LARGE SCALE BEHAVIOR OF A SYSTEM OF INTERACTING DIFFUSIONS

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Abstract. We present the explicit derivation of the asymptotic law of the tagged particle process for a system of interacting Brownian motions in the presence of a diffusive scaling in non-equilibrium. The interaction is local and interpolates between the totally independent case (non-interacting) and the totally reflecting case. It can be viewed as the limiting local version of an interaction through a pair potential as its support shrinks to zero. We also prove the independence of two tagged particles in the limit and analyze the case when the initial profile is singular.

1. Introduction

Assume we have decided to look at a given interacting particles system. There are various scaling options for the same dynamics, but we are interested in the nontrivial ones, i.e. those when certain quantities such as spatial averages are conserved. A system of $n \in Z_+$ independent Brownian motions $(x_1(t), x_2(t), \ldots, x_n(t))_{t\geq 0}$ evolving on a circle of size $N > 0$ can be scaled down to $\frac{1}{N}x_i(t)$, for $i = 1, 2. \ldots, n$ to produce a n-dimensional process on the unit circle. The number of particles can be chosen such that $n = \bar{\rho}N$, for $\bar{\rho} > 0$. The average density on the circle stays $\bar{\rho}$ for any $N > 0$. For a meaningful diffusive evolution we shall speed up time by a factor of N^2. The resulting dynamics is a set of $n = \bar{\rho}N$ independent Brownian motions $(x_1^N(t), x_2^N(t), \ldots, x_n^N(t))_{t\geq 0}$ on the n-dimensional unit torus Γ^n denoted by P^N.

The scaled process represents a *microscopic* realization of the interaction. By contrast, we shall look at *macroscopic* quantities. The average density $\bar{\rho}$ is the only preserved macroscopic quantity in our system. At time $t = 0$ we assume the existence of a measure $\mu(0, dx)$ on the unit circle with total mass $\mu(0, \Gamma) = \bar{\rho}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{n} \delta_{x_i(0)} = \mu(0, dx) \tag{1.1}$$

in weak sense. This macroscopic quantity is the initial profile of the system. A natural quantity to look at is the *hydrodynamic* limit:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{n} \delta_{x_i(t)} = \mu(t, dx) \quad . \tag{1.2}$$

For independent Brownian motions this amounts to the classic law of large numbers. The measure-valued process $\{\mu(t, dx)\}_{t\geq 0}$ will be concentrated on the unique

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deterministic solution of the heat equation $\rho_t = \frac{1}{2}\rho_{xx}$ in the distribution sense with initial condition $\lim_{t\to 0}\rho(t,x)dx = \mu(0,dx)$. Large deviations from the macroscopic profile are studied for this model in [7]. We are also interested in the existence of a limiting measure Q^μ on the path space $\Omega = C([0,\infty),\Gamma)$ with initial distribution $Q^\mu(\omega(0)\in dx) = \mu(0,dx)$ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{n} \delta_{x_i(\cdot)} = Q^\mu \quad . \tag{1.3}$$

Again, for the non-interacting case, the limit Q^μ is identical to a Wiener measure on the unit circle starting at $\mu(0,dx)$.

A third quantity we may look at is the fluctuations from the hydrodynamic limit. Let $\varphi(x)$ and $\psi(x)$ be smooth test functions on Γ. The random field $\{\zeta_N(t)\}_{t\geq 0}$ with values in $H^{-2}(\Gamma)$, defined in equilibrium by

$$\langle \varphi, \zeta_N(t)\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{n}(\varphi(x_i(t)) - \int_\Gamma \varphi(x)dx) \tag{1.4}$$

converges to a generalized Ornstein-Uhlenbeck process $\{\xi(t)\}_{t\geq 0}$ satisfying

$$d\zeta(t) = \frac{1}{2}\partial_{xx}\zeta(t)dt + \sqrt{\bar{\rho}}(d\partial_x W(t)) \quad . \tag{1.5}$$

Here $\partial_x W(t)$ is the Gaussian random field with covariance

$$E[\langle \varphi, \partial_x W(t)\rangle\langle \psi, \partial_x W(s)\rangle] = \min\{s,t\}\int_\Gamma \varphi'(x)\psi'(x)dx \ .$$

The averaging taking place in the law of large numbers (1.3) may wipe out a lot of features of the interaction between particles. However, in the independent case we have in mind, the limit Q^μ is essentially the same as the law of each particle considered by itself, for a fixed label, (say equal to one) $\{x_1(t)\}_{t\geq 0}$. This is the *tagged particle* process or the so-called tracer particle. The individual particles have trajectories supported on the set of continuous paths on the unit circle Γ. Every such path can pe lifted in a canonical way to the real line. Theorems 2.3, 2.4 are true for the lifted process. However we shall not differentiate between the two in the context of this paper.

The outlook of these considerations changes radically as soon as we allow some interaction between the particles in the system. Suppose $V(x)$ is a smooth, positive, even potential with compact support. Let's consider the n-dimensional interacting Brownian motions process P_V^N with generator

$$\mathcal{L}f(x) = \frac{1}{2}\Delta f(x) - \sum_i (\sum_{j\neq i} V'(x_i - x_j))\partial_i f(x) \text{ with } \mathcal{D}(\mathcal{L}) = C^2(\Gamma^n, R) \quad . \tag{1.6}$$

The presence of the Lapalcian induces a diffusive scaling $t_{micro} = N^2 t_{macro}$ and $x_{micro} = N x_{macro}$. This amounts to the scaling of the potential $V_N(x) = V(Nx)$. The equilibrium measure is given by the density

$$\frac{1}{Z_N}\exp\left[-\frac{1}{2}\sum_{i,j} V(N(x_i - x_j))\right] \quad . \tag{1.7}$$

Under the initial condition (1.1) one can obtain a hydrodynamic limit for the macroscopic density of the system. The bulk diffusion appears in [5]. See [18] for the nonequilibrium case. The limit for the fluctuation field from the macroscopic density in equilibrium is a Gaussian random field of the same nature as (1.5), shown in [13]. (See [1] for the Ginzburg-Landau model in nonequilibrium.)

The tagged particle process has been studied in equilibrium by Guo. The self-diffusion coefficient has a variational formula ([4] and [6]). The self-diffusion coefficient in nonequilibrium has not been done.

Another type of interaction we may be interested in studying is an extreme case of (1.6). At this level we are not interested in the scaling for the interacting particle system. The primary fact is that for a fixed n and any $\epsilon > 0$ we can consider a potential $V_\epsilon(x)$ of the same nature but with support included in the interval $[-\epsilon, \epsilon]$. Assume there exists a positive λ such that

$$\lim_{\epsilon\longrightarrow 0}\int_\Gamma (\exp{(2V_\epsilon(x))}-1)dx = \frac{1}{\lambda} \quad . \tag{1.8}$$

Then $\{P^N_{V_\epsilon}\}_\epsilon$ is tight and has a unique limit P^N_λ which is described below.

2. The interaction model

Consider a positive integer n and $\lambda \geq 0$. Let Γ^n be the n-dimensional torus. We define $F^{ij} = \{ x \in \Gamma^n : x_i = x_j\}$ for any i, j in $\{1, \dots, n\}$ and $F = \cup_{1\leq i<j\leq n} F_{ij}$. With this notation we introduce the following class of functions.

Definition 2.1. Let $\bar{C}(\Gamma^n, F) = \{ f : \Gamma^n \longrightarrow R : f \in C^2(\Gamma^n \setminus F)$ with $f^{ij}(x_0)$ and $D^{ij}f(x_0)$ finite for any $x_0 \in F$ and any $(i,j)\}$ - the set of smooth functions up to the boundary F, where f^{ij} and $D^{ij}f$ are defined as

$$\begin{aligned} & f^{ij}(x_1, x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_{j-1}, x, x_{j+1}, ..., x_n) \\ & \quad = f(x_1, x_2, ..., x_{i-1}, x+0, x_{i+1}, ..., x_{j-1}, x-0, x_{j+1}, ..., x_n) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} & D^{ij}f(x_1, x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_{j-1}, x, x_{j+1}, ..., x_n) \\ & \quad = (\partial_i - \partial_j)f(x_1, x_2, ..., x_{i-1}, x+0, x_{i+1}, ..., x_{j-1}, x-0, x_{j+1}, ..., x_n) \quad . \end{aligned} \tag{2.2}$$

We are now in a position to define the generator of the process

$$x^n(t) = (x_1(t), \dots, x_n(t))$$

on Γ^n. For a real $\lambda \geq 0$ we define the boundary conditions:

$$(BC) \qquad D^{ij}f(x) + \lambda(f^{ji}(x) - f^{ij}(x)) = 0 \quad \forall i, j \in \{1, \dots, n\} \quad . \tag{2.3}$$

The operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ with

$$\mathcal{L}f = \frac{1}{2}\Delta f \quad , \qquad \mathcal{D}(\mathcal{L}) = \{f \in \bar{C}(\Gamma^n, F) : \text{(BC) are satisfied}\} \tag{2.4}$$

is the infinitesimal generator of a process P^n_λ on Γ^n.

This interaction model can be more easily understood by considering a set of canonical local times $A^{ij}(t)$, defined for any oriented pair (i,j), $i \neq j$, $(i,j) \in \{1, \dots, n\}^2$ which measure the collision time of two particles x_i and x_j when x_i approaches x_j in positive trigonometric order $x_i > x_j$ on the unit circle. The ordering is nonambiguous because we are only interested in the circumstance when the particles are close to each other. In this framework, we shall consider increasing sequences of Poisson events of intensity λ running according to the local times for each oriented pair (i, j). These will be the "jump" times, or, more accurately, the times of switching of labels. Two particles $x_i(t)$ and $x_j(t)$ with distinct labels i and j will perform independent Brownian motions until they meet. Let τ be an exponential time with intensity λ, independent from the process. We define the stopping time $t_{ij} = \inf_{t>0}\{A^{ij}(t) > \tau\}$. The two particles will bounce back from each other according to a reflected Brownian motion. However, once the time t_{ij} is reached, they exchange labels or, in other words, cross each other. This construction repeats itself in the larger setting of the n particle process. The dynamics is that of reflected Brownian motions until the first "jump" time occurs. The particles will continue to perform a n-dimensional reflected Brownian motion until the next exchange of labels. This evolution interpolates between the reflected Brownian motion ($\lambda = 0$) and the independent case ($\lambda = \infty$). The diffusive scaling for the present interaction model amounts to blowing up the parameter λ by a factor of N to $\lambda_N = N\lambda$.

The new scaled process, denoted by P^N, allows us to calculate explicitly the self-diffusion coefficient in *nonequilibrium.* In the equilibrium case the self-diffusion coefficient can be expressed in terms of a variational formula for various interacting particle systems, as in [8], [14], [16] and [17].

Let $\mu(s, dx) = \rho(s, x)dx$ for $s > 0$ be the solution to the heat equation, with initial conditions (1.1). The next theorem is a version of the hydrodynamic limit valid uniformly in time.

Theorem 2.2. *For any smooth $f : \Gamma \longrightarrow R$ and any $t > 0$*

$$\lim_{N\to\infty} E^N \sup_{0\le s\le t} \left| \frac{1}{N} \sum_{k=1}^{n=N\bar{\rho}} f(x_k^N(s)) - \int_0^1 f(x)\mu(s, dx) \right|^2 = 0 \quad . \tag{2.5}$$

The hydrodynamic limit of the system is identical to the macroscopic profile of the independent Brownian motions (1.2). The symmetry of the boundary conditions cancels out any trace of the switching of labels occurring microscopically.

The invariant measure of the process is the product uniform distribution on the torus Γ^n. In equilibrium the macroscopic profile will be constant equal to $\bar{\rho}$ and the tagged particle process $\{x_1^N(t)\}_{t\ge 0}$ converges weakly, as $N \to \infty$, to a Brownian motion with diffusion coefficient $D_{self}(\lambda, \bar{\rho}) = \lambda/(\lambda + \bar{\rho})$. This is the self-diffusion coefficient for the local interaction (2.4) in the diffusive scaling. We can see that, consistent with the intensity of the interaction,

$$\lim_{\lambda\to 0} \frac{\lambda}{\lambda + \bar{\rho}} = 0 \text{ and } \lim_{\lambda\to\infty} \frac{\lambda}{\lambda + \bar{\rho}} = 1 \quad , \tag{2.6}$$

the first case corresponding to the pure reflection setting, leading to a deterministic profile (constant, in equilibrium) and the second case corresponding to the noninteracting case, when $D_{self}(\lambda, \bar{\rho}) = 1$ as in (1.3). The following theorem states the main result concerning the tracer particle in nonequilibrium (see [2]).

Theorem 2.3. *If the initial density profile $\mu(0, dx)$ has a bounded initial density $\rho_0(x)$, i.e. $\mu(0, dx) = \rho_0(x)dx$ and $P^N(x_1^N(0) = x_1) = 1$ is satisfied for all $N > 0$, then the family of measures $P^N \circ (x_1^N(\cdot))^{-1}$ has a weak limit Q^{x_1} as $N \to \infty$ and*

Q^{x_1} is the unique solution to the martingale problem given by

$$\mathcal{A}_t = \frac{1}{2}\left(\frac{\lambda}{\lambda+\rho(t,x)}\right)\frac{d^2}{dx^2} - \frac{1}{2}\left(\partial_x\rho(t,x)\frac{2\lambda+\rho(t,x)}{(\lambda+\rho(t,x))^2}\right)\frac{d}{dx} \tag{2.7}$$

starting at $(0, x_1)$.

The condition that the initial profile have a bounded density is not technical, as we shall explain later. Yet our interest is focused on the form of the diffusion process Q^{x_1} described in the theorem. The self-diffusion coefficient is consistent with the prediction for the equilibrium case $D_{self}(\lambda, \rho(t,x)) = \lambda/(\lambda+\rho(t,x))$. The drift depends on the spatial derivative of the density (vanishes in equilibrium) and is negative, reflecting the repulsive nature of the interaction. From a heuristic point of view, as soon as we have determined the macroscopic density $\rho(t,x)$ and $D_{self}(\lambda,\rho)$ we should be able to predict the drift term $b(t,x)$ by solving the equation

$$\rho_t = \mathcal{A}_t^* \rho \tag{2.8}$$

which yields exactly

$$b(t,x) = -\frac{1}{2}\left(\partial_x\rho(t,x)\frac{2\lambda+\rho(t,x)}{(\lambda+\rho(t,x))^2}\right) \quad . \tag{2.9}$$

For any smooth test function $\phi(x)$ on Γ we know that

$$\frac{1}{N}\sum_{i=1}^{n}\phi(x_i^N(t)) \Rightarrow \int_0^1 \phi(x)\rho(t,x)dx$$

from (2.5). In the same time, if the particles decouple in the limit, the sum $(1/N)\sum_{i=1}^{n}\phi(x_i^N(t))$ should converge to $\bar{\rho}E^{Q^\mu}[f(\omega(t)]$, where Q^μ is the law of the diffusion described in (2.7) starting from the initial distribution $\mu(0,dx)$. This implies (2.8). For the simple exclusion model this is explained in [15]. The crucial assumption we have made is that two distinct particles become independent in the limit. The next theorem and (2.11) state the propagation of chaos for our model.

Theorem 2.4. *Let $(x_1^N(\cdot), x_2^N(\cdot))$ be a pair of tagged particle processes with distinct labels. Assume that each starts at x_1, respectively x_2 on the unit circle, that is $P^N(x_1^N(0) = x_1) = 1$ and $P^N(x_2^N(0) = x_2) = 1$ and that the initial profile $\mu(0,dx)$ has bounded density $\rho_0(x)$. Let $x_1(\cdot)$ and $x_2(\cdot)$ be the two processes such that $x_1^N(\cdot) \Rightarrow x_1(\cdot)$ and $x_2^N(\cdot) \Rightarrow x_2(\cdot)$, that is there exists a measure $Q^{(x_1,x_2)}$ on $\Omega_2 = C([0,\infty), R^2)$ such that*

$$P^N \circ (x_1^N(\cdot), x_2^N(\cdot))^{-1} \Rightarrow Q^{(x_1,x_2)} \quad . \tag{2.10}$$

Then $x_1(\cdot)$ and $x_2(\cdot)$ are independent with respect to $Q^{(x_1,x_2)}$, or equivalently $Q^{(x_1,x_2)} = Q^{x_1} \otimes Q^{x_2}$.

The dynamics controlled by the parameter λ provides an example of hydrodynamic limit when the interaction is undetectable seen from the level of the marginals at a given time t. This is further illustrated by the fact that both the fluctuation field and the large deviations from the hydrodynamic limit behave identically as in the case of independent Brownian motions. The self-diffusion detected already in equilibrium shows that at the level of the path $\omega(\cdot) \in \Omega$ the amount of interaction is visible. In that sense we can prove the next theorem.

Theorem 2.5.

$$\frac{1}{N}\sum_{i=1}^{n}\delta_{x_i^N(\cdot)} \Rightarrow \bar{\rho}Q^{\mu} \tag{2.11}$$

where Q^{μ} is the law of tagged particle process (2.7) averaged over the initial profile $\mu(0,dx)$.

In equilibrium this law of large numbers is an immediate consequence of Theorems 2.3 and 2.4. In nonequilibrium one cannot assume that the particles converge to their limit uniformly with respect to their label. The proof is part of a work in progress concerning the fluctuation field (2.13) which requires stronger estimates than Theorem 2.3 and Theorem 2.4. For symmetric simple exclusion processes Theorem 2.5 is proved in [12].

A way to point out the emergence of the interaction when we pass to the law of large numbers (2.11) from the hydrodynamic limit (1.2) is to think of the multicolor process. Theorem 2.3 derives the law of large numbers for the marginal at time t of the tagged particle process whereas Theorem 2.5 must take into account any finite collection of marginals at times $t_0 < t_1 < \ldots < t_m$, for any $m \in Z_+$. We may want to see what happens to just two consecutive times $0 \leq s < t$. We look at the average

$$\frac{1}{N}\sum_{i=1}^{n} g(x_i^N(s), x_i^N(t)) \tag{2.12}$$

as $N \to \infty$, for an arbitrary $g(\cdot,\cdot) \in C_0^{\infty}(R^2)$. Let's consider that the particles have the same color at time $t = 0$, say blue. Also, let's pick $A \in \mathcal{B}(\Gamma)$. The particle crossing the set A at time s will change color into green. Due to the Markov property, proving the LLN for (2.12) is equivalent to finding the hydrodynamical limit of the two color process. The interaction becomes apparent because the symmetry (cancellations in the boundary conditions when averaging) is lost to the presence of the two colors. The particles will not exchange freely their labels; it matters where they come from, or, in other words, what is their history. See [10]. A large deviations principle from Q^{μ} for the empirical random measures in (2.11) for the symmetric exclusion process is obtained in [11].

In the future one would also like to give a meaning to the random field on the path space Ω of the fluctuations from the limit Q^{μ} in equilibrium

$$\zeta^N = \frac{1}{\sqrt{N}}\sum_{i=1}^{n}(\delta_{x_i^N(\cdot)} - Q^{\mu}) \quad . \tag{2.13}$$

We shall sketch the proof of Theorem 2.3 and Theorem 2.4.

3. Proof of Theorems 2.3 and 2.4

It is known (see [9]) from the definition of the process $\{x^N(t)\}_{t\geq 0}$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t = \sigma(x^N(s) : 0 \leq s \leq t)$ that there exist $n^2 - n$ local times

$\{A_N^{ij}(t)\}_{t\geq 0}$ for $i \neq j$ in the set $\{1, 2, \ldots, n\}$ such that for any $f \in \bar{C}(\Gamma^n, F)$

$$\begin{aligned}\mathcal{M}_f(t) \quad &:= \quad f(x^N(t)) - f(x^N(0)) - \frac{1}{2}\int_0^t \Delta f(x^N(s))ds - \\ &- \sum_{i\neq j}\int_0^t \left(D^{ji}f(x^N(s)) + (\lambda N)[f^{ij}(x^N(s)) - f^{ji}(x^N(s))]\right)dA_N^{ji}(s)\end{aligned} \tag{3.1}$$

is a $(P^{x^N}, \{\mathcal{F}_t\}_{t\geq 0})$ - martingale. More precisely,

$$\mathcal{M}_f(t) = \sum_{k=1}^{n}\int_0^t \partial_{x_k} f(x^N(s))d\beta_k(s) + \sum_{i\neq j}\int_0^t [f^{ij}(x^N(s)) - f^{ji}(x^N(s))]dM_N^{ji}(s)$$

where $\{\beta_k(t)\}$ $(k = 1, \ldots, n)$ is a family of independent Brownian motions and $M_N^{ij}(t)$, $M_N^{ji}(t)$ are the jump martingales corresponding to the interaction along the boundary F^{ij} such that $[M_N^{ij}(t)]^2 - (\lambda N)A_N^{ij}(t)$ is also a martingale.

A direct attempt to write the stochastic integral equation for the tagged particle $\{x_1(t)\}_{t\geq 0}$ leads to

$$x_1(t) - x_1(0) - \sum_{k\neq 1}\left[A_N^{1k}(t) - A_N^{k1}(t)\right] = \beta_1(t) \tag{3.2}$$

for the function $f_1(x) = x_1$. The term $\sum_{k\neq 1}[A_N^{1k}(t) - A_N^{k1}(t)]$ is very hard to evaluate.

Let $\nu : R \longrightarrow R$ be the periodic extension on the real line of the function $\nu(x) = x$ on $[0, 1]$. For two points x' and x'' on the unit circle $\nu(x' - x'')$ is the distance between the two in positive trigonometric sense.

In general, the process $\{((x_2^N(t) - x_1^N(t)), \ldots, (x_n^N(t) - x_1^N(t)))\}_{t\geq 0}$ is called the environment process for the tagged particle $\{x_1^N(t)\}_{t\geq 0}$. We generate a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t = \sigma(x^N(s) : 0 \leq s \leq t)$ and the measure P^N denoted by

$$z_1^N(t) := x_1^N(t) + \frac{1}{\lambda + \bar{\rho}} \cdot \frac{1}{N}\sum_{k\neq 1}\nu(x_k^N(t) - x_1^N(t)) \tag{3.3}$$

as a linear combination of the two. There are several steps to follow.

3.1. **Step 1: The asymptotic average collision time per particle.** The quadratic variation of the martingale $z_1^N(t)$ is

$$\sigma^2(N, \lambda)t + \frac{1}{(\lambda + \bar{\rho})^2}\left(\frac{\lambda}{N}\right)\sum_{k\neq 1}\left(A_N^{1k}(t) + A_N^{k1}(t)\right) \tag{3.4}$$

where

$$\sigma^2(N, \lambda) = \left[1 - \frac{n-1}{N(\lambda + \bar{\rho})}\right]^2 + \frac{1}{(\lambda + \bar{\rho})^2} \cdot \frac{(n-1)}{N^2} \quad .$$

The average interaction local time per particle

$$A_N^1(t) := \frac{1}{N}\sum_{k\neq 1}\left(A_N^{1k}(t) + A_N^{k1}(t)\right) \tag{3.5}$$

is the main quantity to evaluate in the limit.

Theorem 3.1. *For any initial profile $\mu(0,dx)$*
1) the average interaction local time per particle $\{A_N^1(\cdot)\}_N$ is tight and
2) $dA_N^1(t)$ is asymptotically equal to $\rho(t,x_1^N(t))dt$, i.e. $\forall\ t\geq 0$

$$\lim_{N\to\infty} E^N\left|A_N^1(t)-\int_0^t \rho(s,x_1^N(s))ds\right| = 0 \quad . \tag{3.6}$$

The hardest part of the proof of theorem 2.3 is to establish (3.6).

3.2. **Step 2: The tightness.** Theorem 3.1 shows that $z_1^N(t)$ is tight. At time $t=0$ the martingale $z_1^N(t)$ takes values in a bounded set $[0,2]$. Doob's inequality and the asymptotic estimate (3.6) take care of the rest. The idea underlying the proof of tightness for the tagged particle process $x_1^N(t)$ is that for $s<t$ the quantity

$$\begin{aligned} &|z_1^N(t)-z_1^N(s)| \\ &= \left|x_1^N(t)-x_1^N(s)+\frac{1}{\lambda+\bar\rho}\left(\frac{1}{N}\sum_{k\neq 1}(\nu(x_k^N(t)-x_1^N(t))-\nu(x_k^N(s)-x_1^N(s)))\right)\right| \end{aligned} \tag{3.7}$$

must be large if $|x_1^N(t)-x_1^N(s)|$ is large. This will show that if $z_1^N(t)$ is tight, then $x_1^N(t)$ is tight as well.

3.3. **Step 3: The connection between the martingale $z_1^N(t)$ and $x_1^N(t)$.** It is important to remark that both Step 1 and Step 2 of the proof make no use of the assumption that $\mu(0,dx)=\rho_0(x)dx$ for $\rho_0(x)$ bounded. Let's denote the path space $\{\omega(\cdot):\omega\in C([0,T])\}$ by Ω. The mapping $\Theta:\Omega\longrightarrow\Omega$

$$\Theta(\omega)(t) := \omega(t)+\frac{1}{\lambda+\bar\rho}\int_0^1 \nu(y-\omega(t))\rho(t,y)dy \tag{3.8}$$

is one-to-one and onto if $\rho_0(x)$ is bounded. We want to have

$$\lim_N E^{P^N}\left|\Big(x_1^N(t)+\frac{1}{\lambda+\bar\rho}\frac{1}{N}\sum_{k\neq 1}\nu(x_k^N(t)-x_1^N(t))\Big)\right. \tag{3.9}$$

$$\left.-\Big(x_1^N(t)+\frac{1}{\lambda+\bar\rho}\int_0^1 \nu(y-x_1^N(t))\rho(t,y)dy\Big)\right| = 0 \quad . \tag{3.10}$$

This is done by extending the hydrodynamic limit (1.2) to functions with a finite number of discontinuities like $\nu(x)$ and introducing $x_1^N(t)$ as a parameter. There are many details to fill in, but in the end (3.9) implies that, if $x_1(\cdot)$ is a limit point for the tight processes $\{x_1^N(\cdot)\}_{N>0}$, as well as $z_1(\cdot)$ is a limit point for $\{z_1^N(\cdot)\}_{N>0}$, then

$$z_1(\cdot)=\Theta(x_1(\cdot)) \quad \text{a.s.}\ . \tag{3.11}$$

For every $t\in[0,T]$ the function

$$z=F(t,x)=x+\frac{1}{\lambda+\bar\rho}\int_0^1 \nu(y-x)\rho(t,y)dy$$

has an inverse $G(t,z)$. With this notation, the asymptotic behavior of $A_N^1(t)$ proves that the martingale $z_1^N(\cdot)$ converges to a diffusion with generator

$$\mathcal{A}_t^z = \frac{1}{2}\frac{\lambda(\lambda+\rho(t,x))}{(\lambda+\bar\rho)^2}\frac{d^2}{dz^2} \quad . \tag{3.12}$$

However, we know $z_1(\cdot)$ from (3.12). It is sufficient to show that if two processes supported on Ω are related as in (3.11) then the tagged particle process $x_1(\cdot)$ is a diffusion with generator (2.7). This fact is shown in the next lemma.

Lemma 3.2. *We assume that the martingale problem is well posed for the pair* $(a(t,y), b(t,y))$, *i.e. for any* $(t,y) \in [0,T] \times R$ *there is a measure* $P^{(s,y)}$ *on the path space* $\Omega = C([0,T], R)$ *such that if* $y(\cdot)$ *denotes an element of* Ω *and*

$$\mathcal{L}_t := \frac{1}{2} a(t,y) \frac{d^2}{dy^2} + b(t,y) \frac{d}{dy}$$

then

1)$P^{(s,y)}(\{y(s) = y\}) = 1$ *and*

2) $\forall f(\cdot,\cdot) \in C_0^\infty([0,T], R)$ *the expression*

$$f(t, y(t)) - f(s, y(s)) - \int_s^t (\partial_u + \mathcal{L}_u) f(u, y(u)) du$$

is a $(P^{(s,y)}, \mathcal{F}_t)$*-martingale, where* $\mathcal{F}_t = \sigma(\omega(s) : 0 \le s \le t)$.

Suppose $\Theta : [0,T] \times R \to R$ *is a* C^2 *mapping such that*

1) $(t, \Theta(t,x)) = (t,y)$ *and*

2) $0 < c_1 \le \partial_x \Theta(t,x) \le c_2 < \infty$ *for any* (t,x). *Then* $x \to \Theta(t,x)$ *has an inverse* $y \to \Psi(t,y)$ *for any fixed* $t \ge 0$. *If we define a mapping on the path space* $\Xi : \Omega \to \Omega$ *by* $[\Xi(y)](t) := \Psi(t, y(t)) = x(t)$, *then* $\hat{P}^{(s,x)} := P^{(s,\Theta(s,x))} \circ \Xi^{-1}$ *solves the martingale problem* $(\hat{a}(t,x), \hat{b}(t,x))$ *with*

$$\hat{a}(t,x) = [a \cdot (\partial_y \Psi)^2] \circ (t, \Theta(t,x)) \tag{3.13}$$

and

$$\hat{b}(t,x) := \left[(\partial_t \Psi) + \frac{1}{2} a \cdot (\partial_{yy} \Psi) + b \cdot (\partial_y \Psi) \right] \circ (t, \Theta(t,x)) \quad . \tag{3.14}$$

4. Relaxing the initial conditions

The mapping Θ defined in (3.8) is not well defined for singular measures on the circle. The limit

$$\lim_{t \to 0} (\Theta \omega)(t) = \lim_{t \to 0} \left[\omega(t) + \frac{1}{\lambda + \bar{\rho}} \int_{[0,1]} \nu(y - \omega(t)) \rho(t,y) dy \right]$$

does not exist for paths starting at $\omega(0) = x_1$ if the $\mu(0, \{x_1\}) > 0$. The tagged particle moves both due to the Brownian character of the evolution and the interaction with the environment process. For $t > 0$ the density $\rho(t,x)$ is smooth for an arbitrary initial profile, which, by Theorem 3.1, grants its uniqueness for $t' > t$. This suggests that the uniqueness of the whole process must be exclusively dependent on the outlook of the environment at time $t = 0$, that is on $((x_2^N(0) - x_1^N(0)), \ldots, (x_n^N(0) - x_1^N(0)))$ at time $t = 0$. A tracer particle will be pushed away from the direction which carries more weight in the interaction, that is, where there is a larger number of particles per volume. Hence we must know the amount of mass on both sides of x_1, even though we already know the total mass piling up at this point, namely $\mu(0, \{x_1\})$. A way of making these considerations precise is the assumption stated in equation (4.3).

We define two periodic functions of period one on R. On the unit interval $[0,1)$ they are equal to

$$\phi_\epsilon(x) = x, \text{ if } x \in [0, 1-\epsilon], \quad x-1, \text{ if } x \in [1-\frac{\epsilon}{2}, 1] \quad \text{and smooth on } [0,1) \tag{4.1}$$

and

$$g_\epsilon(x) = \nu(x) - \phi_\epsilon(x) \quad . \tag{4.2}$$

If $\mu(\{x_1\}) := \pi(x_1) > 0$ we assume that there is a number $\pi_-(x_1)$ in the interval $[0, \pi(x_1)]$ (clearly $\pi_-(x_1) = 0$ if $\mu(\{x_1\}) = 0$) such that

$$\lim_{\epsilon \to 0} \limsup_{N\to\infty} E^N \left| \frac{1}{N} \sum_{k \neq 1} g_\epsilon(x_k^N - x_1^N) - \pi_-(x_1) \right| = 0 \quad . \tag{4.3}$$

For any x_0 let $\Omega_{x_0} = \{\omega : \omega \in \Omega \text{ and } \omega(0) = x_0\}$. The mapping Θ (3.8) will be altered at the starting point. We write the function $F(t,x)$

$$F(t,x) = x + \frac{1}{\lambda + \bar{\rho}} \int_0^1 \nu(y-x)\rho(t,y)dy \tag{4.4}$$

for $t > 0$ and equal to the constant

$$z_1 := x_1 + \frac{1}{\lambda+\bar{\rho}} \int_{[0,1]\setminus\{x_1\}} \nu(y-x_1)\mu(dx) + \frac{1}{\lambda+\bar{\rho}} \pi_-(x_1) \tag{4.5}$$

for $(0, x_1)$. This makes

$$\Theta(\omega)(t) = F(t, \omega(t)) \text{ for } t > 0 \text{ and } \Theta(\omega)(0) = z_1$$

a mapping from Ω_{x_1} into Ω_{z_1}. The point z_1 will be the "adjusted" starting point for (3.3). The main result concerning uniqueness is the following theorem, from [3]. The infinitesimal generator $\mathcal{A}_t$ of the process has been defined in (2.7).

Theorem 4.1. *Under (1.1) and (4.3), provided that $P^N(x_1^N(0) = x_1) = 1$ for all $N > 0$, we have*

1) If $\mu(dx)$ is continuous at x_1, i.e. $\mu(\{x_1\}) = 0$ then the tagged particle process starting at x_1 is unique.

2) If $\mu(\{x_1\}) = \pi(x_1) > 0$, then the family of limit points of the tight family of processes $\{P^N \circ (x_1^N(\cdot))^{-1}\}_N$ is infinite and there is a unique such limit for each value of $\pi_-(x_1) \in [0, \pi(x_1)]$ denoted for simplicity by Q^{x_1}.

3) Q^{x_1} can be characterized as the unique measure on Ω_{x_1} with the properties

(i) For any smooth $f(t,x)$ with $supp(f) \subseteq (0,\infty) \times R$ the expression

$$f(t, x_1(t)) - f(0, x_1(0)) - \int_0^t (\partial_u f + \mathcal{A}_u f)(u, x_1(u))du$$

is a $(Q^{x_1}, \{\mathcal{F}_t\}_{t\geq 0})$ - martingale and

(ii) $\quad Q^{x_1}(\{\omega \in \Omega_{x_1} : \Theta\omega \in \Omega_{z_1}\}) = 1.$

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