UNIQUENESS OF THE TAGGED PARTICLE PROCESS IN A SYSTEM WITH LOCAL INTERACTIONS

ILIE GRIGORESCU

Abstract. It has been shown that for a system of Brownian motions with local interaction considered in a diffusive scaling, under some regularity assumptions on the initial profile, the tagged particle process converges to a diffusion. We provide a sufficient condition for granting both the existence and the uniqueness of the tagged particle process for an arbitrary initial profile.

1. **Introduction**

This paper derives a uniqueness result for the tagged particle process associated to a scaled family of Brownian motions with local interactions. The dynamics can be described as a limiting case of the finite-range interaction considered in [2], recast on the circle in a suitable scaling.

The explicit determination of the tagged particle process for this interaction is given in [1]. It is also proved that two particles of distinct labels become independent in the limit. In Theorem 3 of [1] the tightness of the tagged particle process starting at a particular point of the unit circle is demonstrated for a general initial profile while the uniqueness results are proved for a bounded initial density profile $\mu(dx)$. Our goal is to remove any restriction from the initial condition and prove an uniqueness result for an arbitrary $\mu(dx)$. The price one pays for this is the addition of a further hypothesis describing how much mass can be assumed to lie on each "side" of the tagged particle at time $t = 0$, formally described in (2.3). One can

---

Date: February 12, 2002.

1991 *Mathematics Subject Classification.* Primary: 60K35; Secondary: 82C22, 82C05.

Key words and phrases. tagged particle, martingale problem, arbitrary initial profile.

Research supported by the Courant Institute of Mathematical Sciences at New York University.

explain the motion of the tagged particle by considering its underlying free Brownian motion over which an interaction with the environment is superposed. The interacting forces are repulsive and proportional to the amount of mass on each side of the particle. As long as the density profile is non-singular one can easily tell how much mass (i.e. how much repulsion) lies on both sides. Ambiguity arises only when there is positive mass at the specific point of the trajectory. However, we show that the macroscopic profile is smooth at any positive time ($t > 0$) because it verifies the heat equation. The only time when one has to make a description of the way in which mass gathers on each side of the particle is $t = 0$. Once this mass split has been prescribed, uniqueness follows.

1.1. **The Formal Definition.** Consider a positive integer n and $\lambda \geq 0$. Let Γ^n be the n-dimensional torus. We define $F^{ij} = \{ \xi \in \Gamma^n : \xi_i = \xi_j\}$ for any i, j in $\{1, \ldots, n\}$ and $F = \cup_{1 \leq i < j \leq n} F_{ij}$. We shall denote by $\bar{C}(\Gamma^n, F)$ the set of functions f that are piecewise smooth (up to the boundary F) on $\Gamma_n \setminus F$. For such functions we define functions f^{ij} and $D^{ij} f$ on the boundary F by

$$\begin{aligned} f^{ij}(\xi_1,&\xi_2, ..., \xi_{i-1}, \xi, \xi_{i+1}, ..., \xi_{j-1}, \xi, \xi_{j+1}, ..., \xi_n) = \\ &= f(\xi_1, \xi_2, ..., \xi_{i-1}, \xi + 0, \xi_{i+1}, ..., \xi_{j-1}, \xi - 0, \xi_{j+1}, ..., \xi_n) \end{aligned} \tag{1.1}$$

$$\begin{aligned} D^{ij} f(\xi_1, &\xi_2, ..., \xi_{i-1}, \xi, \xi_{i+1}, ..., \xi_{j-1}, \xi, \xi_{j+1}, ..., \xi_n) = \\ &= (\partial_i - \partial_j) f(\xi_1, \xi_2, ..., \xi_{i-1}, \xi + 0, \xi_{i+1}, ..., \xi_{j-1}, \xi - 0, \xi_{j+1}, ..., \xi_n) \end{aligned} \tag{1.2}$$

We are now in a position to define the generator of the process

$$\xi^n(t) = (\xi_1(t), \ldots, \xi_n(t))$$

on Γ^n.

Let $\bar{C}(\Gamma^n, F) = \{ f : \Gamma^n \longrightarrow R : f \in C^2(\Gamma^n \setminus F)$ and $f^{ij}(\xi_0)$, $D^{ij} f(\xi_0)$ are finite for any $\xi_0 \in F$ and any $(i, j)\}$ - called the set of smooth functions up to the boundary F.

For a real $\lambda \geq 0$ we define the boundary conditions:

$$D^{ij} f(\xi) + \lambda(f^{ji}(\xi) - f^{ij}(\xi)) = 0 \quad \forall i, j \in \{1, \ldots, n\} \quad . \tag{1.3}$$

The operator $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ with

$$\mathcal{L}f = \frac{1}{2}\Delta f \tag{1.4}$$

and

$$\mathcal{D}(\mathcal{L}) = \{f \in \bar{C}(\Gamma^n, F) : \text{ s.t. the boundary conditions 1.3 are satisfied}\}$$

is the infinitesimal generator of a process P_λ^n on Γ^n.

1.2. **The Scaled Model.** The considerations made up to this point regard a process P_λ^n for a given n. Let us consider a large positive N and let's blow up the space scale by a factor of N, such that the particles evolve on a circle of radius N instead of 1; in the scaled version we shall look at $\frac{\xi(t)}{N}$. The time scale will also be amplified by N^2 to produce a diffusive scaling ($\frac{\xi^2}{t}$ is invariant, i.e. the Laplacian is preserved).

The number of particles will be scaled to $N\bar{\rho}$ where $\bar{\rho} > 0$ is a fixed constant; physically this implies the average density of the system stays constant.

Let $\bar{\rho} > 0$ and $\lambda > 0$ be fixed constants. We end up with the scaled process defined by (1.4) with $n := N\bar{\rho}$ (n= the # of particles) and $\lambda_N := N\lambda$, denoted by

$$P_\lambda^N = P_{N\lambda}^{N\bar{\rho}} \quad . \tag{1.5}$$

The new process evolves on the $n = N\bar{\rho}$ - dimensional torus Γ^n. Each particle ξ_k, for $k = 1, \dots, n$, performs a Brownian motion on the unit circle until it collides with some other particle, where the given interaction governed by $\lambda_N = N\lambda$ takes place. As explained in [1] the model interpolates between $\lambda = 0$ (the pure reflection case) and $\lambda = \infty$ (the noninteracting case) by switching the labels of the colliding particles according to a Poisson process ran following the local time of collision. The reflected or switched pair proceeds by performing independent Brownian motions until the next collision and so on.

1.3. **The Lifted Process.**

Definition 1. *We shall denote by Ω_{Γ^n} the space of continuous paths from $[0,\infty)$ on the n - dimensional torus Γ^n and by Ω_{R^n} the space of continuous paths from $[0,\infty)$ on R^n.*

Each continuous path on the unit circle can be lifted in a canonical way to a continuous path on the covering space R. The mapping Λ will be the Cartesian product of the n canonical mappings for each component with the given initial condition

$$\xi_k(0) = \xi_k = x_k \in [0,1] \text{ with } k = 1, \dots n \quad .$$

There is an important distinction to make between the process

$$\xi(\cdot) = (\xi_1(\cdot), \dots, \xi_n(\cdot))$$

with state space the n - dimensional torus Γ^n and the lifted process

$$x(\cdot) = (x_1(\cdot), \dots, x_n(\cdot))$$

with state space R^n given by

$$x(\cdot) = \Lambda(\xi(\cdot))$$

constructed with the lift mapping

$$\Lambda : \Omega_{\Gamma^n} \longrightarrow \Omega_{R^n} \tag{1.6}$$

by lifting each component $\xi_1(\cdot), \dots, \xi_n(\cdot)$.
In the following we use the notation

Definition 2. *Let Λ be the lift mapping for $n = N\bar{\rho}$. Then*

$$P^N := P^{\xi^N} \circ \Lambda^{-1} \tag{1.7}$$

and P^N is a measure on the path space $C([0,\infty), R^n)$.

Definition 3. *The process $\{x_1^N(\cdot)\}_{t\geq 0}$ will be called the tagged particle process.*

Remark : For any function $\Phi \in \bar{C}(\Gamma^n, F)$, $\Phi(x) = \Phi(x_1, \ldots, x_n)$ periodic of period 1 in each variable the mappings $t \longrightarrow \Phi(x^N(t))$ can be identified to $t \longrightarrow \Phi(\xi^N(t))$ by taking the image of $x^N(t)$ on Γ^n. Consequently we may always substitute the original $\xi(\cdot)$ process with the lifted process $x(\cdot)$ as long as the test functions are periodic.

1.4. **The Previous Results.** A natural assumption is the existence of an initial density profile.

Hypothesis 1. *We assume that for any* N

$$P^N(\{x_1^N(0) = x_1\}) = 1 \quad ,$$

and there is a measure $\mu(dx)$ *on* Γ^1 *with* $\mu(\Gamma^1) = \bar{\rho}$ *such that*

$$\frac{1}{N} \sum_{k=1}^{n} \delta_{x_k^N(0)} \Longrightarrow \mu(dx) \quad . \tag{1.8}$$

It has been shown in [1] that as long as the dynamics of the entire system of unlabeled particles is concerned, the behavior of the particles is indistinguishable from the unlabeled independent Brownian motions on the torus. As a consequence there exists a hydrodynamical limit of the empirical density

$$\frac{1}{N}\left(\delta_{x_1^N(t)} + \ldots + \delta_{x_n^N(t)}\right) \Longrightarrow \mu(t, dx)$$

as $N \to \infty$. The macroscopic profile $\mu(t, dx) = \rho(t, x)dx$ is the solution to

$$\begin{cases} \rho_t & = \frac{1}{2}\rho_{xx} \\ \lim_{t \to 0} \mu(t, dx) & = \mu(dx) \end{cases} \tag{1.9}$$

in the sense of distributions with $\mu(dx)$ the initial density profile.

Remarks:

(i) In equilibrium the macroscopic density is constant $= \bar{\rho}$.

(ii) For $t > 0$ the macroscopic profile $\rho(t, dx)$ is absolutely continuous and we shall denote the density profile at time "t" by $\mu(t, dx) = \rho(t, x)dx$.

The limiting behavior of the above process is not interesting in itself since it reduces to the simple independent case; however by studying the particular evolution of the tagged particle one can derive a non-trivial result. The next theorem states formally the existence of the hydrodynamical limit under the hypothesis 1.

Theorem 1. *Let $\mu(t,dx) = \rho(t,x)dx$ be the solution to (1.9). For any smooth periodic $J : R \longrightarrow R$ of period 1 and any $t > 0$*

$$\lim_{N\to\infty} E^N \sup_{0\le s\le t} \left| \frac{1}{N} \sum_{k=1}^{n=N\bar{\rho}} J(x_k(s)) - \int_0^1 J(x)\rho(s,x)dx \right|^2 . \tag{1.10}$$

The main result from [1] is the following theorem. It is significant that the initial profile $\mu(dx)$ has a bounded density $\rho_0(x)$. This paper is dedicated to the removal of this assumption.

Theorem 2. *If the Hypothesis 1 is satisfied and the initial profile has a bounded density i.e. there is a bounded positive function $\rho_0(x)$ such that $\mu(dx) = \rho_0(x)dx$ then the sequence of processes $P^N_{\bar{x}} \circ (x_1^N(\cdot))^{-1}$ has a weak limit Q_{x_1} as $N \to \infty$ and Q_{x_1} is the unique solution to the martingale problem given by*

$$\mathcal{L}_t^x = \frac{1}{2}\left(\frac{\lambda}{\lambda+\rho(t,x)}\right)\frac{d^2}{dx^2} - \left(\frac{1}{2}\partial_x\rho(t,x)\frac{2\lambda+\rho(t,x)}{(\lambda+\rho(t,x))^2}\right)\frac{d}{dx} \tag{1.11}$$

starting at $(0,x_1)$.

Definition 4. *Let's denote by $\nu(x)$ the periodic function of period 1 on R such that $\nu(x) = x$ on $[0,1]$. For two points x and y on the circle $\nu(y-x)$ will represent the distance from x to y in positive trigonometric sense.*

Definition 5. *For any N we shall define a the process $z_1^N(\cdot)$ by*

$$z_1^N(t) = x_1^N(t) + \frac{1}{\lambda+\bar{\rho}}\frac{1}{N}\sum_{k\ne 1}\nu(x_k^N(t) - x_1^N(t))$$

for $t \ge 0$.

Theorem 3 (Theorem 6 in [1]) will be used in the proof of the main result of this paper. It provides a connection between the intermediary process $z_1^N(\cdot)$ and $x_1^N(\cdot)$. It is valid for an arbitrary initial profile.

Theorem 3. *If the Hypothesis 1 is satisfied then the families of processes* $\{x_1(\cdot)\}_N$ *and* $\{z_1(\cdot)\}_N$ *are tight. For any limit process* $\{x_1(\cdot)\}_{t\geq 0}$ *of the family of processes* $\{x_1^N(\cdot)\}_{N>0}$ *there is a limit point* $\{z_1(\cdot)\}_{t\geq 0}$ *of the family of processes* $\{z_1^N(\cdot)\}_{N>0}$ *such that if* $Q^{(x_1,z_1)}$ *is the limit point of* $\{P^N \circ \left(x_1^N(\cdot), z_1^N(\cdot)\right)^{-1}\}_N$ *corresponding to their joint distribution then*

1) $(z_1(t) - z_1(0), \mathcal{F}_t)$ *is a continuous martingale with respect to* $Q^{(x_1,z_1)}$ *and*

2)

$$\left([z_1(t) - z_1(0)]^2 - \int_0^t \frac{\lambda(\lambda + \rho(s, x_1(s)))}{(\lambda + \bar{\rho})^2} ds, \mathcal{F}_t\right)$$

is also a $Q^{(x_1,z_1)}$ *- martingale.*

2. **Uniqueness for an Arbitrary Initial Profile**

Up to this point we have seen that the tagged particle process $x_1(\cdot)$ has a unique limit point - a diffusion process - as soon as the initial profile has a bounded density.

The mass at x_1 will be zero in this case. We shall give a more general condition for uniqueness characterized by the repartition of mass at time $t = 0$ around x_1.

An essential step in establishing the asymptotic form of the tagged particle process (in [1]) is to show that for an initial profile $\mu(dx) = \rho_0(x)dx$ (with $\rho_0(x)$ bounded) and any $T > 0$ the mapping

$$\Theta : \Omega \longrightarrow \Omega, \qquad \Omega = C([0, T], R)$$

$$\Theta\omega(\cdot) = \omega(\cdot) + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y - \omega(\cdot))\rho(\cdot, y)dy$$

is well defined (sends continuous paths into continuous paths) and invertible.

The difficulty to prove the uniqueness for an arbitrary initial profile turns out to originate in the ambiguous nature of the limit

$$\lim_{t\to 0} (\Theta\omega)(t) = \lim_{t\to 0} \left[\omega(t) + \frac{1}{\lambda+\bar\rho} \int_{[0,1]} \nu(y-\omega(t))\rho(t,y)dy \right]$$

for paths $\omega(\cdot)$ starting at $\omega(0) = x_1$ as soon as $\mu(\{x_1\}) > 0$. Once we remove this ambiguity we can prove the uniqueness of the limiting process.

For this purpose we assume that besides the initial profile condition

$$\frac{1}{N}\sum_{j=1}^{n} \delta_{x_i} \Rightarrow \mu(dx)$$

(1.8) the process P^N also satisfies (2.3).

Definition 6. *We define two periodic functions of period* 1 *on* R *equal to*

$$\phi_\epsilon(x) = \begin{cases} x, & \text{if } x \in [0, 1-\epsilon] \\ x-1, & \text{if } x \in [1-\frac{\epsilon}{2}, 1] \\ \text{smooth} & \text{on } [0,1) \end{cases} \tag{2.1}$$

on $[0,1)$ *as well as*

$$g_\epsilon(x) = \nu(x) - \phi_\epsilon(x) \quad . \tag{2.2}$$

Hypothesis 2. *If* $\mu(\{x_1\}) := \pi(x_1) > 0$ *we assume that there is a number* $\pi_-(x_1)$ *in the interval* $[0, \pi(x_1)]$ *(clearly* $\pi_-(x_1) = 0$ *if* $\mu(\{x_1\}) = 0$*) such that*

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} E^N \left| \frac{1}{N}\sum_{k\neq 1} g_\epsilon(x_k^N - x_1^N) - \pi_-(x_1) \right| = 0 \quad . \tag{2.3}$$

Then the tagged particle process Q_{x_1} is unique. In other words, to ensure uniqueness we need to know not only the macroscopic mass at the starting point but also to manage making sense of a breakup of this mass in two terms, the left – and right – hand side amount of mass at this same point. The main result is stated in *Theorem 4*. We need a series of definitions before stating the main result.

Definition 7. *Let*

$$F(t,x) = x + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y-x)\rho(t,y)dy \tag{2.4}$$

for $t > 0$ and the constant

$$z_1 := x_1 + \frac{1}{\lambda + \bar{\rho}} \int_{[0,1]\setminus\{x_1\}} \nu(y - x_1)\mu(dx) + \frac{1}{\lambda + \bar{\rho}} \pi_-(x_1) \quad . \tag{2.5}$$

Proposition 1. *The function $F(t,x)$ is in $C^\infty((0,\infty), R)$ and for any fixed $t > 0$*

$$x \to F(t,x)$$

is a strictly non-decreasing function with

$$0 < \frac{\lambda}{\lambda + \bar{\rho}} \le \partial_x F(t,x) \quad .$$

Proof: It is clear that $\rho(t,x) = \int_0^1 \rho_0(x-y)p(t,y)dy$ where $p(t,x)$ is the fundamental solution to the heat equation $\partial_t \rho = \frac{1}{2}\partial_{xx}\rho$ and as such the smoothness is established. The contents of this proposition is the calculation of $\partial_x F(t,x)$.

$$\partial_x \left(x + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y-x)\rho(t,y)dy \right) =$$

$$= 1 + \frac{1}{\lambda + \bar{\rho}} \partial_x \left(\int_0^1 \nu(y-x)\rho(t,y)dy \right) \quad .$$

We look at the derivative of the integral

$$\partial_x \left(\int_0^1 \nu(y-x)\rho(t,y)dy \right) =$$

$$= \partial_x \int_0^1 \nu(y)\rho(t,y+x)dy = \int_0^1 \nu(y)\partial_x\rho(t,y+x)dy =$$

(since the functions have period 1)

$$= \nu(y)\rho(t,y+x)\Big|_0^1 - \int_0^1 \rho(t,y+x)dy = \rho(t,1+x) - \bar{\rho} = \rho(t,x) - \bar{\rho} \quad .$$

Definition 8. *Proposition 1 allows us to define the inverse function for $F(t,\cdot)$ for any $t > 0$:*

$$G(t,\cdot) := (F(t,\cdot))^{-1} \quad .$$

Definition 9. *Let's denote*

$$\Omega := C([0,T],R) \quad ,$$

$$\Omega_{x_1} = \{\omega : \omega \in C((0,T],R) \ \textit{and} \ \lim_{t\to 0}\omega(t) = x_1\} \quad ,$$

$$\Omega_{z_1} = \{\eta : \eta \in C((0,T],R) \ \textit{and} \ \lim_{t\to 0}\eta(t) = z_1\}$$

and

$$\Omega^* = \{f : [0,T] \longrightarrow R \,|\, f \in C((0,T],R)\} \quad .$$

Moreover we shall need a map on the path space $\Theta : \Omega^* \longrightarrow \Omega^*$ *defined by*

$$(\Theta\omega)(t) = F(t,\omega(t))$$

for $t > 0$ *and*

$$(\Theta\omega)(0) = z_1$$

for $t = 0$.

It will also be convenient to denote

$$\Theta_t(\cdot) := F(t,\cdot) \quad .$$

We naturally have the inverse map $\Theta^{-1} : \Omega^* \longrightarrow \Omega^*$ *, equal to*

$$\left(\Theta^{-1}\eta\right)(t) = (\Theta_t^{-1})(\eta(t))$$

for $t > 0$ *and*

$$\left(\Theta^{-1}\eta\right)(0) = x_1$$

for $t = 0$.

Theorem 4. *Under the Hypothesis 1 and the Hypothesis 2 we have*

1) If $\mu(dx)$ *is continuous at* x_1*, i.e.* $\mu(\{x_1\}) = 0$ *then the tagged particle process starting at* x_1 *is unique.*

2) If $\mu(\{x_1\}) = \pi(x_1) > 0$ *then the family of limit points of the tight family of processes* $\{P^N \circ (x_1^N(\cdot))^{-1}\}_N$ *is infinite and there is a unique such limit for each value of* $\pi_-(x_1) \in [0,\pi(x_1)]$ *denoted for simplicity by* Q_{x_1}.

3) Q_{x_1} *can be characterized as the unique measure on* Ω_{x_1} *with the properties* *(i) For any smooth $f(t,x)$ with supp$(f) \subseteq (0,\infty)\times R$ the expression*

$$f(t,x_1(t)) - f(0,x_1(0)) - \int_0^t (\partial_u f + \mathcal{L}_u^x f)(u,x_1(u))du$$

is a $(Q_{x_1}, \{\mathcal{F}_t\}_{t\geq 0})$ - martingale and

(ii) $\quad Q_{x_1}(\{\omega \in \Omega_{x_1} \,:\, \Theta\omega \in \Omega_{z_1}\}) = 1.$

The next section is dedicated to the proof of this uniqueness result.

3. **The Proof of Theorem 4**

Lemma 1. *Suppose we have a positive measurable function $a : [0,\infty)\times R \longrightarrow R$, smooth for $t > 0$. Also assume that the martingale problem associated to $\mathcal{L}_t = \frac{1}{2}a(t,x)\frac{d^2}{dx^2}$ is well-posed for any (t,x) with $t > 0$.*

Let $\Omega = C([0,\infty), R)$ be the space of continuous paths $x(\cdot)$ and let $\mathcal{F}_t = \sigma(\{x(s) : s \in [0,t]\})$ for any $t \geq 0$. If there exist two probability measures Q^1 and Q^2 on Ω satisfying the Markov property and solving the martingale problem for the operator $\mathcal{L}_t$ starting at $x(0) = x \in R$ i.e. $Q^i(\{x(0) = x\}) = 1$ for $i = 1,2$ then $Q^1 = Q^2$.

Proof: For a fixed $T > 0$ and a given $f(x) \in C_0^\infty(R)$ we set $u(t,x)$ the solution to the equation

$$u_t + \mathcal{L}_t u = 0 \tag{3.1}$$

with the condition

$$u(T,x) = f(x) \quad .$$

Since the martingale problem starting at any positive time is well-posed (the solution is unique) and the two measures solve the martingale problem starting at $x(0) = x$ we get that

$$u(t,x) = E^{Q^1}[f(x(T-t)] = E^{Q^2}[f(x(T-t)]$$

naturally well-defined on $t \in (0,T]$.

1) The function $u(t,\cdot)$ is Lipschitz and the Lipschitz constant is independent of the time $t \in (0,T]$.

Set $v(t,x) = u_x(t,x)$ and verify that v satisfies the equation

$$v_t + \frac{1}{2} a v_{xx} + \frac{1}{2} a_x v_x = 0$$

by differentiating (3.1). This equation satisfies the maximum principle and so

$$\sup_{(t,x)} |v(t,x)| \leq \sup_x |f'(x)| =: c_f$$

which proves our statement.

2) For any measure Q satisfying the Markov property with

$$Q(\{x(0) = x\}) = 1$$

and for any $t > 0$

$$E^Q[f(x(T))] = E^Q[E^Q[f(x(T))|\mathcal{F}_t]] = E^Q[E^{x(t)}[f(x(T-t)]] =$$

$$= \int E^y[f(x(T-t)]\, Q \circ x(t)^{-1}(dy) = \int u(t,y)\, Q \circ x(t)^{-1}(dy) =$$

$$= E^Q[u(t,x(t))]$$

so

$$E^{Q^i}[f(x(T))] = E^{Q^i}\left[E^{Q^i}[f(x(T))|\mathcal{F}_t]\right] =$$

$$= E^{Q^i}[u(t,x(t))] \quad , \quad i = 1,2 \quad .$$

Let's bring in a positive number η and set

$$G_\eta = \{ \sup_{s \in [0,t]} |x(s) - x| \leq \eta \}$$

and

$$g_\eta^x = \mathbf{1}_{G_\eta} \quad .$$

3) We shall couple the two processes Q^1 and Q^2 and pretend that $y(t)$ is an independent identically distributed copy of $x(t)$. For any $t \in (0,T]$

$$|E^{Q^1}[f(x(T))] - E^{Q^2}[f(y(T))]| = |E^{Q^1 \otimes Q^2}[u(t,x(t))] - E^{Q^1 \otimes Q^2}[u(t,y(t))]|$$

$$\leq |E^{Q^1\otimes Q^2}[u(t,x(t))g^x_\eta g^y_\eta] - E^{Q^1\otimes Q^2}[u(t,y(t))g^x_\eta g^y_\eta]|+$$

$$+|E^{Q^1\otimes Q^2}[u(t,x(t))(1-g^x_\eta g^y_\eta)] - E^{Q^1\otimes Q^2}[u(t,y(t))(1-g^x_\eta g^y_\eta)]| \quad .$$

It is clear that the last two terms are bounded by $2\cdot\sup|f(x)|\cdot E^{Q^1}[1-g^x_\eta]$ which converges to 0 as $t\to 0$.

$$|E^{Q^1\otimes Q^2}[u(t,x(t))g^x_\eta g^y_\eta] - E^{Q^1\otimes Q^2}[u(t,y(t))g^x_\eta g^y_\eta]| \leq$$

$$\leq E^{Q^1\otimes Q^2}|u(t,x(t)) - u(t,y(t))|\cdot|g^x_\eta g^y_\eta| \leq$$

$$\leq c_f E^{Q^1\otimes Q^2}[|x(t)-y(t)|\cdot|g^x_\eta g^y_\eta|] \leq$$

$$\leq c_f(E^{Q^1}[|x(t)-x|\cdot|g^x_\eta|] + E^{Q^2}[|y(t)-x|\cdot|g^y_\eta|]) \leq 2c_f\eta \quad .$$

We have shown that

$$|E^{Q^1}[f(x(T)] - E^{Q^2}[f(y(T)]| \leq 2c_f\eta$$

for any $\eta>0$ and any $T>0$. The Markov property implies that $Q^1 = Q^2$. □

The following result (Theorem 7 in [1]) states the existence and uniqueness of the limit for the transformed process $z_1(\cdot)$.

Proposition 2. *Under the Hypothesis 1 and 2 the sequence of martingales* $\{z_1^N(\cdot)\}_N$ *(definition 5) is tight and converges weakly to a process* $z_1(\cdot)$ *with law denoted by* P_{z_1} *solving the martingale problem*

$$\mathcal{L}^z_t = \frac{1}{2}\frac{\lambda(\lambda+\rho(t,G(t,z)))}{(\lambda+\bar\rho)^2}\cdot\frac{d^2}{dz^2} \tag{3.2}$$

starting at z_1 *i.e. with*

$$P_{z_1}(\{z_1(0) = z_1\}) = 1 \quad .$$

Proof:

1. The tightness. We know from [1] (Proposition 3 in [1]) that $z_1^N(\cdot)$ is a martingale and by Doob's inequality we can show that for any $\eta > 0$

$$\lim_{\delta\to 0}\limsup_{N\to\infty} P^N(\{\sup_{t,s\in[0,\delta]} |z_1^N(t) - z_1^N(s)| > \eta\}) = 0 \quad .$$

At time $t = 0$ the martingale $z_1^N(\cdot)$ is the sum of the average of the distances between particles (bounded by one) and $x_1^N(0) = x_1$ almost surely.

2. The process starts at z_1 a.s. We now have to show that for any $\eta > 0$

$$\limsup_{N\to\infty} P^N(\{|z_1^N(0) - z_1| > \eta\}) = 0 \quad .$$

Let's bring in the function ϕ_ϵ and $g_\epsilon = \nu - \phi_\epsilon$ (2.2). We shall break the element $z_1^N(0)$ in two parts

$$F_N^\epsilon(0, x_1^N(0)) = x_1^N(0) + \frac{1}{\lambda+\bar\rho}\frac{1}{N}\sum_{k\neq 1}\phi_\epsilon(x_k^N(0) - x_1^N(0))$$

and

$$\frac{1}{\lambda+\bar\rho} G_N^\epsilon(0, x_1^N(0)) = \frac{1}{\lambda+\bar\rho}\frac{1}{N}\sum_{k\neq 1} g_\epsilon(x_k^N(0) - x_1^N(0)) \quad .$$

It is enough to show that

$$(a)\ \lim_{\epsilon\to 0}\limsup_{N\to\infty} E^N \left| F_N^\epsilon(0, x_1^N(0)) - \left[x_1 + \frac{1}{\lambda+\bar\rho}\int_{[0,1]\setminus\{x_1\}} \nu(y - x_1)\mu(dy)\right]\right| = 0$$

and

$$(b) \qquad \lim_{\epsilon\to 0}\limsup_{N\to\infty} E^N \left| G_N^\epsilon(0, x_1^N(0)) - \pi_-(x_1)\right| = 0 \quad .$$

We shall break up (a) into

$$(a1)\ \left| F_N^\epsilon(0, x_1^N(0)) - \left[x_1(0) + \frac{1}{\lambda+\bar\rho}\int_0^1 \phi_\epsilon(y - x_1^N(0))\mu(dy)\right]\right|$$

and

$$(a2)\ \frac{1}{\lambda+\bar\rho}\left| \int_0^1 \phi_\epsilon(y - x_1^N(0))\mu(dy) - \int_{[0,1]\setminus\{x_1\}} \nu(y - x_1)\mu(dy)\right| \quad .$$

$(a1)$ has limit 0 as $N\to\infty$ as long as ϕ_ϵ is smooth. This proves the iterated limit for $(a1)$.

We recall that $P^N(\{x_1^N(0) = x_1\}) = 1$ by hypothesis.

Hence $\lim_{\epsilon\to 0}[(a2)] = 0$ is a consequence of the dominated convergence theorem on $\Gamma^1 \setminus \{x_1\}$ while (b) is exactly (2.3).

3. $z_1(\cdot)$ is the solution to the martingale problem (3.2) starting at z_1. Let

$$\mathcal{M}^z_f(t) := f(z_1(t)) - f(z_1(0)) - \int_0^t \mathcal{L}^z_u f(u, z_1(u))du \quad .$$

The integral always makes sense because $\rho(u,x)$ is of order $\frac{1}{\sqrt{s}}$.

For a positive $s > 0$ and a smooth function $f(z)$ we have no problem to show that

$$\mathcal{M}^z_f(t) - \mathcal{M}^z_f(s)$$

is a $(P_{z_1}, \mathcal{F}_{t\geq s})$ - martingale. This is because as long as $s > 0$ the coefficients of the generator are smooth and bounded. *Theorem 3* implies that for $s > 0$ the process $\{z_1(t)\}_{t\geq s}$ is a diffusion with coefficient

$$\sigma^2(t,z) = \frac{\lambda(\lambda + \rho(t, G(t,z)))}{(\lambda + \bar{\rho})^2} \quad .$$

We only have to show that

$$E^{P_{z_1}}\left[\mathcal{M}^z_f(s)\right] \to 0$$

as $s \to 0$. The term $f(z_1(s)) - f(z_1(0)) \to 0$ because the measure P_{z_1} is concentrated on the continuous paths starting at $z_1(0) = z_1$.
The integral also tends to zero because (again)

$$\rho(s,x) \leq const \cdot \frac{1}{\sqrt{s}}$$

uniformly in s and x so the integral is of the order of $\sqrt{s}$.

4. The uniqueness. We already know that $z_1^N(\cdot) \Rightarrow z_1(\cdot)$ and the limit solves the martingale problem for

$$\mathcal{L}^z_t = \frac{1}{2}\frac{\lambda(\lambda + \rho(t, G(t,z)))}{(\lambda + \bar{\rho})^2} \cdot \frac{d^2}{dz^2}$$

starting at $z_1(0) = z_1$.
The uniqueness is a corollary of *Lemma 1.* $\square$

Proposition 3. *For any $s > 0$*

$$\lim_{N\to\infty} E^N \left|F(s, x_1^N(s)) - z_1^N(s)\right| = 0 \quad ,$$

or equivalently for any $f \in C_0^\infty(R)$

$$E^N \left|f(F(s, x_1^N(s)) - f(z_1^N(s))\right| \to 0$$

as $N \to \infty$.

Proof: What really matters in the difference $\left|F(s, x_1^N(s)) - z_1^N(s)\right|$ is the absolute value of

$$B^\nu(s) := \frac{1}{N} \sum_{k\neq 1} \nu(x_k(s) - x_1^N(s)) - \int_0^1 \nu(y - x_1^N(s))\rho(s, y)dy \quad .$$

Let $\delta > 0$. We shall consider a smooth function $\tilde{\nu}(x)$ approximating $\nu(x)$ pointwise; we mean that $\nu \equiv \tilde{\nu}$ everywhere except a neighborhood $(-\frac{\delta}{2}, +\frac{\delta}{2})$ of the origin. Now let $\phi_\delta(x)$ be a smooth function bounded by 1 with compact support $|supp(\phi)| \leq \delta$ approximating $|\phi - \tilde{\nu}|$ as $\delta \to 0$.

Hence

$$|B^\nu(s)| \leq \left|B^\nu(s) - B^{\tilde{\nu}}(s)\right| + \left|B^{\tilde{\nu}}(s)\right|$$

and we know that the expected value of the second term tends to zero uniformly in s because $\tilde{\nu}$ is smooth. We only have to take care of the first term which is less than

$$\leq \left|B^{\nu-\tilde{\nu}}(s)\right| \leq \left|\sum_{k\neq 1} (\nu - \tilde{\nu})(x_k(s) - x_1^N(s))\right| +$$

$$+ \left|\int_0^1 (\nu - \tilde{\nu})(y - x_1^N(s))\rho(s, y)dy\right|$$

$$\leq \left|B^{\phi_\delta}(s)\right| + 2 \cdot \int_0^1 \phi_\delta(y - x_1^N(s))\rho(s, y)dy \quad .$$

When we take the expected value E^N , the first term above tends to 0 as $N \to \infty$ uniformly in s because ϕ_δ is smooth while the second needs a change of variable (we don't have to forget that ρ and ν are periodic of period 1) to bring down our proof to

$$\lim_{N\to\infty} E^N \int_0^1 \phi_\delta(y)\rho(s, y + x_1^N(s))dy = O(\delta) \quad . \qquad \square$$

Proposition 4. *Let Q be a limit point of the tight family of processes $\{x_1^N(\cdot)\}_N$. For a smooth $f \in C_0^\infty(R)$ and $t > 0$*

$$\lim_{N\to\infty} E^N[f(F(t, x_1^N(t)))] = E^Q[f(F(t, x_1(t)))] \quad .$$

Proof: For $t > 0$

$$x \longrightarrow \Theta_t(x) = F(t, x)$$

is a smooth and increasing function on R. By denoting

$$Q_t^N := P_N \circ (x_1^N(t))^{-1}$$

the marginal at t of $Q^N = P^N \circ (x_1^N(\cdot))^{-1}$ and recalling that

$$Q_t^N \Rightarrow Q_t$$

we get

$$E^N[f(F(t, x_1^N(t)))] = E^{P^N}[f(F(t, x_1^N(t)))] =$$

$$= E^{P_N}[f(\Theta_t x_1^N(t))] =$$

$$= E^{Q_t^N}[f(\Theta_t x)] \to E^{Q_t}[f(\Theta_t x)] = E^Q[f(F(x_1(t))] \quad .$$

□

Proposition 5. *If Q is a limit point of the tagged particle process then for any $f \in C_0^\infty(R)$*

$$\lim_{t\to 0} E^Q[f(F(t, x_1(t)))] = f(z_1)$$

or equivalently

$$Q(\{\omega \,:\, \lim_{t\to 0} F(t, \omega(t)) = z_1\}) = 1 \quad .$$

Proof:

$$\left|E^Q[f(F(t, x_1(t)))] - f(z_1)\right| \leq$$

$$\leq \left|E^Q[f(F(t, x_1(t)))] - E^N[f(F(t, x_1^N(t)))]\right| +$$
$$+ \left|E^N[f(F(t, x_1^N(t)))] - E^N[f(z_1^N(t))]\right| + \left|E^N[f(z_1^N(t))] - f(z_1)\right| \quad .$$

So

$$\limsup_{N\to\infty} \left|E^Q[f(F(t,x_1(t)))] - E^N[f(F(t,x_1^N(t)))]\right| = 0$$

from the precedent proposition and

$$\limsup_{N\to\infty} \left|E^N[f(F(t,x_1^N(t)))] - E^N[f(z_1^N(t))]\right| = 0$$

from the *Proposition 3*. It follows that

$$\lim_{t\to 0} \left|E^Q[f(F(t,x_1(t)))] - f(z_1)\right| \le \lim_{t\to 0} \limsup_{N\to\infty} \left|E^N[f(z_1^N(t))] - f(z_1)\right| \quad .$$

Since $f \in C_0^\infty(R)$ we may truncate $z_1^N(t)$ and integrate on the two sets $\{|z_1^N(t) - z_1| > \eta\}$ and $\{|z_1^N(t) - z_1| \le \eta\}$; on the first the integral is bounded by $\|f\|$ times the measure of this set (tends to 0 as $\lim_{t\to 0}\limsup_{N\to\infty}$) and on the second it is $\le \eta \cdot \|f'\|$. $\square$

Definition 10. *It is convenient to denote*

$$\Omega'_{x_1} := \{\omega \in \Omega_{x_1} \;:\; \lim_{t\to 0} (\Theta\omega)(t) = z_1\}$$

and

$$\Omega'_{z_1} := \Theta\Omega'_{x_1} \quad .$$

Proposition 6. *Let Q^1 and Q^2 be two limit points of the family of tagged particle processes starting at x_1. Then Q^1 and Q^2 are indistinguishable.*

Proof: The two measures Q^1 and Q^2 have the Markov property, start at $x(0) \equiv x_1$ and are concentrated on the set Ω'_{x_1}. It is sufficient to prove that for any positive integer m and any smooth function f of m variables with compact support

$$E^{Q^1}[f(\omega(t_1), \omega(t_2), \ldots, \omega(t_m))] = E^{Q^2}[f(\omega(t_1), \omega(t_2), \ldots, \omega(t_m))] \quad .$$

The Markov property implies that we can limit ourselves to the case $m = 1$. If $t := t_1 = 0$ we naturally get an identity.

¿From *Propositions 3* and *4* and from the *Definition 9* we know that on the set Ω'_{x_1}

$$E^{Q^i}[f(F(t,\omega(t)))] = E^{P_{z_1}}[f(\eta(t))], \qquad \text{for } i = 1, 2 \quad .$$

For $\omega \in \Omega_{x_1}$ and $\eta \in \Omega_{z_1}$ we may write

$$E^{Q^i}[f(\omega(t))\mathbf{1}_{\Omega'_{x_1}}(\omega)] = E^{P_{z_1}}[f(\Theta^{-1}\eta(t))\mathbf{1}_{\Omega'_{z_1}}(\eta)]$$

for both $i = 1, 2$. To prove the uniqueness it will be enough to write down

$$E^{Q^i}[f(\omega(t))] = E^{Q^i}[f(\omega(t))\mathbf{1}_{\Omega'_{x_1}}(\omega)] + E^{Q^i}[f(\omega(t))\mathbf{1}_{\Omega_{x_1}\setminus\Omega'_{x_1}}(\omega)]$$

and observe that since f is integrable the second term is zero. $\square$

Acknowledgements. This paper gives the final form of an idea from my Ph.D. thesis (1997). For this as well as for all my graduate work at New York University I want to express my gratitude towards my advisor, Professor S.R.S. Varadhan.

References

[1] Grigorescu, I. Self-diffusion for Brownian Motions with Local Interaction. *Preprint.*

[2] Guo, M. Z. Limit Theorems for Interacting Particle Systems. **Thesis**, NYU (1984)

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730

E-mail address: grigores@math.nwu.edu