\infty} P(U_2^N) \,.$$

Finally, according to Proposition 6 applied to $\epsilon \leftrightarrow a/2$ and $h \leftrightarrow t_0$ in equation (54),

$$\begin{aligned} &\lim_{r\to 0} \limsup_{t_0\to 0} \limsup_{N\to\infty} P(U_2^N) \ \le \\ \le\ &\lim_{r\to 0} \limsup_{t_0\to 0} \limsup_{N\to\infty} P\Big(\Big\{ \sup_{0\le t\le t_0} \frac{1}{N} \sum_{i=1}^{N} \gamma_r(x_i(t)) > \frac{a}{2} \Big\} \Big) = 0\,, \end{aligned} \tag{84}$$

which proves the iterated limit (81) for U_2^N. □

5 Tightness.

Let X be a Polish space with norm $\|\cdot\|$ and let $\mathbf{D}([0,T],X)$ be the Skorohod space of functions with left limits and right continuous on $[0,T]$. The following are sufficient conditions for tightness in $\mathbf{D}([0,T],X)$ of the family of processes $\{y^N(\cdot)\}_{N>0} \in \mathbf{D}([0,T],X)$, adapted to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ (the Aldous condition).

Let $\mathcal{T}$ be the collection of all stopping times with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ bounded above by T.

(i) There exists a constant $Y_0 > 0$ such that

$$\limsup_{N\to\infty} P\Big(\sup_{t\in[0,T]} \|y^N(t)\| > Y_0 \Big) = 0 \tag{85}$$

(ii) For any $\epsilon > 0$

$$\lim_{\eta\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathcal{T}\\ s\in[0,\eta]}} P\Big(\|y^N(\tau+s)-y^N(\tau)\|>\epsilon\Big)=0 \tag{86}$$

with the convention that $\tau+s$ stands for $(\tau+s)\wedge T$.

Let r_D be the inner radius of the domain D (from Definition 1). For $\epsilon > 0$ we define $r' > 0$ with the property

$$2r' < r_D \qquad C_{t_0}Vol(D_{r'}\setminus D_{2r'}) < \frac{C_{r_D}\epsilon}{48} \tag{87}$$

where C_{r_D} is the lower bound (47). We shall choose $r > 0$ such that

$$4r < r', \qquad C_{t_0}Vol(D_{r'-r}\setminus D_{r'}) < \frac{C_{r_D}\epsilon}{48}, \qquad \text{and let} \quad \delta = r'-2r\,. \tag{88}$$

As in Corollary 2, the number of particles in the set $F \subseteq D$ at time $t \in [0, T]$ will be denoted as $N(F, t)$.

Proposition 9 *Let $t_0 > 0$ and $\tau \geq t_0$ a stopping time. Then, for any $\epsilon > 0$ and any r as in (88), there exists a sufficiently small $\eta > 0$, such that if we write $\epsilon' = \frac{C_{r_D}\epsilon}{24}$, then*

$$\limsup_{N\to\infty} P\Big(A^N((\tau+\eta)\wedge T)-A^N(\tau)\geq\frac{\epsilon}{2}\,,\; N(D_r^c,\tau)\leq[N\epsilon']\Big)=0\,. \tag{89}$$

Proof. We shall suppress the minimum with the time interval endpoint T for simplification. Define

$$S_\tau=\Big\{A^N(\tau+s)-A^N(\tau)\geq\frac{\epsilon}{2}\;,\;N(D_r^c,\tau)\leq[N\epsilon']\Big\} \tag{90}$$

for $0 \leq s \leq \eta$. We shall partition the domain D into D_r^c, $D_r \setminus D_{r+\delta}$, $D_{r+\delta}\setminus D_{2r+\delta}$ and $D_{2r+\delta}$, where $\delta = r' - 2r$ is defined in equation (88). Notice that r' is fixed according to ϵ and δ will be also of the order of ϵ as $r \to 0$. The estimates obtained in the following are valid for any r less than a critical value depending on ϵ only.

The first set contains at most $[N\epsilon']$ particles at time τ. The third is a buffer zone containing an asymptotically bounded number of particles (49). This implies that S_τ in (90) is a sub-event of the event that a number of particles of order N is transferred from $D_r\setminus D_{r+\delta}$ into $D_{2r+\delta}$ in a time interval no longer than η. The probability of this event tends to zero as $N \to \infty$ because the particles have to cross either one of the two buffer zones D_r^c or $D_{r+\delta}\setminus D_{2r+\delta}$ *as independent Brownian motions* in order to reach $D_{2r+\delta}$. The following makes precise this idea.

We recall the asymptotical lower bound for the number of particles away from the boundary (47). For r' we obtain a lower bound $C_{r'}$ and a set

$$S_L^N(r') = \left\{ \inf_{t\in[t_0,T]} \frac{1}{N}\sum_{i=1}^N \gamma_{r'}^c(x_i(t)) \ \le\ C_{r'} \right\}$$

such that $\lim_{N\to\infty} P(S_{r'}^N) = 0$. Due to the monotonicity of the functions $\gamma_r^c(x)$ in r, according to the definition (48), we derive that $C_{r_D} \le C_{r'}$ for $r' < r_D/2$. We write the Itô formula (58) for r' and a time interval $[t', t'']$ instead of $[0, h]$. The details of this estimation are the same as in Proposition 6. We let

$$a(r', t', t'') = \inf_{u\in[t'\wedge T_{r'}, t''\wedge T_{r'}]} \left\{ \frac{1}{N}\sum_{j=1}^N \gamma_{r'}^c(x_j(u)) \right\}.$$

Then outside the exceptional set $S_L^N(r') \subseteq \{T_{r'} > T\}$,

$$C_{r_D}\Big(A^N(t''\wedge T_{r'}) - A^N(t'\wedge T_{r'})\Big) \le \tag{91}$$

$$a(r',t',t'')\Big(\frac{1}{N-1}\sum_{i=1}^N A_i^N(t''\wedge T_{r'}) - \frac{1}{N-1}\sum_{i=1}^N A_i^N(t'\wedge T_{r'})\Big) \le$$

$$\frac{1}{N}\sum_{i=1}^N \int_{t'\wedge T_{r'}}^{t''\wedge T_{r'}} \Big(\frac{1}{N-1}\sum_{j\neq i}\gamma_{r'}^c(x_j(u)) - \gamma_{r'}^c(x_i(u-))\Big) dA_i^N(u) \ =$$

$$\frac{1}{N}\sum_{i=1}^N \gamma_{r'}^c(x_i(t''\wedge T_{r'})) - \frac{1}{N}\sum_{i=1}^N \gamma_{r'}^c(x_i(t'\wedge T_{r'})) \tag{92}$$

$$-\int_{t'\wedge T_{r'}}^{t''\wedge T_{r'}} \frac{1}{N}\sum_{i=1}^N \frac{1}{2}\Delta_d \gamma_{r'}^c(x_i(u))du \tag{93}$$

$$-\frac{1}{N}\sum_{i=1}^N \int_{t'\wedge T_{r'}}^{t''\wedge T_{r'}} \nabla\gamma_{r'}^c(x_i(u))\cdot d\mathbf{w}_i(u) - \mathcal{M}^{N,J}_{\langle\gamma_{r'}^c,\mu\rangle}(t'\wedge T_{r'}) + \mathcal{M}^{N,J}_{\langle\gamma_{r'}^c,\mu\rangle}(t''\wedge T_{r'}). \tag{94}$$

Applying (91) for $t' = \tau$ and $t'' = \tau + s$, we evaluate S_τ from (90) as a sub-event of the union

$$S_\tau \subseteq S_{\tau 1} \cup S_{\tau 2} \cup S_{\tau 3} \cup S_L^N(r_D) \tag{95}$$

where (we suppress the minimum with $T_{r'}$ when unnecessary)

$$S_{\tau 1} = \left\{ \frac{1}{N}\sum_{i=1}^N \gamma_{r'}^c(x_i(t'')) - \frac{1}{N}\sum_{i=1}^N \gamma_{r'}^c(x_i(t')) \ge \frac{\epsilon C_{r_D}}{6}\ ,\ N(D_r^c, \tau) \le [N\epsilon'] \right\}, \tag{96}$$

$$S_{\tau 2} = \left\{ \Big| \int_{t'}^{t''} \frac{1}{N}\sum_{i=1}^N \frac{1}{2}\Delta_d \gamma_{r'}^c(x_i(u))du \Big| \ge \frac{\epsilon C_{r_D}}{6} \right\}, \tag{97}$$

the event $S_{\tau 3} = S_{\tau 31} \cup S_{\tau 32}$, where

$$S_{\tau 31} = \left\{ \left| \mathcal{M}^{N,B}_{\langle \gamma^c_{r'}, \mu \rangle}(t'') - \mathcal{M}^{N,B}_{\langle \gamma^c_{r'}, \mu \rangle}(t') \right| \geq \frac{\epsilon C_{r_D}}{12} \right\} \tag{98}$$

and

$$S_{\tau 32} = \left\{ \left| \mathcal{M}^{N,J}_{\langle \gamma^c_{r'}, \mu \rangle}(t'' \wedge T_{r'}) - \mathcal{M}^{N,J}_{\langle \gamma^c_{r'}, \mu \rangle}(t' \wedge T_{r'}) \right| \geq \frac{\epsilon C_{r_D}}{12} \right\}. \tag{99}$$

The event $S_{\tau 1}$ from (96) is included in

$$S_{\tau 1} \subseteq \left\{ N(D_{r'}, \tau + s) - N(D_{2r'}, \tau) \geq \frac{N \epsilon C_{r_D}}{6} \, , \, N(D^c_r, \tau) \leq [N\epsilon'] \right\} \subseteq S_{\tau 11} \cup S_{\tau 12}$$

with

$$S_{\tau 11} = \left\{ N(D_{r'}, \tau + s) - N(D_{r'}, \tau) \geq \frac{N \epsilon C_{r_D}}{12} \, , \, N(D^c_r, \tau) \leq [N\epsilon'] \right\} \tag{100}$$

and

$$S_{\tau 12} = \left\{ N(D_{r'}, \tau) - N(D_{2r'}, \tau) \geq \frac{N \epsilon C_{r_D}}{12} \right\} \subseteq S^N_U(D_{r'} \setminus D_{2r'}) \, . \tag{101}$$

From the definition (87) of r' and Corollary 2 of Proposition 5 we see that the probability of $S_{\tau 12}$ tends to zero as $N \to \infty$. For sufficiently small η, $P(S_{\tau 2}$ is zero, due to the boundedness of the derivatives (up to the second order) of $\gamma^c_{r'}(\cdot)$. Doob's inequality shows that $\lim_{N \to \infty} P(S_{\tau 31}) = 0$. Similarly, $\limsup_{N \to \infty} P(S_{\tau 32}) = 0$ due to the fact that the quadratic variation has an integrand of order N^{-1} and the second moment of $A^N(T \wedge T_{r'})$ is uniformly bounded in N.

The remaining event to be evaluated is $S_{\tau 11}$. The only way to increase by $[\frac{N\epsilon C_{r_D}}{12}]$ the number of particles in $D_{r'}$ in a time interval $[\tau, \tau + s']$ if $N(D^c_r, \tau) \leq [N\epsilon'] = [\frac{N\epsilon C_{r_D}}{24}]$ is to bring in at least $[\frac{N\epsilon C_{r_D}}{24}]$ new particles from $D_r \setminus D_{r'}$. Again, since $D_{r'-r} \setminus D_{r'}$ has at most $[\frac{N\epsilon C_{r_D}}{48}]$ particles by construction (88) with the exception of a set $S^N_U(D_{r'-r} \setminus D_{r'})$ with negligible probability as $N \to \infty$ (Proposition 5) we are in the position to evaluate the event $S_{\tau 11}$ from the equation (100). Since $r' = 2r + \delta$,

$$P\Big(S_{\tau 11}\Big) \leq P\Big(S^N_U(D_{r+\delta} \setminus D_{2r+\delta})\Big) \quad +$$

$$P\bigg(\bigg\{ \sum_{i=1}^N \mathbf{1}_{(D_r \setminus D_{r+\delta}) \times D_{2r+\delta}}(x_i(\tau), x_i(\tau + s)) \geq \frac{N\epsilon C_{r_D}}{48} \bigg\}\bigg) \, . \tag{102}$$

The particles situated in $D_r \setminus D_{r+\delta}$ at time τ may reach $D_{2r+\delta}$ at time $\tau + s'$ either directly or by reaching first the boundary ∂D and performing a series of jumps according to the definition of the process. In either case, they first must reach the boundary of $D \setminus \overline{D_{2r+\delta}}$. Before reaching ∂D the particles move independently as Brownian motions. Henceforth (102) is bounded above by

$$\binom{N}{\left[N\epsilon\left(\frac{C_{r_D}}{48}\right)\right]}\left(\sup_{x\in\overline{D}_r\setminus D_{r+\delta}} P_x^W\Big(\tau_{D\setminus\overline{D}_{2r+\delta}}\le\eta\Big)\right)^{\left[N\epsilon(\frac{C_{r_D}}{48})\right]} \tag{103}$$

where P^W denotes a Brownian motion on $\mathbb{R}^d$. For fixed r and δ, as defined in (87) and (88),

$$\sup_{x\in\overline{D}_r\setminus D_{r+\delta}} P_x^W\Big(\tau_{D\setminus\overline{D}_{2r+\delta}}\le\eta\Big)\le 1-\inf_{x\in\overline{D}_r\setminus D_{r+\delta}}\int_{D\setminus\overline{D}_{2r+\delta}} p^{abs}_{D\setminus\overline{D}_{2r+\delta}}(\eta,x,y)dy$$

can be further bounded above by a function $p(\eta)$ depending exclusively on the fixed parameter ϵ, the domain D which has limit zero as $\eta\to 0$. Let $M=\left[N\epsilon(\frac{C_{r_D}}{48})\right]$. Using Stirling's formula the upper bound for (103) is of order

$$\Big(\pi\epsilon'(1-\frac{\epsilon'}{2})N\Big)^{-\frac{1}{2}}\exp\left\{N\left(\frac{M}{N}\ln p(\eta)-\frac{M}{N}\ln\frac{M}{N}-\Big(1-\frac{M}{N}\Big)\ln\Big(1-\frac{M}{N}\Big)\right)\right\}.$$

Since $\lim_{\eta\to 0}p(\eta)=0$ the proof is complete. □

Theorem 3 *$\{A^N(\cdot)\}_{N\in\mathbb{Z}_+}$ is tight in $\mathbf{D}([0,T],\mathbb{R}_+)$.*

Proof. The family of processes $\{A^N(\cdot)\}_{N>0}$ belongs to $\mathbf{D}([0,T],\mathbb{R}_+)$ for any $N\in\mathbb{Z}_+$. Condition (85) results from Proposition 3. For condition (ii) given in (86) we shall use the results of Proposition 9 and Proposition 6. Set $h=t_0$ as in Proposition 6. Let $\tau\in\mathcal{T}$, $\eta\in(0,1)$ and $s\in[0,\eta]$. Since the results of the previous sections are valid for an arbitrary $T>0$ we can extend the time interval to $T'=T+1$ to prevent the possibility that $\tau+s$ exceeds T. The variation $A^N(\tau+s)-A^N(\tau)$ is bounded above by $A^N(h)-A^N(0)$ if $\tau+s\le h$, by the sum of $A^N(\tau+s)-A^N(h)$ and $A^N(h)-A^N(0)$ in case $h\in(\tau,\tau+s]$. This implies that in all cases

$$P\Big(A^N(\tau+s)-A^N(\tau)>\epsilon\Big) \tag{104}$$

$$\le P\Big(A^N(h)-A^N(0)>\frac{\epsilon}{2}\Big) \tag{105}$$

$$+\sup_{\substack{\tau\,\in\,\mathcal{T}\,,\,\tau\,\ge\,h\\ s\,\in\,[0,\eta]}} P\Big(A^N(\tau+s)-A^N(\tau)>\frac{\epsilon}{2}\Big). \tag{106}$$

Recall $\epsilon'=\frac{C_{r_D}\epsilon}{24}$ from Proposition 9. Apply Theorem 2 with $\epsilon\mapsto\epsilon'$ to prove that, outside an event with probability vanishing as $N\to\infty$, for a sufficiently small $r > 0$, the number of particles in the boundary layer D_r^c is at most equal to $[\epsilon' N]$. Then (106) is bounded above by

$$\sup_{\tau \in \mathcal{T}, \tau \geq h} P\Big(A^N(\tau + \eta) - A^N(\tau) > \frac{\epsilon}{2}\Big) \leq$$

$$\sup_{\tau \in \mathcal{T}, \tau \geq h} P\Big(A^N(\tau + \eta) - A^N(\tau) > \frac{\epsilon}{2}\,,\, N(D_r^c, \tau) \leq [\epsilon' N]\Big) + \tag{107}$$

$$\sup_{\tau \in \mathcal{T}, \tau \geq h} P\Big(N(D_r^c, \tau) > [\epsilon' N]\Big)\,. \tag{108}$$

This puts (107) in the setting of Proposition 9, which shows that as we let $N \to \infty$ and $\eta \to 0$ (107) vanishes. The term (108) vanishes after we let $N \to \infty$ followed by $r \to 0$, eliminating (106). Finally, we let $N \to \infty$ and then $h \to 0$ and obtain limit zero for (105). To summarize, we can see that the limit of (104) over N, η, r and h, in this order, is zero. $\square$

6 Proof of Theorem 1.

Tightness. Theorem 3 establishes the tightness of $\{A^N(\cdot)\}_{N \in \mathbb{Z}_+}$. We need to prove that $\{\mu^N(\cdot, dx)\}_{N \in \mathbb{Z}_+}$ is tight in the weak* topology of $\mathbf{D}([0,T], \mathcal{M}(D))$, that is, that for any $\phi \in C_b(D)$ the processes $\{\langle \phi, \mu^N(\cdot, dx)\rangle\}_{N \in \mathbb{Z}_+}$ satisfy (85) and (86). Since $\overline{D}$ is compact we only have to prove tightness for $\phi \in C^\infty(\overline{D})$. Condition (85) is immediate from the boundedness of ϕ. For (86) we look at (13). The martingale part is naturally tight by the optional sampling theorem and the maximal inequality with quadratic variation of order N^{-1}. Lemma 2 completes the argument outside the special set (47). The integrand of the second order term containing the Laplacian has a uniform bound in N given by the supremum norm of the Laplacian of ϕ hence the time integral is of the same order as η. The only difficult term is the summation of the singular integrals with respect to the counting measures $dA_i^N(t)$. However, the integrands are uniformly bounded by $2\|\phi\|$ which reduces the total variation to the total variation of the average number of jumps $A^N(t)$ in (16) which is proven in Theorem 3.

Identification of the limiting profile. The tightness of $\{\nu^N(\cdot, dx)\}_{N \in \mathbb{Z}_+}$ follows from the joint tightness of $\{(\mu^N(\cdot, dx), A^N(\cdot))\}_{N \in \mathbb{Z}_+}$. We notice that Proposition 2 stops short of stating the actual tightness of $\{\nu^N(\cdot, dx)\}_{N \in \mathbb{Z}_+}$ since the test functions ϕ for which the weak tightness was shown were restricted to functions vanishing on the boundary. After proving the tightness of $\{A^N(\cdot)\}_{N \in \mathbb{Z}_+}$ we can make full use of the relationship between $\mu^N(t, dx)$ and $\nu^N(t, dx)$. Let $\nu(\cdot, dx)$, $\mu(\cdot, dx)$ and $A(\cdot)$ be limit points of $\{\nu^N(\cdot, dx)\}_{N \in \mathbb{Z}_+}$, $\{\mu^N(\cdot, dx)\}_{N \in \mathbb{Z}_+}$ and $\{A^N(\cdot)\}_{N \in \mathbb{Z}_+}$, respectively. We can assume that there exists a subsequence

$N' \to \infty$ converging jointly to the limit points. For test functions ϕ vanishing on the boundary we know that $\langle \phi, \nu^{N'}(t,dx)\rangle$ converges in distribution to $\langle \phi, u(t,x)dx\rangle$, where $u(t,x)$ is the unique solution of (18). Convergence in distribution coincides with convergence in probability when the limit is nonrandom (a point in the underlying space). We conclude that if $G \subset\subset D$, then $\nu(t,G) = \int_G u(t,x)dx$ almost surely. Since $\nu(t,dx)$ is well defined as a finite measure on D, we see that $\lim_{G\to D} \nu(t,G) = \nu(t,D)$, which implies that $\nu(t,D) = \int_D u(t,x)dx$ almost surely. This identifies $\nu(t,dx)$ obtained above as an absolutely continuous measure with respect to the Lebesgue measure on D coinciding with $u(t,x)dx$. We integrate $\nu^{N'}(t,dx)$ given in (21) against the constant test function $\phi(x) \equiv 1$ and obtain that $\langle 1, \nu^{N'}(t,dx)\rangle$ converges in distribution to $\nu(t,D)$, a nonrandom limit. We notice that $\langle 1, \mu^{N'}(t,dx)\rangle \equiv 1$ for all N'. Since $\langle 1, \nu^{N'}(t,dx)\rangle = \exp(-A^{N'}(t))\langle 1, \mu^{N'}(t,dx)\rangle$ and $\exp(-A^{N'}(t))$ converges in distribution to $\exp(-A(t))$, we can sum up and verify that,

$$\int_D u(t,x)dx = \nu(t,D) = \lim_{N'\to\infty} \langle 1, \nu^{N'}(t,dx)\rangle =$$

$$\lim_{N'\to\infty} \exp(-A^{N'}(t))\langle 1, \mu^{N'}(t,dx)\rangle = \exp(-A(t))$$

in distribution. Once again, the limit $\int_D u(t,x)dx$ is nonrandom and this implies that $\exp(-A^{N'}(t)) \to \exp(-A(t))$ in probability. Due to the uniqueness of the solution to (18) we conclude that $\mu(t,dx)$ solves (19) with $A(t)$ given by $-\ln z(t)$.

Uniform convergence in time. For a given $T > 0$ the Skorohod metric on the space of left-limit and right continuous functions $\mathbf{D}([0,T],X)$ on a Polish space $(X, \|\cdot\|)$ is given by the distance $d(f,g)$ between two elements of $\mathbf{D}([0,T],X)$

$$d(f,g) = \inf_{\lambda\in\Lambda} \left\{ \|\lambda\| + \sup_{t\in[0,T]} \|f(t) - g(\lambda(t))\| \right\} \tag{109}$$

where Λ is the space of nondecreasing continuous functions $\lambda : [0,T] \to [0,T]$ with $\lambda(0) = 0$ and $\lambda(T) = T$ with the notation

$$\|\lambda\| = \sup_{0\le s\le t\le T} \left| \log\left(\frac{\lambda(t)-\lambda(s)}{t-s} \right) \right| .$$

Let $\phi \in C^2(\overline{D})$ and for $X = R$ and $m \in \mathbf{D}([0,T], \mathcal{M}(D))$ we define the bounded continuous functional

$$\hat{d}(m) = d\Big(\langle \phi, m(\cdot,dx)\rangle - \langle \phi, \mu(\cdot,dx)\rangle \, , \, 0 \Big) . \tag{110}$$

Note that $\hat{d}(\cdot)$ would not be continuous in the Skorohod topology if $\langle \phi, \mu(\cdot,dx)\rangle$ would not be continuous, even though $d(f,0) = \|f\|_\infty$ (the distance (109) is not translation invariant). We know that, in distribution,

$$(\langle \phi, \mu^N(\cdot, dx)\rangle - \langle \phi, \mu(\cdot, dx)\rangle \Rightarrow 0 \tag{111}$$

where the limit is the path identically equal to zero. In other words, if $\tilde{P}^N$ denotes the law of (111), then $\tilde{P}^N \Rightarrow \tilde{P} = \delta_0$. Let $\mathcal{O} = \{m : \hat{d}(m) \geq \epsilon\}$, a closed set in $\mathbf{D}([0,T], \mathcal{M}(D))$ not containing the identically equal to zero element. In general for closed sets $\limsup_{N\to\infty} \tilde{P}^N(\mathcal{O}) \leq \tilde{P}(\mathcal{O}) = 0$, which proves our claim. $\square$

Acknowledgment. Part of the present work was done while the first author was visiting Northwestern University. He would like to thank the Mathematics Department for its hospitality. We are most grateful to the anonymous referee for the valuable suggestions and comments.

References

[1] Billingsley, P. (1999) *Convergence of Probability Measures.* Wiley series in probability and statistics, New York.

[2] Burdzy, K., Hołyst, R., Ingerman, D., March, P. (1996) *Configurational transition in a Fleming-Viot type model and probabilistic interpretation of Laplacian eigenfunctions.* J. Phys. A 29, 2633-2642.

[3] Burdzy, K., Hołyst, R., March, P. (2000)*A Fleming-Viot particle representation of the Dirichlet Laplacian.* Comm. Math. Phys. 214, no. 3.

[4] Grigorescu, I., Kang, M. (2002) *Tagged Particle Limit for a Fleming-Viot Type System.* Preprint.

[5] Jacod, J., Shiryaev, A. (1987) *Limit Theorems for Stochastic Processes.* Springer, Berlin Heidelberg.

[6] Oelschläger, K. (1985) *A law of large numbers for moderately interacting diffusion processes.* Z. Wahrscheinlichkeitstheorie verw. Gebiete, vol 69, 279-322.