MAXIMIZING THE VARIANCE OF THE TIME TO RUIN IN A MULTI-PLAYER GAME WITH SELECTION

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Abstract. A game with $K \geq 2$ players, each having an interger-valued fortune, consists of choosing a pair (i, j) among the players with nonzero fortunes, flipping a fair coin, independent of the process up to that time, and refreshing the fortunes by adding one unit to the winner and subtracting one unit from the loser. All other players' fortunes remain the same. The game continues until only one player wins all. The choices of pairs represent the control present in the problem. While it is known that the expected time to ruin (i.e. expected duration of the game) is independent of the choices of pairs (i, j) (the strategies), our objective is to find a strategy which maximizes the variance of the time to ruin. We show that the maximum variance is uniquely attained by the (optimal) strategy which always selects a pair of players who have currently the largest fortunes. An explicit formula for the maximum value function is derived. Additionally, the maximization problem is solved in a simpler constrained case, where a pair, once chosen, continues to play until one of them is broke. By constructing a simple martingale, we also provide a short proof of a result of S.M. Ross that the expected time to ruin is independent of the strategies. A brief discussion of the (open) problem of minimizing the variance of the time to ruin is given in the end.

1. Introduction and results

A game with $K \geq 2$ players consists of choosing a pair (i, j) among the players with nonzero fortunes, flipping a fair coin, independent of the process up to that time, and refreshing the fortunes by adding one unit to the winner and subtracting one unit from the loser. All other players' fortunes remain the same. The game continues until only one

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player wins all. The choices of pairs represent the control present in the problem. It is known [10] that the expected time to ruin $E(T)$ (i.e. expected duration of the game) is independent of the choices of pairs (i, j). It is then meaningful to investigate the relation among possible strategies of picking the pairs in terms of the variance of the time to ruin.

The Gambler's ruin model (GRM) has been used in genetic algorithms (GA) [8] where T is the *time of convergence* of the GA. Our model is an idealization which applies as well to the estimation of the time to reach fixation in an evolutionary model with multiple genotypes. The players represent the competing genotypes and the time to ruin models the extinction of one type, or equivalently completing an evolutionary step in favor of the type with better fitness. This is irreversible, as in *Muller's ratchet* model [5, 7], motivating the GRM.

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space on which a sequence of i.i.d. Bernoulli random variables with probability of success equal to $1/2$ is defined such that for each $t = 1, 2, \ldots$, the first t Bernoulli random variables are measurable with respect to $\mathcal{F}_t$ and all the later Bernoulli random variables are independent of $\mathcal{F}_t$. (In particular, all of the Bernoulli random variables are independent of $\mathcal{F}_0$.)

Thereafter we shall denote by $\eta = (\eta_1, \ldots, \eta_K)$ a configuration with η_i the fortune of player i and e_i the K-dimensional vector with entries (components) equal to zero except at i, where the entry is one. In this way

$$\eta^{ij} = \eta + e_i - e_j \tag{1.1}$$

is the transformation occurring when we pick the pair (i, j) and player i wins. Thus when the pair (i, j) is chosen, the configuration η will move to either η^{ij} (in case i wins) or η^{ji} (in case j wins) with equal probability.

Any sequence of pairs $\{(i(t), j(t))\}_{t\geq 0}$ designating the pair picked at (the end of) time $t \geq 0$ (to play at time $t+1$) will generate a random process denoted by $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_K(t))$, the vector of fortunes $\eta_r(t)$ of players r, $1 \leq r \leq K$, at time t by updating the configuration $\eta(t)$ to $\eta(t+1) = (\eta(t))^{ij}$ if the pair (i, j) is selected at (the end of) time t, plays at time $t + 1$ and i wins by sampling the $(t + 1)$-st term of the Bernoulli random sequence.

We shall assume that $(i(t), j(t)) \in \mathcal{F}_t$, i.e. the pair to play at time $t + 1$ is selected according to the information available up to and including time t of the game. Additionally we assume that *a zero entry in the vector η cannot be selected.* Such a random sequence is said a *strategy* (or *policy*) and will be generally denoted by S. The set of all strategies is denoted $\mathcal{S}$. For any such strategy, the process $\{\eta(t)\}_{t\geq 0}$ is adapted to the filtration. In case the strategy $(i(t), j(t))$ depends only on $\eta(t)$ for all $t \geq 0$, $\{\eta(t)\}_{t\geq 0}$ is a Markov chain.

For any strategy $S \in \mathcal{S}$ and an initial configuration η we denote by $E_\eta^S[\cdot]$ the expected value of the process starting with fortune $\eta_0 = \eta$ which follows the strategy S.

Let $N = |\eta| := \sum_{i=1}^K \eta_i$ be the sum of all fortunes in configuration η and T be the time to ruin of all but one player (i.e. T is the duration of the game). Note that for any strategy the time to ruin T is a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. We also notice that trivially $N = |\eta(t)|$ remains constant.

Proposition 1. *For any strategy S with T denoting the time to ruin, the process*

$$q_1(t) = \sum_{i=1}^{K} (\eta_i(t))^2 - 2t\,, \qquad K \geq 2\,, \tag{1.2}$$

is an $\mathcal{F}_t$-martingale up to T. For $K = 2$, there is only one possible strategy, to pick both players at all times, denoted by S_0, and the process

$$q_2(t) = (q_1(t))^2 - \frac{1}{6}(\eta_1(t) - \eta_2(t))^4 + \frac{8}{3}t\,, \qquad K = 2\,, \tag{1.3}$$

is an $\mathcal{F}_t$-martingale up to T.

Proof. It is easily shown that $q_1(t)$ is a martingale up to T. Recalling $(i(t), j(t)) \in \mathcal{F}_t$, we have *a.s.* on $\{t < T\}$

$$E_\eta^S[(q_1(t+1))^2|\mathcal{F}_t] = (q_1(t))^2 + 4[\eta_{i(t)}(t) - \eta_{j(t)}(t)]^2, \quad K \geq 2, \tag{1.4}$$

and

$$E_\eta^{S_0}[(\eta_1(t+1) - \eta_2(t+1))^4|\mathcal{F}_t] = (\eta_1(t) - \eta_2(t))^4 + 24(\eta_1(t) - \eta_2(t))^2 + 16, \quad K = 2, \tag{1.5}$$

from which it follows that $q_2(t)$ is a martingale up to T. □

Note that Eq. (1.4) for $K > 2$, while not needed for the proof of Proposition 1, will be called for later. Now for $K \geq 2$, noting that

$$q_1(0) = \sum_{i=1}^{K} (\eta_i)^2 \quad \text{and} \quad q_1(T) = N^2 - 2T = \Big(\sum_{i=1}^{K} \eta_i\Big)^2 - 2T\,,$$

the first martingale gives

$$E^S_\eta[T] = \frac{1}{2}\Big[\Big(\sum_{i=1}^K \eta_i\Big)^2 - \sum_{i=1}^K (\eta_i)^2\Big] = \sum_{1\le i<j\le K} \eta_i\eta_j \,, \tag{1.6}$$

which is *independent of the strategy* (*cf.* [10]). For $K = 2$, the only strategy S_0 is deterministic and the second martingale yields the following formula for the variance of T

$$Var^{S_0}_\eta(T) = \frac{\eta_1\eta_2}{3}((\eta_1)^2 + (\eta_2)^2 - 2)\,,\ K = 2\,. \tag{1.7}$$

Hereafter we will write $V^S(\eta) = Var^S_\eta(T)$ for notational simplicity.

Since the expected value of T is finite, the stopping time T is finite almost surely. While the expected time to ruin is independent of the strategy, the variance depends on S for $K \ge 3$. In this case, we would like to solve the problems

$$\text{find } S_+ \text{ such that} \qquad V^{S_+}(\eta) = \sup_{S\in\mathcal{S}} V^S(\eta)\,, \tag{1.8}$$

$$\text{find } S_- \text{ such that} \qquad V^{S_-}(\eta) = \inf_{S\in\mathcal{S}} V^S(\eta)\,. \tag{1.9}$$

Remark. Since $E^S_\eta(T)$ does not depend on S, optimizing the variance $V^S(\eta)$ is equivalent to optimizing the second moment $E^S_\eta(T^2)$, but it is more convenient to work directly with the variance. Also, it is not difficult to show (*cf.* proof of Lemma 1 of [10]) that there exist $0 < \rho < 1$ and $0 < C < \infty$ such that $P^S_\eta(T > t) \le C\rho^t$ for all $t \ge 0$ and all $S \in \mathcal{S}$, implying that $\sup_{S\in\mathcal{S}} V^S(\eta) < \infty$. Moreover, $V^{S_+}(\eta)$ and $V^{S_-}(\eta)$ are invariant with respect to permutations of η.

Adopting the terminology from the literature of dynamic programming and Markov decision processes (see, e.g. [3]), a *stationary* strategy S is determined by a (deterministic) mapping s from the configuration space to the set of pairs $\{(i,j) : 1 \le i < j \le K\}$ such that the pair $(i(t), j(t))$ is given by $s(\eta(t))$. Then the following recurrence holds

$$V^S(\eta) = \frac{1}{2}(V^S(\eta^{ij}) + V^S(\eta^{ji})) + (\eta_i - \eta_j)^2\,, \quad (i,j) = s(\eta)\,, \tag{1.10}$$

which follows from the strong Markov property and the well-known conditional variance formula

$$Var(X) = E[Var(X|\mathcal{G})] + Var(E[X|\mathcal{G}]) \tag{1.11}$$

for any σ-field $\mathcal{G}$ and random variable X with $E(X^2) < \infty$.

In particular, when $K = 2$ the variance (1.7) satisfies trivially (1.10).

While any stationary strategy satisfies (1.10), only the optimal strategy satisfies Eq. (1.12) below.

Proposition 2. *The dynamic programming equation for the maximization problem (1.8) is*

$$V(\eta) = \max_{(i,j)} \left\{ \frac{1}{2}(V(\eta^{ij}) + V(\eta^{ji})) + (\eta_i - \eta_j)^2 \right\}, \qquad V(\eta^f) = 0\,, \tag{1.12}$$

where the maximum is taken over all pairs (i, j) with $\eta_i\eta_j > 0$, and η^f is any final (terminal) configuration, i.e. with all but one entry equal to zero. The dynamic programming equation for the minimization problem (1.9) is (1.12) with $\max_{(i,j)}$ replaced by $\min_{(i,j)}$.

Proposition 3. *Assume a real-valued function $V(\eta)$ defined on the (finite) configuration space satisfies $V(\eta^f) = 0$ for any final configuration η^f and*

$$V(\eta) \geq \frac{1}{2}(V(\eta^{ij}) + V(\eta^{ji})) + (\eta_i - \eta_j)^2\,, \text{ for any pair } (i,j) \text{ with } \eta_i\eta_j > 0. \tag{1.13}$$

Then $V(\eta) \geq V^S(\eta)$ for any $S \in \mathcal{S}$. If there exists $S' \in \mathcal{S}$ such that $V(\eta) = V^{S'}(\eta)$, then $S_+ = S'$ and V is the solution to the maximization problem (1.8). The same is true for the minimization problem (1.9) by replacing $\geq$ with $\leq$ in all inequalities.

Proof. For any real-valued function $f(\eta)$, by conditional probability, we have

$$E^S_\eta[f(\eta(t+1))|\mathcal{F}_t] = \frac{1}{2}\Big(f((\eta(t))^{i(t)j(t)}) + f((\eta(t))^{j(t)i(t)})\Big)\,, \quad 0 \leq t \leq T-1\,. \tag{1.14}$$

Applying this relation to $f = V$ and using (1.4), the process

$$M(t) = V(\eta(t)) + \frac{1}{4}(q_1(t))^2$$

is a super-martingale up to T and comparison between the expected values at $t = 0$ and $t = T$ shows the claim of the proposition. $\square$

Theorem 1. *Let the stationary Markovian strategy S_+ be defined by $s_+(\eta) = (i, j)$ where η_i and η_j are the largest two values in η. (In case of ties, any pair corresponding to the largest two values may be selected.) Then $(S_+, V^{S_+}(\eta))$ solves the maximization problem (1.8). Furthermore, the maximum variance of the time to ruin cannot be attained by any strategy that ever selects a pair which does not correspond to the largest two values in the current configuration.*

Proof. By Proposition 3, the proposed strategy S_+ solves (1.8) if $V^{S_+}(\eta)$ satisfies (1.13). Note that by the definition of S_+, both sides of (1.13) with $V = V^{S_+}$ are equal if (i,j) with $\eta_i\eta_j > 0$ is such that η_i and η_j are the largest two values in η. Thus it suffices to consider those pairs (i,j) for which $\eta_i\eta_j > 0$ and $\{\eta_i, \eta_j\}$ is not the set of the largest two values in η.

Indeed, we will show that $V^{S_+}(\eta)$ satisfies the following (stronger) strict inequality

$$V^{S_+}(\eta) > \frac{1}{2}(V^{S_+}(\eta^{ij}) + V^{S_+}(\eta^{ji})) + (\eta_i - \eta_j)^2\,, \tag{1.15}$$

for any pair (i,j) with $\eta_i\eta_j > 0$ and $\{\eta_i, \eta_j\} \neq \{\eta_{M1}, \eta_{M2}\}$ where η_{M1} and η_{M2} denote, respectively, the largest and second largest values in η. Here $\{\eta_i, \eta_j\}$ and $\{\eta_{M1}, \eta_{M2}\}$ are interpreted as multisets counting multiplicities of elements (cf. page 483 in [9]). (Note that η_{M1} and η_{M2} are equal if two or more entries are tied for the maximum value in η.) The proof is done by induction on K, and for fixed K, by induction on N as stated in Proposition 7 of Section 5 where the induction step is proven. The most difficult case of the induction step is proven separately in Section 6. This case requires a lemma, proved in Section 7. The verification step corresponding to $K = 2$ is done in Section 4. □

The rest of the paper is organized as follows. Section 2 gives several useful reduction formulas, which provide an effective way to do explicit calculations in Section 3 and to permit the induction argument in Sections 4–7. In particular, *an explicit formula for the maximal value function* is presented in Section 3; see Theorem 2. In Section 8, we consider a constrained version of the maximization problem (1.8) where an (admissible) strategy is subject to the constraint that once a pair is chosen, it must play until one of the two players is defeated and eliminated from the game (whose fortune reaches 0). This constrained version is much easier to solve and the optimal strategy is similar to S_+, which is to select a pair of players with the largest two fortunes every time when a player is defeated and eliminated. The final Section 9 contains brief discussions of the minimization problem (1.9) (which remains open) and related multi-player gamblers' ruin problems.

2. Reduction formulas

Let $\eta = (\eta_1, \ldots, \eta_K)$ be a configuration with K components and total fortune $N = |\eta|$. A configuration is said *extremal* if all except possibly one component are equal and the unequal one (if exists) has a greater value. In other words, a configuration is extremal if either all components are equal or all except the (unique) greatest component are equal.

Since the total number N is known, we shall specify only the common value of the smaller components, so ζ_c designates the configuration with $K-1$ components equal to c and one component equal to $N-(K-1)c$ (which is greater than or equal to c). Thus c must satisfy the condition $Kc \leq N$. In particular, when $c = 0$, the extremal configuration ζ_0 has only one nonzero component and is referred to as a *final* configuration.

For notational simplicity, we shall write P^+ for P^{S_+} and $V^+(\eta)$ for $V^{S_+}(\eta)$ (the variance of T under the strategy S_+ with initial configuration η). Two configurations are said *indistinguishable* if they are identical up to an ordering (i.e. one is a permutation of the other). Note that indistinguishability is an equivalence relation, so that the equivalence class of η is the set of all configurations that are indistinguishable from η. Clearly $V^+(\eta) = V^+(\eta')$ if η and η' are indistinguishable, i.e. V^+ is invariant with respect to permutations. A configuration ζ (or more precisely, the equivalence class of ζ) is said *accessible from* η if under the strategy S_+, it is reached before T with probability one, i.e. the hitting time τ_ζ of (the equivalence class of) ζ has the property $P^+_\eta(\tau_\zeta \leq T) = 1$. In what follows, the words "the equivalence class of" will be omitted unless necessary for clarity purposes.

Among all extremal configurations we single out the one with $c = 1$. We shall deal separately with $V^+(\zeta_1) = V^{S_+}(\zeta_1)$. First, we look at $m(\eta) := \min_i \eta_i$.

Proposition 4. *For any η with $m(\eta) \geq 1$, ζ_1 is accessible from η.*

Proof. The case $K = 2$ is trivial. Below we assume $K \geq 3$. Under S_+, the components with values equal to $m = m(\eta)$ will not be touched as long as there exist two larger components.

If η is such that $M = M(\eta) := \max_i \eta_i = m$ (all flat), we have two possibilities. In case $M = m = 1$, $\eta = \zeta_1$ in the special case when $N = K$. In case $M = m \geq 2$ we play one turn under the strategy S_+ and the two resulting configurations will be indistinguishable, denoted η', for which we have $M(\eta') > m(\eta') \geq 1$.

Thus we can assume without loss of generality that η has $M > m \geq 1$. If there exists exactly one component greater than m, then $\eta = \zeta_m$. If there are two or more components greater than m, we may view m as a baseline and the set of those components greater than the baseline continues to evolve under S_+ until all except one component equal m. The strategy S_+ will simply not look at components equal to m until the process reaches the extremal ζ_m, which shows that ζ_m is accessible. Note that this process up to the hitting time τ_{ζ_m} of ζ_m is exactly the same as the ruin problem with the initial configuration

$\eta - \overline{m} = (\eta_1 - m, \ldots, \eta_K - m)$, where $\overline{m}$ is the K-dimensional configuration with all entries equal to m. (I.e. τ_{ζ_m} has the same distribution as the time to ruin when the initial configuration is $\eta - \overline{m}$.) As such, $P^+_\eta(\tau_{\zeta_m} < \infty) = 1$.

On the other hand, under S_+ with $K \geq 3$, we have $m(\eta(t)) - 1 \leq m(\eta(t+1)) \leq m(\eta(t))$, i.e. $m(\eta(t))$ is nonincreasing and can move down by one unit only. Since ζ_0 (the final configuration when the game stops) has $m(\zeta_0) = 0$, it follows that a configuration with $m = 1$ will be reached a.s., and based on the preceding reasoning on τ_{ζ_m}, the configuration ζ_1 will be reached with probability one as well. $\square$

Proposition 5. *For any η with $m(\eta) \geq 1$*

$$V^+(\eta) = V^+(\eta - \bar{1}) + V^+(\zeta_1)\,. \tag{2.1}$$

Proof. By Proposition 4, for a given initial configuration η with $m(\eta) \geq 1$, we have $0 \leq \tau_{\zeta_1} < T < \infty$ a.s. under S_+. Write $T = \tau_{\zeta_1} + (T - \tau_{\zeta_1})$. It follows from the strong Markov property that τ_{ζ_1} and $T - \tau_{\zeta_1}$ are independent. Moreover, τ_{ζ_1} has the same distribution as the time to ruin when the initial configuration is $\eta - \bar{1}$, while $T - \tau_{\zeta_1}$ has the same distribution as the time to ruin when the initial configuration is ζ_1. So,

$$V^+(\eta) = Var^{S_+}_\eta(T) = Var^{S_+}_\eta(\tau_{\zeta_1}) + Var^{S_+}_\eta(T - \tau_{\zeta_1}) = V^+(\eta - \bar{1}) + V^+(\zeta_1),$$

proving (2.1). $\square$

Let η be a configuration given in ordered form and let $c \geq 0$ with the property

$$\eta_1 \leq \ldots \leq \eta_i \leq c < \eta_{i+1} \leq \ldots \leq \eta_K\,, \tag{2.2}$$

where the strict inequality is to be interpreted that *there exists at least one entry strictly larger than c*. We define the configuration *flattened up to level c*, denoted $\eta^{|c}$, by

$$\begin{aligned} &\eta^{|c}_r = \eta_r\,, \quad 1 \leq r \leq i\,; \qquad \eta^{|c}_r = c\,, \quad i < r \leq K-1\,; \\ &\eta^{|c}_K = \sum_{r=i+1}^{K} \eta_r - (K-i-1)c\,, \end{aligned} \tag{2.3}$$

which results from following S_+ until all of the last $K-i$ entries (except one) are reduced to the level c. (We remark that the exceptional entry in the resulting configuration has a value equal to $\sum_{r=i+1}^K \eta_r - (K-i-1)c > c$, which is not necessarily the K-th entry. Thus we should interpret $\eta^{|c}$ as a configuration up to an ordering.) Let $(\eta - \bar{c})_+ := ((\eta_1 - c)_+, \ldots, (\eta_K - c)_+)$, where $(x)_+ := \max\{x, 0\}$. If the configuration η would be restricted to entries η_r $(r > i)$ and shifted by c to $\eta_r - c$, i.e. $(\eta - \bar{c})_+$, then the configuration $\eta^{|c}$ (restricted to the last $K - r$ entries and shifted by c) coincides, up to an ordering, with the final configuration of the restricted process, while all other entries η_r, $1 \le r \le i$, are left unchanged. This shows that $\eta^{|c}$ is accessible from η.

Proposition 6. *Let η be a configuration and c as in (2.2), such that there is at least one entry greater than c. Then $\eta^{|c}$ is accessible from η and*

$$V^+(\eta) = V^+((\eta - \bar{c})_+) + V^+(\eta^{|c}) \,. \tag{2.4}$$

In particular, for $c = m = m(\eta)$ and η not constant, we have $\eta^{|c} = \zeta_m$ and

$$V^+(\eta) = V^+((\eta - \overline{m})_+) + V^+(\zeta_m) \,. \tag{2.5}$$

Proof. If there is exactly one entry strictly larger than c, then $\eta = \eta^{|c}$ and (2.4) holds since $V^+((\eta - \bar{c})_+) = 0$, $(\eta - \bar{c})_+$ being a final configuration.

Suppose there are at least two entries greater than c. The reasoning is almost identical to the proofs of Propositions 4 and 5. Since $(\eta - \bar{c})_+$ has at least two nonzero components, the process evolving under strategy S_+ will not touch any entry $\eta_r \le c$ until the entries above c will be flattened out, i.e. until the configuration $\eta^{|c}$ is reached, which we know happens with probability one. So $\eta^{|c}$ is accessible from η. Now write $T = \tau + (T - \tau)$ where $\tau = \tau_{\eta^{|c}}$, the hitting time of $\eta^{|c}$. By the strong Markov property, τ and $T - \tau$ are independent. Moreover, τ has the same distribution as the time to ruin with initial configuration $(\eta - \bar{c})_+$, while $T - \tau$ has the same distribution as the time to ruin with initial configuration $\eta^{|c}$, from which (2.4) follows. □

2.1. **A reduction formula that gives insight but we do not use in the proof.** The next lemma shows that we can prove (1.15) for any configuration and pair having at least two entries dominating the members of the pair by proving it for a simplified configuration $\eta^{|c}$. The reader should think of the case $c > \max\{\eta_i, \eta_j\}$ and should understand the condition that there must exist two entries exceeding strictly $\max\{\eta_i, \eta_j\} + 1$, to prevent interference when we commute the operation of "moving" between two entries and "flattening" at level c described formally in (2.6) and (2.7) below.

Recall (1.1) that the transformation of η consisting of a move from entry j to entry i is denoted

$$\eta^{ij} = \eta + e_i - e_j \,.$$

Lemma 1. *Let η, $c \geq 0$ and (i,j) be such that $\max\{\eta_i, \eta_j\} < c$ and there exist at least two entries of η greater than c. Then*

$$(\eta - \bar{c})_+ = (\eta^{ij} - \bar{c})_+ = (\eta^{ji} - \bar{c})_+$$

$$V^+(\eta^{ij}) = V^+((\eta^{ij} - \bar{c})_+) + V^+((\eta^{ij})^{|c}) = V^+((\eta - \bar{c})_+) + V^+((\eta^{|c})^{ij}) \tag{2.6}$$

$$V^+(\eta^{ji}) = V^+((\eta^{ji} - \bar{c})_+) + V^+((\eta^{ji})^{|c}) = V^+((\eta - \bar{c})_+) + V^+((\eta^{|c})^{ji}) \,. \tag{2.7}$$

Proof. The operation $\eta \to \eta^{|c}$ involves only entries exceeding c. The lemma follows from Proposition 6 and the fact that η, η^{ij} and η^{ji} differ only in the i-th and j-th entries which are all less than or equal to c since $\max\{\eta_i, \eta_j\} < c$. □

3. Explicit formula for the maximal value function

For given K (the number of entries) and N (the total sum of entries), for $0 \leq c \leq N/K$, recall that ζ_c is the extremal configuration being all flat at c except possibly one maximal value. We may write $\zeta_c = (N - Kc + c, c, c, \ldots, c)$ up to an ordering. The values of V^+ at these extremal configurations will allow us to calculate $V^+(\eta)$ for general η. In some sense, we need to develop a rudimentary calculus for these structures as presented below.

To make the dependence on K and N explicit, we write

$$\zeta_c = \zeta_{c,K,N} = (N - Kc + c, c, c, \ldots, c) \quad \text{(with K entries summing up to N)},$$

and introduce the convenient notation

$$W_K(N,c) := V^+(\zeta_{c,K,N}) = V^+(N - Kc + c, c, c, \ldots, c)\,, \quad Kc \leq N. \tag{3.1}$$

We start writing a formula for $W_K(N,c)$. Based on (2.1), we have

$$\begin{aligned} W_K(N,c) &= V^+(N - Kc + c, c, \ldots, c) \\ &= V^+(N - Kc + c - 1, c-1, \ldots, c-1) + V^+(N - K + 1, 1, \ldots, 1)\,, \\ &= W_K(N-K, c-1) + W_K(N,1)\,. \end{aligned}$$

Repeating the same argument,

$$W_K(N-K, c-1) = W_K(N-2K, c-2) + W_K(N-K, 1)$$
$$\dots$$
$$W_K(N-(c-2)K, 2) = W_K(N-(c-1)K, 1) + W_K(N-(c-2)K, 1)\,.$$

Summing up yields

$$W_K(N, c) = \sum_{r=0}^{c-1} W_K(N-rK, 1)\,. \tag{3.2}$$

In these formulas the parameter K does not change. The simpler function $W_K(d,1)$ with $d \geq K$ can be obtained by applying the recurrence formula (1.10) for V^S to $V^+ = V^{S_+}$ as follows. For $d \geq K$, we have by (1.10)

$$\begin{aligned} W_K(d,1) &= V^+(d-K+1, 1, \dots, 1) \\ &= \frac{1}{2}V^+(d-K+2, 0, 1, \dots, 1) + \frac{1}{2}V^+(d-K, 2, 1, \dots, 1) + (d-K)^2 \\ &= \frac{1}{2}W_{K-1}(d,1) + \frac{1}{2}V^+(d-K, 2, 1, \dots, 1) + (d-K)^2. \end{aligned} \tag{3.3}$$

By (2.1), we have for $d > K$

$$\begin{aligned} V^+(d-K, 2, 1, \dots, 1) &= V^+(d-K-1, 1, 0, \dots, 0) + V^+(d-K+1, 1, \dots, 1) \\ &= V^+(d-K-1, 1, 0, \dots, 0) + W_K(d,1), \end{aligned}$$

which together with (3.3) implies that

$$W_K(d,1) = W_{K-1}(d,1) + V^+(d-K-1, 1, 0, \dots, 0) + 2(d-K)^2. \tag{3.4}$$

By formula (1.7) for the two-player case for which there is only one strategy denoted S_0, we have

$$\begin{aligned} V^+(d-K-1, 1, 0, \dots, 0) = V^{S_0}(d-K-1, 1) &= \frac{1}{3}(d-K-1)((d-K-1)^2 + 1 - 2) \\ &= \frac{1}{3}(d-K)(d-K-1)(d-K-2). \end{aligned}$$

It follows from (3.4) that for $d > K$,

$$W_K(d,1) = W_{K-1}(d,1) + Q(d-K), \tag{3.5}$$

where

$$Q(x) = \frac{1}{3}x(x-1)(x-2) + 2x^2 = \frac{1}{3}x(x+1)(x+2). \tag{3.6}$$

Note that for $d = K$, we have by (3.3)

$$\begin{aligned} W_K(K,1) &= \frac{1}{2}W_{K-1}(K,1) + \frac{1}{2}W_{K-1}(K,1) + (K-K)^2 \\ &= W_{K-1}(K,1) = W_{K-1}(K,1) + Q(K-K), \end{aligned}$$

so that (3.5) also holds for $d = K$. Applying (3.5) repeatedly yields

$$W_K(d,1) = W_2(d,1) + \sum_{r=3}^{K} Q(d-r) = \sum_{r=2}^{K} Q(d-r)\,, \tag{3.7}$$

where we have used the fact that

$$W_2(d,1) = V^{S_0}(d-1,1) = Q(d-2).$$

Remark. For convenience, we define

$$W_1(d,1) := 0 \text{ for all } d, \quad \text{and} \quad V^+(\eta) := 0 \quad \text{for all } \eta \text{ of dimension } 1, \tag{3.8}$$

which is consistent with (3.1).

Remark. We may derive (3.7) alternatively by considering $K-1$ subgames each involving two players. Specifically, let T be the time to ruin under S_+ with initial configuration $\eta = (\eta_1, \ldots, \eta_K) = (d-K+1, 1, \ldots, 1) = \zeta_{1,K,d}$. We take the convention that in case of ties, S_+ picks the lower-indexed players. Since $\eta_3 = \ldots = \eta_K = 1$, under S_+ players 1 and 2 continue to play until one of them has fortune 0 and is eliminated. Then the survivor (with fortune $d-K+2$) plays with player 3 until one of them has fortune 0, and so on. Thus the original game is decomposed into $K-1$ subgames where the i-th game involves two players with fortunes $d-K+i$ and 1 ($i = 1, \ldots, K-1$) for which the time to ruin T_i has variance $V^{S_0}(d-K+i,1) = Q(d-K+i-1)$. Since $T = T_1 + \ldots + T_{K-1}$ and the $T_i's$ are independent, the variance of T is $V^+(\zeta_{1,K,d}) = W_K(d,1) = Q(d-K) + \ldots + Q(d-2)$, agreeing with (3.7).

We are now ready to derive a formula for $V^+(\eta)$ for general η. Let $0 < \eta'_1 < \eta'_2 < \ldots < \eta'_p$ be the distinct values present in η, in increasing order. Let $1 \le p = p(\eta) \le K$ be the total number of such values and let ℓ_k, $1 \le k \le p$ be the multiplicities of the values η'_k. Note that

$$\sum_{k=1}^{p} \ell_k = K\,, \quad |\eta| := \sum_{k=1}^{p} \ell_k \eta'_k = N. \tag{3.9}$$

Here we have assumed that η has no zero entries, i.e. $m(\eta) \ge 1$. In case $m(\eta) = 0$, we simply reduce η to a lower-dimensional configuration by deleting all zero entries. By (2.5) with $m = \eta'_1$ and (3.1), we have

$$V^+(\eta) = V^+(\eta - \overline{\eta'_1}) + V^+(\zeta_{\eta'_1,K,N}) = V^+(\eta - \overline{\eta'_1}) + W_K(N, \eta'_1).$$

Since with all zero entries removed, $\eta - \overline{\eta'_1}$ reduces to a lower-dimensional configuration with $K - \ell_1$ entries summing up to $N - K\eta'_1$ and the minimal entry value being $\eta'_2 - \eta'_1$, we have by (2.5) with $m = \eta'_2 - \eta'_1$

$$\begin{aligned} V^+(\eta - \overline{\eta'_1}) &= V^+((\eta - \overline{\eta'_2})_+) + V^+(\zeta_{\eta'_2-\eta'_1,K-\ell_1,N-K\eta'_1}) \\ &= V^+((\eta - \overline{\eta'_2})_+) + W_{K-\ell_1}(N - K\eta'_1, \eta'_2 - \eta'_1). \end{aligned}$$

Repeating this argument, we have for $r = 0, \dots, p-1$

$$\begin{aligned} V^+((\eta - \overline{\eta'_r})_+) &= V^+((\eta - \overline{\eta'_{r+1}})_+) + V^+(\zeta_{\eta'_{r+1}-\eta'_r,K_r,N_r}) \\ &= V^+((\eta - \overline{\eta'_{r+1}})_+) + W_{K_r}(N_r, \eta'_{r+1} - \eta'_r), \end{aligned} \tag{3.10}$$

where $\eta'_0 := 0$, $V^+((\eta - \overline{\eta'_p})_+) := 0$, and

(3.11)
$$K_r := K - \sum_{i=1}^{r} \ell_i, \quad N_r := N - \sum_{i=1}^{p} \ell_i \min\{\eta'_i, \eta'_r\} = \sum_{i=r+1}^{p} \ell_i(\eta'_i - \eta'_r), \quad r = 0, \dots, p-1.$$

Note that $K_0 = K$ and $N_0 = N$. Summing up (3.10) over $r = 0, \dots, p-1$ yields the following formula for $V^+(\eta)$.

Theorem 2. *The maximal value function is given by (1.7) if $K = 2$. For $K \ge 3$ and for η with p distinct nonzero values $0 < \eta'_1 < \dots < \eta'_p$ and multiplicities $\ell_r, r = 1, \dots, p$ satisfying (3.9), we have*

$$V^+(\eta) = \sum_{r=0}^{p-1} W_{K_r}(N_r, \eta'_{r+1} - \eta'_r)\,, \tag{3.12}$$

where K_r and N_r are given in (3.11) and $W_K(N,c)$ is given in (3.2), which can be reduced to the special case $c=1$, as shown in formula (3.7) for $W_K(d,1)$.

4. Cases $K=2$ and $K=3$

Case $K=2$. In this case, (1.15) is trivially satisfied, since there is only one pair $(1,2)$ and hence there is no (i,j) such that $\{\eta_i,\eta_j\} \neq \{\eta_{M1},\eta_{M2}\} = \{\eta_1,\eta_2\}$. It is useful to look at the next simplest case $K=3$ which can be calculated explicitly.

Case $K=3$. Let $\eta=(a,b,c)$ with $\min\{a,b\} \geq c \geq 1$ and $N=a+b+c$. By (2.5) and (3.2),

$$\begin{aligned} V^+(a,b,c) &= V^+(a-c,b-c,0) + W_3(N,c) \\ &= V^+(a-c,b-c,0) + \sum_{r=0}^{c-1} W_3(N-3r,1)\,. \end{aligned} \tag{4.1}$$

By (1.7) and (3.7),

$$\begin{aligned} V^+(a-c,b-c,0) &= \frac{1}{3}(a-c)(b-c)[(a-c)^2+(b-c)^2-2]\,, \\ W_3(N-3r,1) &= Q(N-3r-2)+Q(N-3r-3) \\ &= \frac{1}{3}(N-3r-2)(N-3r-1)(2N-6r-3)\,, \end{aligned}$$

implying by (4.1) that for $\min\{a,b\} \geq c \geq 1$ and $N=a+b+c$,

$$V^+(a,b,c) = U(a-c,b-c) + \frac{1}{3}\sum_{r=0}^{c-1}(N-3r-2)(N-3r-1)(2N-6r-3)\,, \tag{4.2}$$

where

$$U(x,y) := \frac{1}{3}xy(x^2+y^2-2)\,. \tag{4.3}$$

For $a \geq b \geq c > 0$, let

$$\Delta_{23} := V^+(a,b,c) - \frac{1}{2}[V^+(a,b+1,c-1)+V^+(a,b-1,c+1)] - (b-c)^2,$$

which is the difference between the two sides of (1.15) with $(i,j)=(2,3)$. Now for $a \geq b \geq c+2$, letting $\alpha := a-c$ and $\beta := b-c$, we have by (4.2)

$$
\begin{aligned}
\Delta_{23} =& U(\alpha,\beta) - \frac{1}{2}\Big(U(\alpha+1,\beta+2)+U(\alpha-1,\beta-2)\Big) - \beta^2 \\
&+ \frac{1}{6}\Big((N-3c+1)(N-3c+2)(2N-6c+3) - (N-3c-2)(N-3c-1)(2N-6c-3)\Big) \\
=& U(\alpha,\beta) - \frac{1}{2}\Big(U(\alpha+1,\beta+2)+U(\alpha-1,\beta-2)\Big) - \beta^2 \\
&+ \frac{1}{6}\Big((\alpha+\beta+1)(\alpha+\beta+2)(2\alpha+2\beta+3) - (\alpha+\beta-2)(\alpha+\beta-1)(2\alpha+2\beta-3)\Big) \\
=& \alpha^2 + \alpha\beta > 0.
\end{aligned}
$$

For $a \geq b = c+1$ and $a > b = c$, it can be shown that

$$\Delta_{23} = (a-c)(a-c+1) > 0.$$

Let

$$\Delta_{13} := V^+(a,b,c) - \frac{1}{2}[V^+(a+1,b,c-1) + V^+(a-1,b,c+1)] - (a-c)^2,$$

which is the difference between the two sides of (1.15) with $(i,j)=(1,3)$. Similarly, by (4.2), for $a \geq b > c$, it can be shown that $\Delta_{13} = (a-c)(b-c)+(b-c)^2 > 0$. This proves that (1.15) holds for $K=3$. (It should be noted that the induction step in the next section covers $K=3$.)

5. The induction step

Let $\mathbb{S}(K)$ be the following induction statement.

$\mathbb{S}(K)$: For any $K' \leq K$ and any $N > 0$, the function $V^+(\cdot)$ satisfies

$$V^+(\eta) > \frac{1}{2}\Big(V^+(\eta^{ij}) + V^+(\eta^{ji})\Big) + (\eta_i - \eta_j)^2\,, \tag{5.1}$$

for any pair (i,j) with $\eta_i\eta_j > 0$ and $\{\eta_i,\eta_j\} \neq \{\eta_{M1},\eta_{M2}\}$ where η_{M1} and η_{M2} denote the largest two values in η. Note that (5.1) is (1.15) where $V^+ = V^{S_+}$. (As remarked before, by the definition of strategy S_+, both sides of (5.1) are equal if (i,j) is such that $\eta_i\eta_j > 0$ and $\{\eta_i,\eta_j\} = \{\eta_{M1},\eta_{M2}\}$.)

For $K=2$, $\mathbb{S}(K)$ is true trivially. The next proposition concludes the proof of Theorem 1.

Proposition 7. *For each $K \geq 3$, if $\mathbb{S}(K')$ is true for all $K' < K$, then it is true for K.*

Proof. For $K \geq 3$ fixed, we start an induction on N. Note that for $m = m(\eta) = 0$, η has at least one zero entry, which reduces to a K'-dimensional configuration for some $K' < K$, so that (5.1) holds by the induction hypothesis. Furthermore, if $m = m(\eta) = M = M(\eta) := \max_r \eta_r$, then all entries in η are equal, so that there is no (i,j) such that $\{\eta_i, \eta_j\} \neq \{\eta_{M1}, \eta_{M2}\}$, implying that (5.1) holds trivially. Thus it suffices to consider the case $M > m \geq 1$ (implying that $N > K$). In light of (2.1), we shall prepare by observing that for pair (i,j), $\eta^{ij} + \bar{1} = (\eta + \bar{1})^{ij}$ is well defined if $m \geq 1$, and $\eta^{ij} - \bar{1} = (\eta - \bar{1})^{ij}$ is well defined if $m \geq 2$. With this in mind we now proceed by induction on N (with $K \geq 3$ fixed).

Assume that (5.1) holds for all $N' < N$ ($N > K$) and we want to prove it for N.

Case $m \geq 2$**.** We have $m(\eta^{ij}) \geq 1$ and $m(\eta^{ji}) \geq 1$ for any pair (i,j), so that by (2.1)

$$V^+(\eta) = V^+(\eta - \bar{1}) + V^+(\zeta_1) \tag{5.2}$$

$$V^+(\eta^{ij}) = V^+(\eta^{ij} - \bar{1}) + V^+(\zeta_1) = V^+((\eta - \bar{1})^{ij}) + V^+(\zeta_1) \tag{5.3}$$

$$V^+(\eta^{ji}) = V^+(\eta^{ji} - \bar{1}) + V^+(\zeta_1) = V^+((\eta - \bar{1})^{ji}) + V^+(\zeta_1)\,. \tag{5.4}$$

This gives, for any pair (i,j) with $\{\eta_i, \eta_j\} \neq \{\eta_{M1}, \eta_{M2}\}$,

$$\begin{aligned} V^+(\eta) - \frac{1}{2}\Big(V^+(\eta^{ij}) + V^+(\eta^{ji})\Big) &= V^+(\eta - \bar{1}) - \frac{1}{2}\Big(V^+((\eta - \bar{1})^{ij}) + V^+((\eta - \bar{1})^{ji})\Big) \\ &> ((\eta_i - 1) - (\eta_j - 1))^2 = (\eta_i - \eta_j)^2\,, \end{aligned} \tag{5.5}$$

where the inequality is due to the induction hypothesis applied to the configuration $\eta - \bar{1}$ for which the total fortune is $N - K < N$ and

$$\{(\eta - \bar{1})_i, (\eta - \bar{1})_j\} = \{\eta_i - 1, \eta_j - 1\} \neq \{\eta_{M1} - 1, \eta_{M2} - 1\} = \{(\eta - \bar{1})_{M1}, (\eta - \bar{1})_{M2}\}.$$

Case $m = 1$**.** If the pair (i,j) does not contain any minimum (i.e. $\min\{\eta_i, \eta_j\} \geq 2$), the above argument for $m \geq 2$ (i.e. (5.2)–(5.5)) works identically. Also recall that it suffices to consider $M > m \geq 1$. Thus it is the case $M > m = 1$ and $\min\{\eta_i, \eta_j\} = 1$ that will be done separately in Section 6. $\square$

Remark. The case $m = 1$ stands apart from all others because either (5.3) or (5.4) cannot be applied when the pair (i,j) contains a minimal value, say $\eta_i = m = 1$. It is perfectly correct to drop one unit on all entries based on (2.1) which is done in (5.2). Applying the transformation η^{ij} which increases the i-th entry to 2 and shifting by $\bar{1}$ is possible as they clearly commute. However, the transformation η^{ji} which lowers the i-th entry to zero would not commute with the one unit shift as $(\eta - \bar{1})^{ji}$ is not properly defined. Moreover, (2.1) does not apply to η^{ji} since $m(\eta^{ji}) = 0$. Thus (5.4) does not hold. As a result, the induction step (5.5) breaks down.

6. The case $m = 1 < M$ and $\min\{\eta_i, \eta_j\} = 1$

In this section we consider the case $m = 1 < M$ and $\min\{\eta_i, \eta_j\} = 1$. We shall assume without loss of generality that $i = 1$ and $\eta_1 = 1$. Thus $j \geq 2$ and $\eta_j \geq 1$. With ℓ denoting the multiplicity of 1 in η, we write the configuration $\eta = (\bar{1}_\ell, \xi)$, where the subscript to the vector of ones marks its dimension. (This notation is sometimes suppressed when no danger of confusion can arise.) Then ξ is a $(K - \ell)$-dimensional vector with the total sum of entries $|\xi| = |\eta| - |\bar{1}_\ell| = N - \ell$ and the minimum value $m(\xi) \geq 2$. (Note that $\ell < K$ since $m(\eta) = 1 < M(\eta)$.)

We write $V_K^+(\eta) = V^+(\eta)$ with the subscript K denoting the dimension of the argument η. This is necessary in order to keep track of the reduction formulas of the type (6.4) below.

Case $\ell \geq 2$ and $\eta_j = 1$. This treats the case when we pick two minima. Without loss of generality, assume $\eta_1 = \eta_2 = 1$, $j = 2$. (Note that $\{\eta_1, \eta_2\} = \{1, 1\} \neq \{\eta_{M1}, \eta_{M2}\}$ since $M = \eta_{M1} > 1$.)

We need to prove inequality (5.1), i.e.

$$V_K^+(\eta) - \frac{1}{2}\Big(V_K^+(\eta - e_1 + e_2) + V_K^+(\eta + e_1 - e_2)\Big) > 0\,. \tag{6.1}$$

The inequality contains no $(\eta_i - \eta_j)^2$ term since the two entries are equal. As $\eta - e_1 + e_2$ and $\eta + e_1 - e_2$ are indistinguishable, (6.1) reduces to

$$V_K^+(\bar{1}_\ell, \xi) > V_K^+(2, 0, \bar{1}_{\ell-2}, \xi) = V_{K-1}^+(2, \bar{1}_{\ell-2}, \xi)\,, \tag{6.2}$$

where we have used the projection identity $V_K^+(\xi, \bar{0}_\ell) = V_{K-\ell}^+(\xi)$ which removes the zero entries by lowering the dimension K correspondingly.

We have by (2.1) (recalling $W_K(N, c) := V^+(\zeta_{c,K,N})$ in (3.1))

$$\begin{aligned} V_K^+(\eta) = V_K^+(\bar{1}_\ell, \xi) &= V_K^+(\bar{0}_\ell, \xi - \bar{1}) + W_K(N, 1) \\ &= V_{K-\ell}^+(\xi - \bar{1}) + W_K(N, 1)\,, \end{aligned} \tag{6.3}$$

where for notational simplicity we have suppressed the subscript $K-\ell$ to the vector $\bar{1}$ in $\xi - \bar{1}$. Similarly, by (2.1),

(6.4)

$$\begin{aligned}
V_K^+(2,0,\bar{1}_{\ell-2},\xi) &= V_{K-1}^+(2,\bar{1}_{\ell-2},\xi) = V_{K-1}^+(1,\bar{0}_{\ell-2},\xi-\bar{1}) + W_{K-1}(N,1) \\
&= V_{K-\ell+1}^+(1,\xi-\bar{1}) + W_{K-1}(N,1) \\
&= V_{K-\ell+1}^+(0,\xi-\bar{2}) + W_{K-\ell+1}(N-K+1,1) + W_{K-1}(N,1) \\
&= V_{K-\ell}^+(\xi-\bar{2}) + W_{K-\ell+1}(N-K+1,1) + W_{K-1}(N,1) \\
&= V_{K-\ell}^+(\xi-\bar{1}) - W_{K-\ell}(N-K,1) + W_{K-\ell+1}(N-K+1,1) + W_{K-1}(N,1)\,.
\end{aligned}$$

The last line has replaced $V_{K-\ell}^+(\xi-\bar{2})$ by $V_{K-\ell}^+(\xi-\bar{1}) - W_{K-\ell}(N-K,1)$. This follows from (2.1) applied to $V_{K-\ell}^+(\xi-\bar{1})$ where $|\xi-\bar{1}| = |\xi| - (K-\ell) = (N-\ell) - (K-\ell) = N-K$. Note that if $K-\ell = 1$, we have $V_{K-\ell}^+(\xi-\bar{2}) = V_{K-\ell}^+(\xi-\bar{1}) = W_{K-\ell}(N-K,1) := 0$ (cf. (3.8)).

The advantage is that both last lines in (6.3) and (6.4) contain $V_{K-\ell}^+(\xi-\bar{1})$, and the rest are known computable quantities. Now (6.2) is equivalent to

$$W_K(N,1) - W_{K-1}(N,1) > W_{K-\ell+1}(N-K+1,1) - W_{K-\ell}(N-K,1)\,,$$

which by (3.7) is equivalent to

$$Q(N-K) > Q(N-K-1).$$

This is true since $Q(x)$ is an increasing function for $x \geq 0$.

Case $\ell \geq 2$ and $\eta_j > 1$.

Note that $(i,j) = (1,j), \eta = (\bar{1}_\ell,\xi), \eta^{ij} = (\bar{1}_\ell,\xi) - e_j + e_1$ and $\eta^{ji} = (\bar{1}_\ell,\xi) + e_j - e_1$. Note also that the entry $\eta_j > 1$ is an entry of ξ. We show inequality (5.1) using the following terms. First

$$\begin{aligned}
V_K^+(\bar{1}_\ell,\xi) &= V_K^+(\bar{0}_\ell,\xi-\bar{1}) + W_K(N,1) \\
&= V_{K-\ell+1}^+(0,\xi-\bar{1}) + W_K(N,1)\,,
\end{aligned} \tag{6.5}$$

where we have intentionally kept one zero entry resulting in the dimension $K-\ell+1$. This equals

$$V_K^+(\bar{1}_\ell,\xi) = V_{K-\ell+1}^+(1,\xi) - W_{K-\ell+1}(N-\ell+1,1) + W_K(N,1)\,, \tag{6.6}$$

where we have used the identity $V^+_{K-\ell+1}(1,\xi) = V^+_{K-\ell+1}(0,\xi-\bar{1}) + W_{K-\ell+1}(N-\ell+1,1)$ which follows from (2.1).

Second, with e'_j denoting the $(K-\ell)$-dimensional vector of zeros except for a one at the location where η_j appears in ξ,

(6.7)
$$\begin{aligned}
V^+_K((\bar{1}_\ell,\xi)-e_j+e_1) &= V^+_K(2,\bar{1}_{\ell-1},\xi-e'_j)\\
&= V^+_K(1,\bar{0}_{\ell-1},\xi-e'_j-\bar{1}) + W_K(N,1)\\
&= V^+_{K-\ell+1}(1,\xi-e'_j-\bar{1}) + W_K(N,1)\\
&= V^+_{K-\ell+1}(2,\xi-e'_j) - W_{K-\ell+1}(N-\ell+1,1) + W_K(N,1)\\
&= V^+_{K-\ell+1}((1,\xi)-e''_j+e_1) - W_{K-\ell+1}(N-\ell+1,1) + W_K(N,1),
\end{aligned}$$

where $e''_j = (0,e'_j)$ (the $(K-\ell+1)$-dimensional vector of zeros except for a one at the location where η_j appears in $(1,\xi)$), and the fourth equality follows from the identity

$$V^+_{K-\ell+1}(2,\xi-e'_j) = V^+_{K-\ell+1}(1,\xi-e'_j-\bar{1}) + W_{K-\ell+1}(N-\ell+1,1).$$

Third,

(6.8)
$$\begin{aligned}
V^+_K((\bar{1}_\ell,\xi)+e_j-e_1) &= V^+_K(0,\bar{1}_{\ell-1},\xi+e'_j)\\
&= V^+_{K-1}(\bar{1}_{\ell-1},\xi+e'_j) = V^+_{K-1}(\bar{0}_{\ell-1},\xi+e'_j-\bar{1}) + W_{K-1}(N,1)\\
&= V^+_{K-\ell}(\xi+e'_j-\bar{1}) + W_{K-1}(N,1)\\
&= V^+_{K-\ell}(\xi+e'_j) - W_{K-\ell}(N-\ell+1,1) + W_{K-1}(N,1)\\
&= V^+_{K-\ell+1}(0,\xi+e'_j) - W_{K-\ell}(N-\ell+1,1) + W_{K-1}(N,1)\\
&= V^+_{K-\ell+1}((1,\xi)+e''_j-e_1) - W_{K-\ell}(N-\ell+1,1) + W_{K-1}(N,1)\,,
\end{aligned}$$

where $e''_j = (0,e'_j)$. Notice that we have brought the same terms in terms of $K \to K-\ell+1$. Since the pair $(i,j)=(1,j)$ satisfies $\{\eta_1,\eta_j\} \neq \{\eta_{M1},\eta_{M2}\}$, it follows that the two entries 1 and η_j in $(1,\xi)$ are not the pair consisting of the largest two values in $(1,\xi)$. By the induction hypothesis applied to the configuration $(1,\xi)$ of dimension $K-\ell+1<K$, we have

$$V^+_{K-\ell+1}(1,\xi) - \frac{1}{2}\Big(V^+_{K-\ell+1}((1,\xi)-e''_j+e_1) + V^+_{K-\ell+1}((1,\xi)+e''_j-e_1)\Big) - (1-\eta_j)^2 > 0.$$

It remains to show that the extra terms that appear in (6.6)–(6.8) add up to a nonnegative term, which means that

$$[-W_{K-\ell+1}(N-\ell+1,1)+W_K(N,1)]$$
$$-\frac{1}{2}\Big([-W_{K-\ell+1}(N-\ell+1,1)+W_K(N,1)]+[-W_{K-\ell}(N-\ell+1,1)+W_{K-1}(N,1)]\Big)\geq 0\,,$$

or equivalently,

$$W_K(N,1)-W_{K-1}(N,1)\geq W_{K-\ell+1}(N-\ell+1,1)-W_{K-\ell}(N-\ell+1,1). \tag{6.9}$$

By (3.5), the left and right hand sides of (6.9) both equal $Q(N-K)$. Thus (6.9) holds as an equality.

Case $\ell=1$ and $\eta_j\geq 3$. This treats a subcase of $\eta_r\geq 2$ for all $r>1$. The subcase $\eta_j=2$ is treated in the next subsection. The reason we adopt $\eta_j\geq 3$ is (6.11) where we reduce the configuration by two units. Writing $\eta=(1,\xi)$, note that all entries of ξ are greater than 1. Since $\eta_j\geq 3$, all entries in $\xi-e'_j-\bar{2}$ are nonnegative where e'_j denotes the vector of all entries equal to zero except a one at the location where η_j appears in ξ.

First,

$$\begin{aligned} V_K^+(1,\xi) &= V_K^+(0,\xi-\bar{1})+W_K(N,1)\\ &= V_{K-1}^+(\xi-\bar{1})+W_K(N,1)\,. \end{aligned} \tag{6.10}$$

Second, using $\eta_j\geq 3$,

$$\begin{aligned} V_K^+((1,\xi)-e_j+e_1) &= V_K^+(2,\xi-e'_j)\\ &= V_K^+(0,\xi-e'_j-\bar{2})+W_K(N,2) \quad \text{(by (2.5) with } m=2)\\ &= V_{K-1}^+(\xi-e'_j-\bar{2})+W_K(N,2)\\ &= V_{K-1}^+(\xi-\bar{1}-e'_j)-W_{K-1}(N-K-1,1)+W_K(N,2)\,, \end{aligned} \tag{6.11}$$

where we have used the identity

$$V_{K-1}^+(\xi-\bar{1}-e'_j)=V_{K-1}^+(\xi-e'_j-\bar{2})+W_{K-1}(N-K-1,1).$$

Third,

$$\begin{aligned} V_K^+((1,\xi)+e_j-e_1) &= V_K^+(0,\xi+e'_j)\\ &= V_{K-1}^+(\xi+e'_j)=V_{K-1}^+(\xi-\bar{1}+e'_j)+W_{K-1}(N,1)\,. \end{aligned} \tag{6.12}$$

We need to establish

$$V_K^+(1,\xi) - \frac{1}{2}\Big(V_K^+((1,\xi)-e_j+e_1) + V_K^+((1,\xi)+e_j-e_1)\Big) - (\eta_j-1)^2 > 0\,. \tag{6.13}$$

By (6.10)–(6.12), the left hand side of (6.13) equals $A+B$ where

$$A = V_{K-1}^+(\xi-\bar{1}) - \frac{1}{2}\Big(V_{K-1}^+(\xi-\bar{1}+e_j') + V_{K-1}^+(\xi-\bar{1}-e_j')\Big) - (\eta_j-1)^2 \tag{6.14}$$

and

$$B = W_K(N,1) - \frac{1}{2}\Big(-W_{K-1}(N-K-1,1) + W_K(N,2) + W_{K-1}(N,1)\Big). \tag{6.15}$$

By (3.2), (3.7) and (6.15),

$$\begin{aligned}
2B &= 2W_K(N,1) - \Big(-W_{K-1}(N-K-1,1) + W_K(N,1) + W_K(N-K,1) + W_{K-1}(N,1)\Big)\\
&= (W_K(N,1) - W_{K-1}(N,1)) - (W_K(N-K,1) - W_{K-1}(N-K-1,1))\\
&= Q(N-K) - Q(N-K-2) = 2(N-K)^2.
\end{aligned}$$

So we have

$$B = (N-K)^2. \tag{6.16}$$

By (6.14) and (6.16), (6.13) is equivalent to

$$V_{K-1}^+(\xi-\bar{1}) - \frac{1}{2}\Big(V_{K-1}^+(\xi-\bar{1}+e_j') + V_{K-1}^+(\xi-\bar{1}-e_j')\Big) - (\eta_j-1)^2 + (N-K)^2 > 0,$$

which follows from Lemma 2 in Section 7 (and concludes the proof of the most difficult case). Note that $N-K, K-1, \xi-\bar{1}, \eta_j-1$ and e_j' here should be identified, respectively, with N, K, η, η_j and e_j in (7.1) of Lemma 2. Note also that $m(\xi)\geq 2$ implies that $m(\xi-\bar{1})\geq 1$ and $N-K\geq K-1$ as required by Lemma 2.

We are left with the case $\ell=1$ and $\eta_j=2$. Without loss of generality, assume $j=2$. Let ℓ' be the multiplicity of $\eta_2=2$.

Case $\ell=1$, $\eta_j=2$ and $\ell'=1$.

Write $\eta = (1, 2, \xi)$ where all entries in ξ are greater than 2. (Note that ξ cannot be vacuous since $K \geq 3$.) We have by (2.1)

$$\begin{aligned} (6.17) \qquad V_K^+(1,2,\xi) &= V_K^+(0,1,\xi - \bar{1}) + W_K(N,1) \\ &= V_{K-1}^+(1,\xi - \bar{1}) + W_K(N,1) \\ &= V_{K-1}^+(0,\xi - \bar{2}) + W_{K-1}(N-K,1) + W_K(N,1) \\ &= V_{K-2}^+(\xi - \bar{2}) + W_{K-1}(N-K,1) + W_K(N,1), \end{aligned}$$

and by (2.5) (with $m = 3$)

$$\begin{aligned} (6.18) \qquad V_K^+(0,3,\xi) &= V_{K-1}^+(3,\xi) = V_{K-1}^+(0,\xi - \bar{3}) + W_{K-1}(N,3) \\ &= V_{K-2}^+(\xi - \bar{3}) + W_{K-1}(N,3) \\ &= V_{K-2}^+(\xi - \bar{2}) - W_{K-2}(N-2K+1,1) + W_{K-1}(N,3), \end{aligned}$$

where the last line follows from the identity

$$V_{K-2}^+(\xi - \bar{2}) = V_{K-2}^+(\xi - \bar{3}) + W_{K-2}(N-2K+1,1).$$

Since $V_K^+((1,2,\xi) - e_1 + e_2) = V_K^+(0,3,\xi)$ and $V_K^+((1,2,\xi) + e_1 - e_2) = V_K^+(1,2,\xi)$, the inequality (5.1) is equivalent to $V_K^+(1,2,\xi) > V_K^+(0,3,\xi) + 2$. By (3.2), we have

$$W_{K-1}(N,3) = W_{K-1}(N,1) + W_{K-1}(N-K+1,1) + W_{K-1}(N-2K+2,1)\,.$$

By (3.7), (6.17) and (6.18),

$$\begin{aligned} V_K^+(1,2,\xi) - V_K^+(0,3,\xi) =& [W_K(N,1) - W_{K-1}(N,1)] \\ & - [W_{K-1}(N-K+1,1) - W_{K-1}(N-K,1)] \\ & - [W_{K-1}(N-2K+2,1) - W_{K-2}(N-2K+1,1)] \\ =& Q(N-K) - [Q(N-K-1) - Q(N-2K+1)] - Q(N-2K) \\ =& [Q(N-K) - Q(N-K-1)] + [Q(N-2K+1) - Q(N-2K)] \\ \geq& 12 > 2, \end{aligned}$$

since $Q(x) - Q(x-1) = x(x+1) \geq 12$ for $x \geq 3$ and $N - K \geq 3$.

Case $\ell = 1$, $\eta_j = 2$ and $\ell' \geq 2$. Let $\eta = (1, 2, \bar{2}_{\ell'-1}, \xi)$ where ξ is possibly vacuous. We single out one "2" to explicitly carry out the transform corresponding to the pair $(1, 2)$. Note that ξ is of dimension $K - \ell' - 1$ with $|\xi| = N - 2\ell' - 1$. We have by (2.1)

$$
\begin{aligned}
V_K^+(1, 2, \bar{2}_{\ell'-1}, \xi) &= V_K^+(0, 1, \bar{1}_{\ell'-1}, \xi - \bar{1}) + W_K(N, 1) \\
&= V_{K-1}^+(1, \bar{1}_{\ell'-1}, \xi - \bar{1}) + W_K(N, 1) \\
&= V_{K-1}^+(0, \bar{0}_{\ell'-1}, \xi - \bar{2}) + W_{K-1}(N - K, 1) + W_K(N, 1) \\
&= V_{K-\ell'-1}^+(\xi - \bar{2}) + W_{K-1}(N - K, 1) + W_K(N, 1)\,,
\end{aligned} \tag{6.19}
$$

and

$$
\begin{aligned}
V_K^+(0, 3, \bar{2}_{\ell'-1}, \xi) &= V_{K-1}^+(3, \bar{2}_{\ell'-1}, \xi) \\
&= V_{K-1}^+(1, \bar{0}_{\ell'-1}, \xi - \bar{2}) + W_{K-1}(N, 2) \\
&= V_{K-\ell'}^+(1, \xi - \bar{2}) + W_{K-1}(N, 2) \\
&= V_{K-\ell'}^+(0, \xi - \bar{3}) + W_{K-\ell'}(N - 2K + 2, 1) + W_{K-1}(N, 2) \\
&= V_{K-\ell'-1}^+(\xi - \bar{3}) + W_{K-\ell'}(N - 2K + 2, 1) + W_{K-1}(N, 2) \\
&= V_{K-\ell'-1}^+(\xi - \bar{2}) - W_{K-\ell'-1}(N - 2K + 1, 1) + W_{K-\ell'}(N - 2K + 2, 1) + W_{K-1}(N, 2),
\end{aligned} \tag{6.20}
$$

where the last equality follows from the identity

$$
V_{K-\ell'-1}^+(\xi - \bar{2}) = V_{K-\ell'-1}^+(\xi - \bar{3}) + W_{K-\ell'-1}(N - 2K + 1, 1).
$$

(Note that if ξ is vacuous, $V_K^+(1, 2, \bar{2}_{\ell'-1}, \xi)$ and $V_K^+(0, 3, \bar{2}_{\ell'-1}, \xi)$ reduce to $W_{K-1}(N - K, 1) + W_K(N, 1)$ and $W_{K-1}(N, 2)$, respectively.) As in the preceding case, the inequality (5.1) is equivalent to

$$
V_K^+(1, 2, \bar{2}_{\ell'-1}, \xi) > V_K(0, 3, \bar{2}_{\ell'-1}, \xi) + 2. \tag{6.21}
$$

Noting by (3.2) that $W_{K-1}(N,2) = W_{K-1}(N,1) + W_{K-1}(N-K+1,1)$, we have by (6.19) and (6.20) that

$$
\begin{aligned}
&V_K^+(1,2,\bar{2}_{\ell'-1},\xi) - V_K^+(0,3,\bar{2}_{\ell'-1},\xi) = \\
&\qquad\qquad = [W_K(N,1) - W_{K-1}(N,1)] + [W_{K-1}(N-K,1) - W_{K-1}(N-K+1,1)] \\
&\qquad\qquad\quad - [W_{K-\ell'}(N-2K+2,1) - W_{K-\ell'-1}(N-2K+1,1)] \\
&\qquad\qquad = Q(N-K) + [Q(N-2K+1) - Q(N-K-1)] - Q(N-2K) \quad \text{by (3.7)} \\
&\qquad\qquad \geq Q(N-K) - Q(N-K-1) = (N-K)(N-K+1) \geq 6, \quad \text{since } N-K \geq 2,
\end{aligned}
$$

establishing (6.21). This completes the proof.

7. Lemma 2

In this section we have the same notation $V_K^+(\eta)$ for the value function under S_+ in dimension K as in Section 6.

Lemma 2. *For $N \geq K \geq 2$, for configuration η with $m(\eta) \geq 1$ and $N = |\eta|$,*

$$
V_K^+(\eta) - \frac{1}{2}\Big(V_K^+(\eta+e_j) + V_K^+(\eta-e_j)\Big) - \eta_j^2 + N^2 > 0\,. \tag{7.1}
$$

Proof. We first consider the special case $N = K \geq 2$ and $\eta = \bar{1}_K$. By (3.7),

$$
\begin{aligned}
&2V_K^+(\eta) - V_K^+(\eta+e_j) - V_K^+(\eta-e_j) = \\
&\qquad\qquad = 2W_K(K,1) - W_K(K+1,1) - W_{K-1}(K-1,1) \\
&\qquad\qquad = [W_K(K,1) - W_{K-1}(K-1,1)] - [W_K(K+1,1) - W_K(K,1)] \\
&\qquad\qquad = Q(K-2) - Q(K-1) = -K(K-1) > 2(1-K^2) = 2(\eta_j^2 - N^2),
\end{aligned}
$$

establishing (7.1) for this case. Next for the case $K = 2$, note by (1.7) that $V_K^+(\eta_1,\eta_2) = U(\eta_1,\eta_2)$ where $U(x,y) = (1/3)xy(x^2+y^2-2)$ as defined in (4.3). Then for $N \geq K = 2$, the left hand side of (7.1) (with $j = 1$) equals

$$
\begin{aligned}
&V_K^+(\eta) - \frac{1}{2}\Big(V_K^+(\eta+e_1) + V_K^+(\eta-e_1)\Big) - \eta_1^2 + N^2 \\
&\quad = U(\eta_1,\eta_2) - \frac{1}{2}\Big(U(\eta_1+1,\eta_2) + U(\eta_1-1,\eta_2)\Big) - \eta_1^2 + (\eta_1+\eta_2)^2 \\
&\quad = -\eta_1\eta_2 - \eta_1^2 + (\eta_1+\eta_2)^2 = \eta_1\eta_2 + \eta_2^2 > 0,
\end{aligned}
$$

establishing (7.1). (The case with $j = 2$ is done by symmetry.) Thus we have shown that (7.1) holds for all $2 \le K \le N$ with either $K = 2$ or $N = K$.

To prove the general case with $N > K \ge 3$, in similar fashion as in the proof of Proposition 7, we do induction on K, and for fixed K, induction on N. Specifically, with $3 \le K < N$ fixed, suppose (7.1) holds for all $2 \le K' \le N'$ with either $K' < K$ or $K' = K$ and $N' < N$. Then we need to prove that (7.1) holds for (K, N).

Case $m = m(\eta) \ge 2$ **or** $m = 1 < \eta_j$**.** Like in (5.2)–(5.4), if $m \ge 2$ or $m = 1 < \eta_j$, we may apply the induction step immediately for $\eta - \bar{1}$ for which K' (the number of nonzero entries in $\eta - \bar{1}$) is at most K and $N' := |\eta - \bar{1}| = N - K < N$. Clearly, $N' \ge K' \ge 1$. In the special case $K' = 1$ (arising when $m = 1 < \eta_j$ and $\eta_i = 1$ for all $i \ne j$) for which the induction hypothesis does not apply, we note that $\eta, \eta + e_j$ and $\eta - e_j$ are respectively the extremal configurations $\zeta_{1,K,N}, \zeta_{1,K,N+1}$ and $\zeta_{1,K,N-1}$, so that

$$V_K^+(\eta) = W_K(N, 1),\ V_K^+(\eta + e_j) = W_K(N + 1, 1),\ V_K^+(\eta - e_j) = W_K(N - 1, 1)\,.$$

The left hand side of (7.1) equals $C + D$ where

$$\begin{aligned} C &:= -\eta_j^2 + N^2 = -(N - K + 1)^2 + N^2 \quad \text{(since } \eta_j = N - K + 1\text{)}, \\ D &:= W_K(N, 1) - \frac{1}{2}\Big(W_K(N + 1, 1) + W_K(N - 1, 1)\Big) \qquad (7.2) \\ &= \frac{1}{2}\Big([W_K(N, 1) - W_K(N - 1, 1)] - [W_K(N + 1, 1) - W_K(N, 1)]\Big) \\ &= \frac{1}{2}\Big([Q(N - 2) - Q(N - K - 1)] - [Q(N - 1) - Q(N - K)]\Big) \quad \text{(by (3.7))} \\ &= \frac{1}{2}\Big([Q(N - K) - Q(N - K - 1)] - [Q(N - 1) - Q(N - 2)]\Big) \\ &= \frac{1}{2}(N - K)(N - K + 1) - \frac{1}{2}N(N - 1). \qquad (7.3) \end{aligned}$$

It follows that $C + D = g(N) - g(N - K + 1) > 0$, where

$$g(x) := x^2 - \frac{1}{2}x(x - 1), \quad \text{an increasing function in } x > 0. \qquad (7.4)$$

This establishes (7.1) for $K' = 1$.

We now consider $N' \ge K' \ge 2$. Let η' denote the K'-dimensional vector derived from $\eta - \bar{1}$ by deleting all zero entries, and let e_j' be the K'-dimensional vector of zeros except for a one at the location where $\eta_j - 1$ appears in η'. Then $m(\eta') \ge 1$ and $|\eta'| = N - K = N' \ge K' \ge 2$. By the induction hypothesis applied to η' (for which either $2 \le K' < K$ or $K' = K$ and

$N' < N$), we have

$$\begin{aligned}
\text{(7.5)} \qquad \alpha := {} & V_K^+(\eta - \bar{1}) - \frac{1}{2}\Big(V_K^+(\eta - \bar{1} + e_j) + V_K^+(\eta - \bar{1} - e_j)\Big) \\
= {} & V_{K'}^+(\eta') - \frac{1}{2}\Big(V_{K'}^+(\eta' + e_j') + V_{K'}^+(\eta' - e_j')\Big) > (\eta_j - 1)^2 - (N-K)^2.
\end{aligned}$$

Since by (2.1)

$$\begin{aligned}
V_K^+(\eta) &= V_K^+(\eta - \bar{1}) + W_K(N,1) \\
V_K^+(\eta + e_j) &= V_K^+(\eta - \bar{1} + e_j) + W_K(N+1,1) \\
V^+(\eta - e_j) &= V_K^+(\eta - \bar{1} - e_j) + W_K(N-1,1)\,,
\end{aligned}$$

the left hand side of (7.1) equals

$$\begin{aligned}
&\alpha + W_K(N,1) - \frac{1}{2}\Big(W_K(N+1,1) + W_K(N-1,1)\Big) - \eta_j^2 + N^2 \\
&\quad > \Big((\eta_j - 1)^2 - (N-K)^2\Big) + D - \eta_j^2 + N^2 \quad \text{(by (7.2) and (7.5))} \\
&\quad = D + N^2 - (N-K)^2 - 2\eta_j + 1 \\
&\quad \geq D + N^2 - (N-K)^2 - 2(N-K+1) + 1 \quad (\text{since } \eta_j \leq N - K + 1) \\
&\quad = \frac{1}{2}(N-K)(N-K+1) - \frac{1}{2}N(N-1) + N^2 - (N-K)^2 - 2(N-K+1) + 1 \quad \text{(by (7.3))} \\
&\quad = \frac{1}{2}(N-K)(N-K+1) - \frac{1}{2}N(N-1) + N^2 - (N-K+1)^2 \\
&\quad = g(N) - g(N-K+1) > 0 \quad \text{(by (7.4))}.
\end{aligned}$$

This completes the proof for the case $m \geq 2$ or $m = 1 < \eta_j$.

Case $m = 1 = \eta_j$**.** Let ℓ be the multiplicity of the minimum value 1. Without loss of generality, assume $j = 1$. Write $\eta = (\bar{1}_\ell, \xi) = (1, \bar{1}_{\ell-1}, \xi)$ where ξ is a vector of dimension $K - \ell$ with $m(\xi) \geq 2$. Note that ξ cannot be vacuous (since $N > K$), so $K - \ell > 0$.

By (2.1),

$$
\begin{aligned}
V_K^+(1,\bar{1}_{\ell-1},\xi) &= V_K^+(0,\bar{0}_{\ell-1},\xi-\bar{1}) + W_K(N,1) \\
&= V_{K-\ell}^+(\xi-\bar{1}) + W_K(N,1)\,, \qquad (7.6)\\
V_K^+(0,\bar{1}_{\ell-1},\xi) &= V_{K-1}^+(\bar{1}_{\ell-1},\xi) \\
&= V_{K-1}^+(\bar{0}_{\ell-1},\xi-\bar{1}) + W_{K-1}(N-1,1) \\
&= V_{K-\ell}^+(\xi-\bar{1}) + W_{K-1}(N-1,1)\,, \qquad (7.7)\\
V_K^+(2,\bar{1}_{\ell-1},\xi) &= V_K^+(1,\bar{0}_{\ell-1},\xi-\bar{1}) + W_K(N+1,1) \\
&= V_{K-\ell+1}^+(1,\xi-\bar{1}) + W_K(N+1,1) \\
&= V_{K-\ell+1}^+(0,\xi-\bar{2}) + W_{K-\ell+1}(N-K+1,1) + W_K(N+1,1) \\
&= V_{K-\ell}^+(\xi-\bar{2}) + W_{K-\ell+1}(N-K+1,1) + W_K(N+1,1) \\
&= V_{K-\ell}^+(\xi-\bar{1}) - W_{K-\ell}(N-K,1) + W_{K-\ell+1}(N-K+1,1) + W_K(N+1,1)\,, \qquad (7.8)
\end{aligned}
$$

where the last equality follows from the identity

$$V_{K-\ell}^+(\xi-\bar{1}) = V_{K-\ell}^+(\xi-\bar{2}) + W_{K-\ell}(N-K,1).$$

(Note by (3.8) that if $K-\ell=1$, we have $V_{K-\ell}^+(\xi-\bar{2}) = V_{K-\ell}^+(\xi-\bar{1}) = W_{K-\ell}(N-K,1) = 0$.) By (7.6)–(7.8),

$$
\begin{aligned}
(7.9)\qquad \beta :&= V_K^+(1,\bar{1}_{\ell-1},\xi) - \frac{1}{2}\Big(V_K^+(0,\bar{1}_{\ell-1},\xi) + V_K^+(2,\bar{1}_{\ell-1},\xi)\Big) \\
&= W_K(N,1) - \frac{1}{2}\Big(W_{K-1}(N-1,1) - W_{K-\ell}(N-K,1) \\
&\qquad + W_{K-\ell+1}(N-K+1,1) + W_K(N+1,1)\Big) \\
&= D + \frac{1}{2}\Big(W_K(N-1,1) - W_{K-1}(N-1,1)\Big) \\
&\qquad - \frac{1}{2}\Big(W_{K-\ell+1}(N-K+1,1) - W_{K-\ell}(N-K,1)\Big) \quad \text{(by (7.2))} \\
&= D + \frac{1}{2}Q(N-K-1) - \frac{1}{2}Q(N-K-1) = D.
\end{aligned}
$$

Thus, the left hand side of (7.1) equals

$$\begin{aligned}
\beta - \eta_1^2 + N^2 &= D - 1 + N^2 \\
&\geq D - (N-K+1)^2 + N^2 \\
&= \frac{1}{2}(N-K)(N-K+1) - \frac{1}{2}N(N-1) - (N-K+1)^2 + N^2 \quad \text{(by (7.3))} \\
&= g(N) - g(N-K+1) > 0 \quad \text{(by (7.4))}.
\end{aligned}$$

This concludes the proof of the lemma.

□

8. The constrained case

In this section we shall consider only strategies where once a pair is chosen, it must play until one of the two players is defeated and eliminated from the game (whose fortune reaches zero). After that, another pair is chosen and the game continues with the new pair until one is defeated, and so on, until all but one are defeated, at time T. We are once again interested in maximizing $Var(T)$, the variance of the duration of the game.

Recall (1.7) for the variance of the duration of the game with only two players. Let $U(x,y) := \frac{1}{3}xy(x^2+y^2-2)$ (cf. (4.3)).

We shall denote by S^* the "constrained maximal" strategy, given by choosing the pair with the largest two values, letting them play, and continuing with the new pair with largest values at the end of the round between the first pair and so on. Let $V_K^*(\eta)$ be the corresponding value function. We shall prove that S^* is optimal and unique to achieve the maximum variance.

Let $\eta = (\eta_1, \eta_2, \ldots, \eta_K)$ be an initial configuration, where K is the number of players at $t=0$. Plainly, $V_K^*(\eta)$ is invariant with respect to permutations of η. With the players' fortunes arranged in descending order $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_K$, strategy S^* first picks the pair $(1,2)$ for the first round and then pick the winner (survivor of the first round between the pair $(1,2)$) together with player 3 for the second round and so on. It follows easily that

$$V_K^*(\eta) = V_\ell^*(\eta_1,\ldots,\eta_\ell) + V_{K-\ell+1}^*(\eta_1+\ldots+\eta_\ell,\eta_{\ell+1},\ldots,\eta_K)\,,\quad 2\le \ell\le K-1,$$

$$\begin{aligned} V_K^*(\eta) &= U(\eta_1,\eta_2) + V_{K-1}^*(\eta_1+\eta_2,\eta_3,\ldots,\eta_K) \\ &= U(\eta_1,\eta_2) + U(\eta_1+\eta_2,\eta_3) + V_{K-2}^*(\eta_1+\eta_2+\eta_3,\eta_4,\ldots,\eta_K) \\ &= \sum_{r=1}^{K-1} U(\eta_1+\ldots+\eta_r,\eta_{r+1})\,. \end{aligned} \tag{8.1}$$

Proposition 8. *For any pair (i,j) with $\{\eta_i,\eta_j\}\neq\{\eta_{M1},\eta_{M2}\}$,*

$$V_K^*(\eta) > V_{K-1}^*(\eta^{(ij)}) + U(\eta_i,\eta_j)\,, \tag{8.2}$$

where η_{M1} and η_{M2} denote the largest two values in η, and $\eta^{(ij)}$ is the $(K-1)$-dimensional vector which is derived from η by adding an entry $\eta_i+\eta_j$ and deleting two entries η_i and η_j.

Proof. Note that the two sides of (8.2) are equal if $\{\eta_i,\eta_j\}=\{\eta_{M1},\eta_{M2}\}$. The proof of the proposition is done by induction on K. For $K=3$, it is done by direct computation. Specifically, for $\eta=(\eta_1,\eta_2,\eta_3)$ with $\eta_1\ge\eta_2\ge\eta_3$, we have

$$V_{K-1}^*(\eta^{(13)}) = U(\eta_1+\eta_3,\eta_2)\,,\quad \text{and}\quad V_{K-1}^*(\eta^{(23)}) = U(\eta_2+\eta_3,\eta_1)\,.$$

It is readily verified that

$$\begin{aligned} V_K^*(\eta) - [V_{K-1}^*(\eta^{(13)}) + U(\eta_1,\eta_3)] &= \eta_1\eta_1\eta_3(\eta_2-\eta_3) > 0\,,\quad \text{if}\quad \eta_2>\eta_3\,, \\ V_K^*(\eta) - [V_{K-1}^*(\eta^{(23)}) + U(\eta_2,\eta_3)] &= \eta_1\eta_2\eta_3(\eta_1-\eta_3) > 0\,,\quad \text{if}\quad \eta_1>\eta_3\,, \end{aligned}$$

implying that (8.2) holds for $K=3$.

Now suppose (8.2) holds for $K-1$ ($K\ge4$). We need to prove that it holds for K.

Without loss of generality, assume that $\eta_1\ge\eta_2\ge\ldots\ge\eta_K$. Let (i,j) be a pair with

$$i<j \quad\text{and}\quad \{\eta_i,\eta_j\}\neq\{\eta_{M1},\eta_{M2}\}=\{\eta_1,\eta_2\}. \tag{8.3}$$

Necessarily, $j\ge3$.

Case 1. Suppose $\{\eta_i,\eta_j\}\cap\{\eta_1,\eta_2\}\neq\emptyset$. By (8.3), we have $\eta_i\in\{\eta_1,\eta_2\}$ and $\eta_1>\eta_j$. (Recall that $\{\eta_i,\eta_j\}$ is interpreted as a multiset counting multiplicities. It is possible that both η_i and η_j belong to $\{\eta_1,\eta_2\}$ but $\{\eta_i,\eta_j\}\neq\{\eta_1,\eta_2\}$, which arises when $\eta_1>\eta_2=\eta_i=\eta_j$.)

The left hand side of (8.2) satisfies

$$
\begin{aligned}
&V_K^*(\eta_1, \eta_2, \ldots, \eta_K) = V_{K-1}^*(\eta_1 + \eta_2, \eta_3, \ldots, \eta_K) + U(\eta_1, \eta_2) \\
&\geq V_{K-2}^*(\eta_1 + \eta_2 + \eta_j, \eta_3, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_K) + U(\eta_1 + \eta_2, \eta_j) + U(\eta_1, \eta_2)\,, \qquad (8.4)
\end{aligned}
$$

where the equality is by (8.1) and the inequality is true from the induction hypothesis (noting that the inequality becomes an equality if $\eta_1 + \eta_2$ and η_j are the largest two values in $\eta^{(12)} = (\eta_1 + \eta_2, \eta_3, \ldots, \eta_K)$).

Subcase 1(i). Suppose $\{\eta_i, \eta_j\} \cap \{\eta_1, \eta_2\} = \{\eta_1\}$, i.e. $\eta_i = \eta_1$. Without loss of generality, assume $i = 1$. Then the right hand side of (8.2) satisfies

$$
\begin{aligned}
&V_{K-1}^*(\eta^{(1j)}) + U(\eta_1, \eta_j) = V_{K-1}^*(\eta_1 + \eta_j, \eta_2, \eta_3, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_K) + U(\eta_1, \eta_j) \\
&= V_{K-2}^*(\eta_1 + \eta_j + \eta_2, \eta_3, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_K) + U(\eta_1 + \eta_j, \eta_2) + U(\eta_1, \eta_j)\,, \qquad (8.5)
\end{aligned}
$$

where the first equality is just re-stating the definition of $\eta^{(1j)}$ and the second equality uses the definition of the constrained maximal strategy S^* (observing that $\eta_1 + \eta_j$ and η_2 are the largest two values in $\eta^{(1j)}$). By (8.4) and (8.5), to prove (8.2), it suffices to show

$$
U(\eta_1 + \eta_2, \eta_j) + U(\eta_1, \eta_2) > U(\eta_1 + \eta_j, \eta_2) + U(\eta_1, \eta_j)\,. \qquad (8.6)
$$

Letting

$$
\eta' = (\eta'_1, \eta'_2, \eta'_3) := (\eta_1, \eta_2, \eta_j), \qquad (8.7)
$$

the left and right hand sides of (8.6) are, respectively, $V_3^*(\eta')$ and $V_2^*(\eta'^{(13)}) + U(\eta'_1, \eta'_3)$. The inequality (8.6) is true since (8.2) holds for $K = 3$ and $\{\eta'_1, \eta'_3\} \neq \{\eta'_1, \eta'_2\}$.

Subcase 1(ii). Suppose $\{\eta_i, \eta_j\} \cap \{\eta_1, \eta_2\} = \{\eta_2\}$. Then $\eta_i = \eta_2$. Without loss of generality, assume $i = 2$. We further assume that $\eta_1 > \eta_2$ (otherwise, it reduces to the preceding subcase). The right hand side of (8.2) satisfies

$$
\begin{aligned}
&V_{K-1}^*(\eta^{(2j)}) + U(\eta_2, \eta_j) = V_{K-1}^*(\eta_1, \eta_2 + \eta_j, \eta_3, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_K) + U(\eta_2, \eta_j) \\
&= V_{K-2}^*(\eta_1 + \eta_2 + \eta_j, \eta_3, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_K) + U(\eta_2 + \eta_j, \eta_1) + U(\eta_2, \eta_j)\,, \qquad (8.8)
\end{aligned}
$$

where the second equality follows from (8.1) upon observing that η_1 and $\eta_2 + \eta_j$ are the largest two values in $\eta^{(2j)}$. (We remark that we do not need to know which is larger between η_1 and $\eta_2 + \eta_j$.)

By (8.4) and (8.8), to prove (8.2), it suffices to show

$$U(\eta_1+\eta_2,\eta_j)+U(\eta_1,\eta_2)>U(\eta_2+\eta_j,\eta_1)+U(\eta_2,\eta_j)\,.$$

With η' as defined in (8.7), this inequality is stating that

$$V_3^*(\eta')>V_2^*(\eta'^{(23)})+U(\eta_2',\eta_3')\,,$$

which is true as (8.2) holds for $K=3$ and $\{\eta_2',\eta_3'\}\neq\{\eta_1',\eta_2'\}$.

Case 2. Suppose $\{\eta_i,\eta_j\}\cap\{\eta_1,\eta_2\}=\emptyset$. Then $\eta_1\geq\eta_2>\eta_i\geq\eta_j$.

Subcase 2(i). $\eta_i+\eta_j\leq\eta_2$. By (8.1), the left hand side of (8.2) equals

$$V_K^*(\eta)=V_{K-1}^*(\eta^{(12)})+U(\eta_1,\eta_2), \tag{8.9}$$

while the right hand side satisfies

$$V_{K-1}^*(\eta^{(ij)})+U(\eta_i,\eta_j)=V_{K-2}^*(\eta^{(12)(ij)})+U(\eta_1,\eta_2)+U(\eta_i,\eta_j), \tag{8.10}$$

where $\eta^{(12)(ij)}$ denotes the $(K-2)$-dimensional vector which is derived from η by adding two entries $\eta_1+\eta_2$ and $\eta_i+\eta_j$ and deleting four entries η_1,η_2,η_i and η_j. Note that (8.10) follows from (8.1) upon observing that η_1 and η_2 are the largest two values in $\eta^{(ij)}$. By (8.9) and (8.10), (8.2) is equivalent to

$$V_{K-1}^*(\eta^{(12)})>V_{K-2}^*(\eta^{(12)(ij)})+U(\eta_i,\eta_j),$$

which is true by the induction hypothesis applied to the $(K-1)$-dimensional vector $\eta^{(12)}$. (Note that the largest value in $\eta^{(12)}$ is $\eta_1+\eta_2>\eta_i\geq\eta_j$ and that $\eta^{(12)(ij)}$ is the $(K-2)$-dimensional vector which is derived from the $(K-1)$-dimensional vector $\eta^{(12)}$ by adding an entry $\eta_i+\eta_j$ and deleting two entries η_i and η_j.)

Subcase 2(ii). Suppose $\eta_i+\eta_j>\eta_2$. First consider the pair $(i',j')=(1,i)$ for which (8.2) holds since this corresponds to subcase 1(i). So,

$$V_K^*(\eta)>V_{K-1}^*(\eta^{(i'j')})+U(\eta_{i'},\eta_{j'})=V_{K-1}^*(\eta^{(1i)})+U(\eta_1,\eta_i). \tag{8.11}$$

By the induction hypothesis applied to $\eta^{(1i)}$, we have

$$V_{K-1}^*(\eta^{(1i)})>V_{K-2}^*(\eta^{(1ij)})+U(\eta_1+\eta_i,\eta_j), \tag{8.12}$$

where $\eta^{(1ij)}$ is the $(K-2)$-dimensional vector which is derived from η by adding an entry $\eta_1+\eta_i+\eta_j$ and deleting three entries η_1,η_i,η_j (which can also be defined as derived from

$\eta^{(1i)}$ by adding an entry $(\eta_1 + \eta_i) + \eta_j$ and deleting two entries $\eta_1 + \eta_i$ and η_j.) Note that in $\eta^{(1i)}$, $\eta_j < \eta_2$ so η_j is not the second largest value.

Noting that η_1 and $\eta_i + \eta_j$ are the largest two values in $\eta^{(ij)}$ (since $\eta_i + \eta_j > \eta_2$), the right hand side of (8.2) (corresponding to the pair (i, j)) satisfies by (8.1)

$$V^*_{K-1}(\eta^{(ij)}) + U(\eta_i, \eta_j) = V^*_{K-2}(\eta^{(1ij)}) + U(\eta_1, \eta_i + \eta_j) + U(\eta_i, \eta_j). \tag{8.13}$$

By (8.11)–(8.13), to prove (8.2), it suffices to show

$$U(\eta_1 + \eta_i, \eta_j) + U(\eta_1, \eta_i) > U(\eta_1, \eta_i + \eta_j) + U(\eta_i, \eta_j). \tag{8.14}$$

Letting $\eta'' = (\eta''_1, \eta''_2, \eta''_3) := (\eta_1, \eta_i, \eta_j)$, (8.14) is equivalent to

$$V^*_3(\eta'') > V^*_2(\eta''^{(23)}) + U(\eta''_2, \eta''_3),$$

which is true since (8.2) holds for $K = 3$ and $\eta''_1 > \eta''_2 \geq \eta''_3$. □

Theorem 3. *The strategy S^* is optimal and unique to achieve the maximum variance in the constrained case.*

Proof. Inequality (8.2) is the analogue of (1.15) for the constrained strategies. This shows that V^*_K is an upper bound for the value function of the game. Since V^*_K is realized for the strategy S^*, V^*_K is equal to the value function of the game. The optimal strategy is unique since (8.2) is a strict inequality.. □

Remark. Since in the constrained game, a pair (i, j) continues to play until one player is defeated, there is no intrinsic randomness in the game. The control (choice of pairs) is the only variable, and as such, the optimization is a finite dimensional problem unlike the main problem in the paper. The value of the game is an explicit function calculated in $K - 1$ recursive steps, written in terms of $U(x, y)$. Then $V^*_K(\eta)$ is the maximum over sequences of pairs (i, j) of length $K-1$. It is of interest to note that in order to maximize the variance of the duration of the game in both constrained and unconstrained cases, one should always pick a pair of players who have currently the largest fortunes whenever selection of a pair is called for.

9. Concluding remarks

We introduced the "maximal" strategy S_+ which is stationary Markovian, defined by $s_+(\eta) := (i, j)$ where η_i and η_j are the largest two values in η. We showed that S_+

uniquely attains the maximum variance of the duration of the game, i.e. S_+ along with the corresponding value function $V^{S_+}(\eta)$ solves the maximization problem (1.8). However, for the minimization problem (1.9), we have yet to find a strategy that attains the minimum variance of the duration of the game. A natural candidate strategy is the "minimal" strategy $\tilde{S}$ which is stationary Markovian, defined by $\tilde{s}(\eta) = (i, j)$ where η_i and η_j are the smallest two values in η. Unfortunately, this strategy does not yield the minimum variance. As an example, consider the case $\eta = (\eta_1, \eta_2, \eta_3) = (1, 2, 2)$. Under $\tilde{S}$, players 1 and 2 continue to play until one of them has fortune 0 (and is out of the game). It follows that $V^{\tilde{S}}(\eta) = U(1, 2)+U(3, 2) = 24$. (For this special configuration $\eta = (1, 2, 2)$, $\tilde{S}$ happens to be an "admissible" strategy in the constrained case.) On the other hand, consider the strategy that selects players 2 and 3 to play at $t = 1$ and then selects the loser and player 1 to play at $t = 2$. At the end of $t = 2$, only two players survive with fortunes 2 and 3. Thus the variance of the duration of the game under this strategy equals $U(2, 3) = 22 < 24 = V^{\tilde{S}}(\eta)$.

It may be instructive to consider the minimization problem for the constrained case as in Section 8, which is relatively easier to deal with. Again, a natural candidate strategy is to select the two players with smallest fortunes every time when a player is defeated and eliminated from the game. We refer to this strategy as $\tilde{S}'$. While it can be readily shown that $\tilde{S}'$ attains the minimum variance in the constrained case for $K = 3$, the following example demonstrates that $\tilde{S}'$ does not attain the minimum variance in general. For $K = 4$ and a configuration $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$, consider strategies S_1 and S_2 in the constrained case where S_1 (S_2, resp.) selects players 3 and 4 (2 and 3, resp.) to play in the initial round (until one of them is defeated), and then selects players 1 and 2 (1 and 4, resp.) for the second round. At the end of the second round, only two players survive who then play till the end of the game. It can be shown that

$$V^{S_1}(\eta) - V^{S_2}(\eta) = (\eta_1 - \eta_3)(\eta_2 - \eta_4)(\eta_1\eta_3 + \eta_2\eta_4).$$

If $\eta_1 \geq \eta_2 > \eta_3 \geq \eta_4$, then $V^{S_1}(\eta) > V^{S_2}(\eta)$. If, in addition, $\eta_3 + \eta_4 \geq \eta_1$, then $S_1 = \tilde{S}'$, which yields a larger variance than S_2. In fact, in this case S_2 attains the minimum variance in the constrained case. It is also worth noting that

$$V^{S_1}(\eta) - V^{S_3}(\eta) = (\eta_1 - \eta_4)(\eta_2 - \eta_3)(\eta_1\eta_4 + \eta_2\eta_3) > 0,$$

where S_3 is the strategy that selects players 2 and 4 in the initial round and selects players 1 and 3 in the second second. Even in the constrained case, the minimization problem (1.9) does not seem to admit a simple solution.

Finally, we conclude the paper by reviewing some relevant literature. The so-called K-tower problem is concerned with the strategy S_R that at each time t, a pair is chosen at random among all players remaining in the game and the game stops as soon as one player's furtune drops to 0. For $K = 3$, Engel [6] obtained a simple formula for the expected duration with the help of extensive computer calculations, while Stirzaker [12] used martingale theory to drive the formula. Bruss et al. [4] later derived the variance and the probability distribution of the duration for $K = 3$, and also argued convincingly that no simple formula for the expected duration can be expected for $K \geq 4$. Engel [6] and Stirzaker [12] also considered the ruin problem where the game stops when one player wins all, and found the expected duration under S_R for general K (*cf.* (1.6)). Later Ross [10] showed among other things that the expected duration is the same for all strategies. We gave a short proof of this result by constructing a simple martingale (*cf.* (1.2)).

There are other versions of the multi-player gamblers' ruin problem. In particular, the so-called multi-player ante one game consists of K players each with initial (interger-valued) fortune $\eta_i, i = 1, \ldots, K$. At each time $t = 1, 2, \ldots$, each player with positive fortune puts one unit in a pot, which is then won (with equal probability) by one of them. Players whose fortunes drop to 0 are eliminated. Let $T^{(i)}$ be the total time player i stays in the game. (Equivalently, $T^{(i)}$ is the first time when player i's fortune either drops to 0 or reaches the maximum $|\eta|$.) Let $T = \max_i T^{(i)}$, the duration of the game. Let T_j be the total time when exactly j players are in the game. Note that $T = T_K + \ldots + T_2$ and that T_K is equivalent to the first time when at least one player's fortune drops to 0. Martingale theory has been used to derive $E(T)$, $E(T^{(i)})$ and $E(T_j)$ for $K = 3$; see [2, 6, 11]. See also [1] for related results. No simple formulas are available for $K \geq 4$.

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