QUASI-STATIONARITY FOR A NON-CONSERVATIVE EVOLUTION SEMIGROUP

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Abstract. We investigate a non-conservative branching particle process on an open domain $D \subseteq \mathbb{R}^d$ and its marginal evolution semigroup in sub- and super- critical regimes. Between branching the particles are driven by a diffusion killed at the boundary. The expected value of its empirical measure is the stochastic representation of the solution to a heat equation with mass creation at a random source with distribution $\gamma(dx)$ on D. Normalized to have total mass equal to one, it is the hydrodynamic limit of the Fleming-Viot type branching particle system from [13]. A limit theorem identifies the quasi-stationary distribution as the normalized resolvent kernel of the killed process at α^*, a number uniquely determined by $\bar{K}E_\gamma[e^{-\alpha^*\tau^D}] = 1$, where τ^D is the hitting time of the boundary and $\bar{K} > 0$ is the branching intensity.

1. Introduction

Let $D \subseteq \mathbb{R}^d$ an open domain with piecewise C^1 boundary, which will be divided in two subsets $(\partial D)_r$, the reflecting part, which is relatively open, and $(\partial D)_a$, which is relatively closed, the absorbing part. The reflective part is not essential to our setup, but the absorbing one is, and we impose a positive exponential moment for its hitting time.

Likewise, the smoothness of the boundary can be significantly relaxed to Lipschitz regularity. Let L be a second order strongly elliptic operator defining a diffusion on D generated by $(L, \mathcal{D}(L))$ (1.4), where the domain satisfies boundary conditions defined as reflecting on $(\partial D)_r$ and absorbing on $(\partial D)_a$.

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Based on this *underlying* (also known as *driving*) diffusion, in Section 2, we shall construct a branching process $(\zeta_t)_{t\geq 0}$ (2.2), closely related to models in genome population dynamics. Here *mutation* is represented by the diffusive term (the Brownian term), *selection* is represented by drift (in the probabilistic, not geneticists' sense) and *recombination* is represented by the redistribution at a random point $\sim \gamma(dx)$ where the new mass is born. Genetic recombination can be seen as a repair to damaged DNA. If artificial, it is under the effect of a *catalyst*, many times simplified as contact with a portion of the boundary.

The process has a finite number of particles at all times, with probability one. Branching occurs when the boundary $(\partial D)_a$ is hit. At that point, the particle is removed and an independent random integer number K of particles is born at a location with distribution $\gamma(dx)$ concentrated on D. Between boundary hits the particles follow independent diffusions driven by L.

1.1. **Conditions on the underlying process.** The usual notation $M_F(D)$ designates the finite measures on $D \subset \mathbb{R}^d$, $M_1(D)$ the space of probability measures on D (distributions), both with the topology of convergence in distribution (weak convergence). The most general class of test functions $\mathcal{T}$ used will be the set of time-space bounded and smooth up to the boundary functions

$$\mathcal{T} = \{\phi \,|\, \phi \in C_b^{1,2}([0,\infty) \times \bar{D}, \mathbb{R})\}\,. \tag{1.1}$$

For test functions $\phi(t,x) = \psi(x)$ in the space variable only and all test functions $\phi(t,\cdot)$, with $t \geq 0$ fixed, we write $\langle \psi(\cdot), m\rangle$ for the integral against a finite measure $m(dx)$ on D.

For a test function $\phi \in \mathcal{T}$, we shall denote

$$\phi \in (BC)_r \qquad \text{if} \qquad \nabla\phi(t,x)\cdot \mathbf{n} = 0\,, \quad x \in (\partial D)_r\,,\ \ t \geq 0\,, \tag{1.2}$$

where $\mathbf{n}$ is the normal to the boundary $(\partial D)_r$, which will be prescribed for the underlying diffusion, respectively

$$\phi \in (BC)_a \quad (\in (BC)_{ac}) \quad \text{if } \phi(t,x) \text{ vanishes (is constant) on } x \in (\partial D)_a\,, \forall t \geq 0\,. \tag{1.3}$$

The boundary $(\partial D)_a$ can be assimilated to the cemetery state $\mathfrak{b}$ and a function $\phi \in (BC)_{ac}$ will take constant value $\phi(\mathfrak{b})$ on $(\partial D)_a$, and that constant will be zero if $\phi \in (BC)_a$.

The diffusion described solves the martingale problem $(L, \mathcal{D}(L))$ with

$$\mathcal{D}(L) = \{\phi \in C^1(\bar{D}) \cap C^2(D) \,|\, \phi \in (BC)_r \cap (BC)_a\}\,. \tag{1.4}$$

It is assumed that it defines a *strongly continuous Feller-Dynkin semigroup* in the sense of [16], Chapter III.6, that is, the transition probabilities satisfy

$$(1.5)\quad S_t^D\phi(x) = E_x[\phi(x_t)] = \int_D p^D(t,x,dy)\phi(y) \in C_b(D)\,, \quad \phi \in C_b(D) \ \text{(Feller property)}$$

and determine a C_0 (i.e. strongly continuous) semigroup on $C_0(D)$, the space continuous functions vanishing at infinity on D with the supremum norm.

Let τ^D be the hitting time of the absorbing boundary $(\partial D)_a$, $f_D(t,x)$, $F_D(t,x)$ its density, respectively distribution functions for initial state $x \in D$ and $t > 0$, defined by

$$(1.6)\qquad P_x(\tau^D > t) = 1 - F_D(t,x) = \int_t^\infty f_D(s,x)\,ds\,,$$

having Laplace transform

$$(1.7)\qquad \alpha \to \hat{f}_D(\alpha,x) = E_x[\exp(-\alpha\tau^D)]$$

defined for all real α where the integral is finite.

It will be assumed that there exists a spectral gap, more precisely,

$$(1.8)\qquad \exists\ \tilde{\alpha} < 0 \qquad \lim_{\alpha\downarrow\tilde{\alpha}} \hat{f}_D(\alpha,x) = +\infty$$

which also implies that $(\tilde{\alpha},+\infty) \subseteq Res(L)$. Since $\hat{f}_D(\alpha,\gamma)$ is decreasing in α and vanishes at infinity, for any $\bar{K} > 0$ there exists a unique $\alpha^* \in (\tilde{\alpha},+\infty)$ such that

$$(1.9)\qquad 1 - \bar{K}\hat{f}_D(\alpha^*,\gamma) = 0\,.$$

Moreover, due to $\hat{f}_D(0,\gamma) = 1$, $\alpha^* > 0$ (< 0) for $\bar{K} > 1$ (< 1), with equality $\alpha^* = 0$ when $\bar{K} = 1$.

In addition, even though it is strictly needed only in the proof of the uniqueness part of the strong solution in Theorem 3, Section 4, we also require that the heat kernel be sufficiently smooth to have

$$(1.10)\qquad S_t^D\phi(x)\,,\ f_D(t,x) \in C_b^{1,2}([t_0,\infty)\times\bar{D})\,, \qquad \phi \in C_b(\bar{D})\,, \quad t_0 > 0\,.$$

Remark. Some of these conditions are redundant if D is bounded. Even for unbounded domains, in most applications, when the coefficients of L are C_b^∞, then functions and thus (1.10) and stronger conditions would immediately hold, with precise bounds on the heat kernel when $t \to 0$ and $x, y \to \infty$.

The next element is $\gamma \in M_1(D)$, a probability measure on D. It is important to emphasize that γ does not charge the boundary, i.e. $\gamma(D) = 1$. By construction, the number of individuals born at the point with distribution $\gamma(dx)$ is is a random non-negative integer K with mean value $\bar{K}$ from (2.1).

When the boundary condition on $(\partial D)_a$ is replaced by $(BC)_{ac}$ we denote the set $\mathcal{D}_c(L)$ (the subscript "c" from constant)

$$\mathcal{D}_c(L) = \{\phi \in C^1(\bar{D}) \cap C^2(D) \,|\, \phi \in (BC)_r \cap (BC)_{ac}\}\,. \tag{1.11}$$

1.2. **Summary of the results.** Section 2 has Proposition 1 at the center, establishing an exponential bound on the total number of particles in the branching process.

Section 3 uses these bounds to define rigorously the branching process $(\zeta_t)_{t\geq 0}$, which is is Markovian and has a transition semigroup defined on the space of continuous bounded functions on $M_F(D)$, the space of finite measures on D. Considering only linear functionals $\zeta \to \langle \zeta, \phi \rangle$, ϕ an appropriate test function, we obtain the *marginal transition semigroup* as defined in (5.4). This is a non-conservative semigroup, allowing for sub- and super- critical behavior. Theorems 1 and 2 identify its quasi stationary distribution (qsd) via a Yaglom limit. It is equal to the resolvent kernel of the underlying semigroup, calculated at a real value uniquely determined by the mass creation.

Section 4 is one of the main motivations of this work. Theorem 3 solves the *heat equation with mass creation*, giving its representation as the expected value of the empirical measure of the branching process (4.4). The hydrodynamic limit of the conservative particle system studied in [13] and [17] referred to as the Bak-Sneppen Branching Diffusions or BSBD, is exactly the normalization, with mass one, of this solution. Additionally, since, in principle, the qsd of the branching system in this paper is the asymptotic profile of the BSBD equilibrium, these results permit the simulation ([14] and references within) for the resolvent kernels of the underlying diffusion.

This feature is exactly the pattern followed by the dissipative process (thus always subcritical case) present in the standard Fleming-Viot particle system, for example when driven by the Dirichlet Laplacian [5, 12, 15] or a dissipative random walk [7, 2], having roots in the renewal dynamic approach from [8]. The difference here is that one can represent all resolvents corresponding to nonsingular values (1.8) $\alpha^* > \tilde{\alpha}$, according to the value of $\bar{K}$, with a bijection obtained by solving (1.9). For the Dirichlet Laplacian, $\tilde{\alpha}$ is the first eigenvalue, and the qsd is the normalized first eigenfunction, obtained for $\bar{K} \downarrow 0$.

Section 5 gives a brief summary of concepts needed for non-contractive semigroups in Section 3 and a one dimensional, computable example.

2. The branching processes Z_t and ζ_t

We introduce a branching process with *mutation* represented by diffusion and *resampling/recombination* represented by the distribution $\gamma(dx)$ where the new mass is born. First, we construct it as a particle system Z_t having a random total mass changing by branching and allowing for possible extinction. The plan is to prove that the expected value of its empirical distribution, divided by the expected value of its total mass, is equal to the hydrodynamic limit of the empirical profile of the N particle Bak-Sneppen Branching (BSB) diffusion process [13].

With the same notations as in Section 1, a single particle is placed at a random point with initial distribution $\nu_0(dx)$ at $t = 0$ and starts moving according to $(L, \mathcal{D}(L))$ until it dies. At that moment, instantaneously, a random number K of particles are born at a specific point in D distributed according to $\gamma(dx)$. The random number K is independent of the process up to that time and has a distribution $\pi(dk)$ on non-negative integers,

$$\pi(\{k\}) = p_k\,, \qquad P(K = k) = p_k\,, \quad k \geq 0\,, \qquad \bar{K} = \sum_{k=0}^{\infty} k p_k < +\infty\,. \tag{2.1}$$

All particles restart afresh, independently, and move in D until the first one dies and the branching is repeated. The procedure is continued indefinitely. It is shown below that the total mass has an exponential bound in expected value, showing that it is not explosive. In agreement to (1.9) we see three regimes: subcritical ($\bar{K} < 1$), critical ($\bar{K} = 1$) and supercritical ($\bar{K} > 1$).

To fix ideas, we shall exemplify with the Poisson distribution $\pi = Poisson(a)$, i.e. $p_k = e^{-a}a^k/k!$ for $k \geq 0$; the Bernoulli distribution as in Corollary 1, part 3); or simply a delta function $\pi(dk) = \delta_{\bar{K}}(dk)$. The last case gives the process from [10, 12, 4] when $\bar{K} = 1$ (critical case) and from [13, 17] when $\bar{K} = 2$ - the arguments are valid for any $\bar{K} > 1$ (supercritical case).

The process is defined constructively up to extinction or up to a possible explosion, when mass equals infinity. The former may occur with positive probability as long as $p_0 > 0$, as one can see from killing the one particle process after the very first boundary hit. The latter cannot occur if $\bar{K}$ is finite, as proven in Proposition 1.

Let N_t be the number of particles in the system at time $t \geq 0$. Define the stopping times $T_m \in [0, +\infty]$ as the first time $N_t \geq m$, $m \in \mathbb{Z}_+$. The process $t \to N_t$ is rcll (cadlag) and piecewise constant, so T_m is a stopping time and T_m is nondecreasing in m. The stopping time $T_\infty = \lim_{m\to\infty} T_m$ is the *time of explosion* which is the life time of the process.

By construction, the process starts and preserves a finite number of particles during its lifetime with probability one.

Let $\tilde{D} = D \cup \{\mathfrak{o}, \mathfrak{b}\}$ be an extension of the usual compactification of D with $D \cup \{\mathfrak{o}\}$, where $\mathfrak{o}$ is the point at infinity for D and $\mathfrak{b}$ is another isolated point. Denote by Z_t^i, $i \in \mathbb{Z}_+$ the i-th particle born in the process. Here $\mathfrak{o}$ is the cemetery point as usual and $Z_t^i = \mathfrak{b}$ prior to birth. When at time τ a number $j \geq 1$ of individuals are born, their birth being simultaneous and at the same point, their ordering is not relevant. They are simply labeled $i = N_{\tau-} + l$, $1 \leq l \leq j$. If $j = 0$, no new label is added.

Since the number of particles is a nonnegative integer at all times, with probability one, we could adopt $\tilde{D}_0^\infty$, the subspace of $\tilde{D}^{\mathbb{Z}_+}$ with only finitely many components in D as the state space of the process $Z_t = (Z_t^1, \ldots, Z_t^{N_t})$, $t \geq 0$. It follows by construction that its law is a probability measure on the Skorokhod space $\mathbf{D}([0,\infty), \tilde{D}_0^\infty)$ of *right continuous with left limits paths* (RCLL) on $\tilde{D}_0^\infty$.

It is more convenient to work directly with the formalism of measure valued processes, using the state space $M_F(D)$, the space of finite measures on D. The two descriptions are related by introducing the notation

$$\zeta_t = \sum_{i=1}^{N_t} \delta_{Z_t^i}\,, \quad \text{if} \quad N_t > 0 \tag{2.2}$$

and $\zeta_t = \delta_{\mathfrak{o}}$, the cemetery state on $M_F(D)$ for the empirical measure of the process on $\tilde{D}$. Since test functions will vanish at $\mathfrak{o}$ and $\mathfrak{b}$, without loss of generality, the sum can be considered only over particles alive at time t. In this case, the law of ζ_t, $t \geq 0$ is a probability measure on $\mathbf{D}([0,\infty), M_F(D))$.

Proposition 1. *The number of particles N_t of the process (Z_t) starting with a finite number of particles has finite expectation for any $t > 0$, assuming that $\bar{K}$ is finite. More precisely, there exists $C(\gamma, \bar{K}) > 0$ and $\alpha^* \geq 0$ depending only on γ, both independent of t and x, such that*

$$\sup_{x\in D} E_x[N(t)] \leq C(\gamma, \bar{K}) e^{\alpha^* t}\,. \tag{2.3}$$

More precisely, α^* is the solution to (1.9) depending on γ and the set D via the distribution of the first hitting time τ^D.

Remark. We shall analyze α^* in relation to one in the context of criticality giving more precise bounds. Proposition 1 only shows $\alpha^* = 0$ when K is Bernoulli, and $\alpha^* > 0$ when $\bar{K} > 1$, more precisely the true bound is no greater than α^* which is the solution of (1.9), but it is an exact exponential bound as we see in Proposition 3.

Proof. We first remark that if all $p_k = 0$, $k \geq 2$ then $0 \leq N_t \leq 1$ almost surely. In case at least one of these probabilities is non-zero, then we proceed to Step 1.

Step 1. First we couple the process with a new process $\tilde{Z}_t$ having the same evolution mechanism as Z_t with the exception that the number of particles born at a boundary hit is $\tilde{K}$ with $P(\tilde{K} = k) = p_k$, $k \geq 2$, $P(\tilde{K} = 0) = 0$ and $P(\tilde{K} = 1) = p_0 + p_1$. The processes are identical up to the first boundary hit, starting with the same number of particles at the same locations and following the same Brownian paths. If Z_t draws a sample K of the number of particles to be born and $K \geq 1$ then the two processes continue to be identical until the corresponding K equals zero. At that moment, $\tilde{Z}_t$ will continue with an additional particle born at location chosen with the same distribution $\gamma(dx)$. The offspring of this particle follows the dynamics using the distribution of $\tilde{K}$ for the numbers of births upon each boundary hit and will be independent forever of Z_t. The rest of the process continues its evolution. This coupling will follow the same paths for the original particles or the particles born when $K \neq 0$, while the other particles of $\tilde{Z}_t$ not belonging to Z_t follow independent paths from an infinite supply of Brownian paths on D. It is important that, path-by-path, $N_t \leq \tilde{N}_t$ and $\tilde{N}_t$ is non decreasing in $t \geq 0$. We notice that $E[\tilde{K}] = p_0 + E[K]$. We know that at least one of p_k, $k \geq 2$ is positive, and then $E[\tilde{K}] > 1$.

Step 2. Without loss of generality, we proceed to prove the Proposition assuming $p_0 = 0$ and $\bar{K} = E[K] > 1$. Recall T_m is the first time N_t exceeds m particles. Let N_t^x denote the number of particles at time $t \geq 0$ of the process starting with exactly one particle at $x \in D$ and $E_x[\cdot]$ the expectation with respect to this initial state. Let τ^D be the first boundary hit. Since $p_0 = 0$, the process has a non-decreasing number of particles, we have the time shift inequality holding for all ω in the sample space

$$\tau^D + T_{m-1} \circ \theta_{\tau^D}(\omega) \geq T_m(\omega)\,, \quad T_m(\omega) \geq T_{m-1}(\omega)\,, \quad m \geq 2\,. \tag{2.4}$$

We remark that this is the only reason we needed a coupling in Step 1. Moreover, if $m \geq 2$ then $T_m \geq \tau^D$ implying that $\{\tau^D > t'\} = \{\tau^D > t\}$ when $t' = t \wedge T_m$.

With $t' = t \wedge T_m$, we have $t' \leq t \wedge [(T_{m-1} \circ \theta_{\tau^D}) + \tau^D]$. And on the event $\{\tau^D < t\}$,

$$t' \leq [(t - \tau^D) \wedge (T_{m-1} \circ \theta_{\tau^D})] + \tau^D \,. \tag{2.5}$$

We now write

$$E_x[N_{t'}] = E[N^x_{t'}] = E[\mathbf{1}_{\{t'<\tau^D\}} + \mathbf{1}_{\{t'\geq\tau^D\}} \sum_{j=1}^{K} N^{Z^j_{\tau^D}}_{t'-\tau^D}] \tag{2.6}$$

$$\leq E[\mathbf{1}_{\{t<\tau^D\}} + \mathbf{1}_{\{t\geq\tau^D\}} \sum_{j=1}^{K} N^{Z^j_{\tau^D}}_{(t-\tau^D)\wedge(T_{m-1}\circ\theta_{\tau^D})}] \tag{2.7}$$

$$= P_x(t < \tau^D) + \bar{K} \int_0^t E_\gamma[N_{(t-s)\wedge T_{m-1}}] f_D(s,x)\, ds \tag{2.8}$$

where $E_\gamma[\cdot] = \int_D E_{x'}[\cdot]\gamma(dx')$ and $f_D(t,x)$ is the density function of τ^D when the particle starts at $x \in D$. On line (2.7) we used (2.5) and on line (2.8) we used the strong Markov property and the independence of K from the past of the process.

Let $a_m(t) = E_\gamma[N_{t\wedge T_m}]$, $m \geq 0$, $t \geq 0$. By integrating $E_x[N_{t'}]$ over $\gamma(dx)$ we have

$$a_m(t) \leq \int_D P_x(t < \tau^D)\, \gamma(dx) + \bar{K} \int_0^t a_{m-1}(t-s) f_D(s,\gamma)\, ds \tag{2.9}$$

where $f_D(t,\gamma) = \int_D f_D(t,x')\gamma(dx')$. The unknown expected values satisfy the bounds $0 \leq a_m(t) \leq m$. It follows that their Laplace transforms $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha s} g(s) ds$, for an integrable function $g : [0,\infty) \to \mathbb{R}$, satisfy

$$\hat{a}_m(\alpha) \leq \frac{1}{\alpha}(1 - \hat{f}_D(\alpha,\gamma)) + \bar{K}\hat{a}_{m-1}(\alpha)\hat{f}_D(\alpha,\gamma) \tag{2.10}$$

A simple estimate is to bound further line (2.8) using $T_{m-1} \leq T_m$ and the monotonicity of N_t to obtain $\hat{a}_{m-1}(\alpha) \leq \hat{a}_m(\alpha)$.

For any $\alpha > \alpha^*$ we have

$$\hat{a}_m(\alpha) \leq \frac{1}{\alpha}\Big(\frac{1 - \hat{f}_D(\alpha,\gamma)}{1 - \bar{K}\hat{f}_D(\alpha,\gamma)}\Big) \tag{2.11}$$

The Tauberian theorem proves that there exists a constant $C_1 > 0$, dependent of γ but independent of m and t such that $\lim_{t\to\infty} e^{-\alpha^* t} a_m(t,\gamma) \leq C_1$, which implies that for a constant C depending only on γ

$$a_m(t,\gamma) \leq C e^{\alpha^* t} \,. \tag{2.12}$$

Letting $m \to \infty$ and the monotonicity of $\hat{a}_m$ in m, we obtain the same inequality uniformly in m. Plugging into (2.8) we obtain

$$E_x[N_{t\wedge T_m}] \le P_x(t < \tau^D) + C\bar{K} \int_0^t e^{\alpha^*(t-s)} f_D(s,x)\, ds$$

Note that T_m goes to infinity as $m \to \infty$, so monotone convergence shows that the bound is uniform in m, which proves $E_x[N_t] < \infty$ and a fortiori the claim that N_t is finite. To estimate its growth rate we factor out e^t and then we bound the integral all by itself by letting $t \to \infty$, which will give the Laplace transform of f_D at α^*. Then

$$E_x[N_t] \le P_x(t < \tau^D) + C\bar{K} e^{\alpha^* t} \hat{f}_D(\alpha^*, x) \le 1 + C\bar{K} e^{\alpha^* t}\,. \tag{2.13}$$

This proves the claim that, uniformly in $x \in D$, there exists a constant depending only on γ and $\tilde{K}$, inequality (2.3) is true. □

Let $\Theta_K(s) = \sum_{k=0}^{\infty} p_k s^k$, $s \le 1$ be the generating function of K. Denote $u_x(t) = E[e^{-\lambda N_t^x} \mathbf{1}_{[0,\infty)}(N_t^x)]$ for $\lambda \ge 0$ and $u_\gamma(t) = \langle \gamma, u_\cdot(t) \rangle$.

Proposition 2. *The following convolution formulas hold*

$$u_x(t) = e^{-\lambda} P_x(\tau^D > t) + \int_0^t \Theta_K(u_\gamma(t-s)) f_D(s,x)\, ds\,; \tag{2.14}$$

when $\lambda \to \infty$ we obtain for the probability $v_x(t) = P(N_t^x = 0) = P(T_{ext} \le t)$

$$v_x(t) = \int_0^t \Theta_K(v_\gamma(t-s)) f_D(s,x)\, ds\,; \tag{2.15}$$

and for $n_x(t) = E[N_t^x]$

$$n_x(t) = P_x(\tau^D > t) + \bar{K} \int_0^t n_\gamma(t-s) f_D(s,x)\, ds\,. \tag{2.16}$$

Proof. In analogous fashion with (2.6), now knowing that N_t^x is finite almost surely, we obtain (2.14). The generating function appears because at the time $\tau^D = s$ (under conditioning) in the integral, the process will instantaneously have $K \sim \pi(dk)$ independent copies of itself. Relation (2.15) follows as $\lambda \to \infty$ by dominated convergence. Finally (2.16) can be obtained from (2.6) or by differentiating (2.14) and taking $\lambda = 0$. □

It is almost immediate that we can gather together the following conclusions, not requiring a proof.

Corollary 1. *1) When $\lambda = 0$ in (2.14) the trivial solution gives that $N_t^x = +\infty$ has zero probability. 2) The probability of extinction $v_x(t) = P(N_t^x = 0)$ has trivial solution equal to zero if $p_0 = 0$. In general, it can be calculated based on the fixed point of the equation (2.15) after integrating over $\gamma(dx)$*

$$v_\gamma = \Theta_K(v_\gamma(\cdot)) \star f_D(\cdot, \gamma)\,, \quad v_\gamma(t) = \langle \gamma, v_\cdot(t) \rangle\,. \tag{2.17}$$

3) In the subcritical Bernoulli case it is equal to

$$\Theta_K(s) = (1-p) + ps\,, \quad p_0 = 1-p \quad \text{and} \quad p_1 = p\,, \qquad 0 \le p \le 1 \tag{2.18}$$

$$\hat{v}_\gamma(\alpha) = \frac{1-p}{\alpha} \frac{\hat{f}_D(\alpha,\gamma)}{1 - p\hat{f}_D(\alpha,\gamma)} \tag{2.19}$$

and the Laplace transform of the tail $P_\gamma(T_{ext} > t) = 1 - v_\gamma(t)$

$$P_\gamma(\widehat{T_{ext} >} \cdot)(\alpha) = \frac{1}{\alpha} - \hat{v}_\gamma(\alpha) = \frac{1}{\alpha} \frac{1 - \hat{f}_D(\alpha,\gamma)}{1 - p\hat{f}_D(\alpha,\gamma)} \tag{2.20}$$

showing that $P_\gamma(T_{ext} > t) \sim e^{\alpha^* t}$ with $1 - p\hat{f}_D(\alpha^*, \gamma) = 0$, $\alpha^* < 0$ (note that $p = \bar{K}$), and extinction occurs in finite time almost surely. If γ is quasi-stationary, we have $\tau^D \sim exp(\theta)$, then $T_{ext} \sim exp((1-p)\theta)$, the thinned exponential.

Remarks.

1) When γ is a quasi stationary distribution for underlying dynamics L, the process N_t^γ is Markovian, equal to a birth-death chain in continuous time.
2) The process N_t^γ is not Markovian in general, since the holding times between branchings are not exponential. However, this is a branching process where each particle branches a after i.i.d. times τ^D (starting from a point $Z \sim \gamma$).
3) The sub case $p = 1$ is the *Brownian motion with rebirth* from [10, 11, 4, 3] and this is the only case when the particle process is Markovian on D. It was shown in the same paper that it is exponentially ergodic with invariant measure with density equal to the normalized Green function integrated against γ.
4) In the critical Bernoulli case

$$\Theta_K(s) = (1-p) + ps^2\,, \quad p = \frac{1}{2}\,. \tag{2.21}$$

We now concentrate on (2.16). After taking the Laplace transform of (2.16)

$$\hat{n}_x(\alpha) = \frac{1}{\alpha}(1 - \hat{f}_D(\alpha, x)) + \bar{K}\hat{n}_\gamma(\alpha)\hat{f}_D(\alpha, x) \tag{2.22}$$

and integrating over γ

$$\hat{n}_\gamma(\alpha) = \frac{1}{\alpha}\frac{1 - \hat{f}_D(\alpha, \gamma)}{1 - \bar{K}\hat{f}_D(\alpha, \gamma)} \tag{2.23}$$

we can plug back into the first equation to conclude the proof of the next proposition.

Proposition 3. *The Laplace transform in time of the expected number of particles satisfies*

$$\hat{n}_x(\alpha) = \frac{1}{\alpha}\Big[1 + (\bar{K} - 1)\frac{\hat{f}_D(\alpha, x)}{1 - \bar{K}\hat{f}_D(\alpha, \gamma)}\Big] \tag{2.24}$$

and has asymptotic growth rate at $t \to \infty$

$$n_x(t) \sim e^{\alpha^* t} \quad \textit{with} \quad 1 - \bar{K}\hat{f}_D(\alpha^*, \gamma) = 0\,. \tag{2.25}$$

Remark. In the subcritical $\bar{K} < 1$, critical $\bar{K} = 1$ and critical $\bar{K} > 1$ we have $\alpha^* < 0$, $\alpha^* = 0$ and $\alpha^* > 0$, respectively, as can be checked from the properties of the Laplace transform of the first hitting time τ^D.

3. The branching process (ζ_t) semigroup and its marginal

The process (ζ_t) in (2.2) has state space $M_F(D)$, which is a Polish space. We then define the continuous, bounded functions $F \in C_b(M_F(D))$ of the form

$$\mu \in M_F(D) \to F(\mu) = \varphi(\langle \mu, \phi_1 \rangle, \dots, \langle \mu, \phi_l \rangle)\,, \quad l \in \mathbb{N} \tag{3.1}$$

where $(\phi_i)_{1 \le i \le l}$ are test functions in $\mathcal{T}$ and $\varphi \in C_b(\mathbb{R}^d)$. A class with smooth components including $C_c^\infty(D)$ of such test functions is sufficient to determine the law of the process (2.2) as the solution to the martingale problem, see [6].

Due to Proposition 1 we can extend φ to polynomial growth functions (and more). In fact, we shall be only interested in the functionals $\mu \to F(\mu) = \langle \mu, \phi \rangle$, for some test function ϕ, in other words a *linear functional*, when $\varphi(u) = u$ and $l = 1$. In that sense we refer to the restriction of the semigroup as *the marginal transition semigroup*, formally defined in (3.3), as already mentioned in Subsection 1.2.

The first result is valid for arbitrary test functions F.

Proposition 4. *Let a non-random initial finite point measure be* $\mu = \sum_{k=1}^{N} \delta_{x_k} \in M_F(D)$, *with* N *a nonrandom positive integer. Then, the process* (ζ_t) *defined in (2.2) is a* pure branching process, *in the sense that*

$$E_\mu[F(\zeta_t)] = \sum_{i=1}^{N} E_{x_k}[F(\zeta_t)] = \langle E_\cdot[F(\zeta_t)], \mu\rangle = \int_D E_x[F(\zeta_t)]\mu(dx)\,. \tag{3.2}$$

Proof. The relation is a consequence of the construction of the process. Particles independent at time $s \geq 0$ remain independent forever. The only dependence is through the ancestry tree. Particles distributed deterministically at time $t = 0$ are independent. Hence the result. $\square$.

Now consider a mapping defined for $\phi \in C_b(D)$,

$$t \to E_x[\,\langle\, \psi, \zeta_t \,\rangle\,] = S_t\phi(x) \tag{3.3}$$

Proposition 5. *The mapping (3.3) defines a strongly continuous semigroup on* $C_0(D)$, *the set of continuous functions vanishing at infinity, with resolvent*

$$R_\alpha\phi(x) = R_\alpha^D\phi(x) + \frac{\bar{K}\hat{f}_D(\alpha, x)}{1 - \bar{K}\hat{f}_D(\alpha,\gamma)}\gamma R_\alpha^D\phi\,, \quad \Re(\alpha) > \alpha^*\,, \tag{3.4}$$

where α^* *is defined in (1.9).*

Proof. Step 1 - Semigroup property Using (3.2) and the Markov property for $\zeta_\cdot$,

$$E_\mu[F(\zeta_{s+t})] = E_\mu[E_{\zeta_s}[F(\zeta_t)]] = E_\mu[\langle E_\cdot[F(\zeta_t)], \zeta_s\rangle]\,. \tag{3.5}$$

When F is linear, i.e. $F(\mu) = \langle\phi, \mu\rangle$, $\phi \in C_0(D)$, we have

$$\begin{aligned} E_\mu[\langle E_\cdot[F(\zeta_t)], \zeta_s\rangle] &= E_\mu[\langle E_\cdot[\langle\phi, \zeta_t\rangle], \zeta_s\rangle] = E_\mu[\langle S_t\phi(\cdot), \zeta_s\rangle] \\ &= E_\mu[S_s(S_t\phi(\cdot))] = \int_D S_sS_t\phi(x)\mu(dx)\,. \end{aligned} \tag{3.6}$$

Step 2 - Strong continuity. The strong continuity in t derives from the renewal equation and the continuity of the underlying semigroup killed at the boundary.

Let R_α^D and R_α be the resolvents of $(L, \mathcal{D}(L))$ and the semigroup S_t defined in (3.3).

Then, following the reasoning used to establish (2.14),

$$S_t\phi(x) = S_t^D\phi(x) + \bar{K}\int_0^t\int_D S_{t-s}\phi(x')\gamma(dx')dF_D(s,x)\,, \tag{3.7}$$

with resolvents obtained by Laplace transforms in time and using the definition (1.6) given by

$$R_\alpha \phi(x) = R_\alpha^D \phi(x) + \bar{K}(\gamma R_\alpha \phi)\hat{f}_D(\alpha, x)\,. \tag{3.8}$$

We apply γ on both, then

$$\gamma R_\alpha \phi = \gamma R_\alpha^D \phi + \bar{K}(\gamma R_\alpha \phi)\hat{f}_D(\alpha, \gamma)\,, \tag{3.9}$$

solving

$$\gamma R_\alpha \phi = \frac{\gamma R_\alpha^D \phi}{1 - \bar{K}\hat{f}_D(\alpha, \gamma)}\,, \tag{3.10}$$

and plugging back in (3.8) we establish (3.4).

Notice that

$$\hat{f}_D(\alpha, x) = 1 - \alpha R_\alpha^D \mathbf{1}(x)\,, \tag{3.11}$$

where $\mathbf{1}(x)$ is the constant function equal to one. This, together with (3.4) prove the strong continuity of the semigroup. The domain $\alpha > \alpha^*$ is a consequence of the fact that if $\{\alpha \in \mathbb{C} | \Re(\alpha) > \tilde{\alpha}\} \subseteq Res(L)$, $\tilde{\alpha} < \alpha^*$; all functions in the formula (3.4) of the resolvent are meromorphic on $Res(L)$; and, finally, the definition of α^*. □

We remind the reader that quasi stationarity and Yaglom limits are defined in the Appendix (Section 5). Definition (5.5) corresponds to the special case of linear test functions $F(\mu) = \langle \phi, \mu \rangle$, $\phi \in C_b(D)$.

Theorem 1. *When $\bar{K} > 0$, the Yaglom limit (5.7) exists and is equal to $\gamma R_{\alpha^*}^D$ for $\alpha^* > \tilde{\alpha}$ solving (2.25), modulo a normalization constant. More precisely*

$$\nu(dx) = C(\alpha^*) \int_D \gamma(dx') R_{\alpha^*}^D (x', dx)\,, \qquad C(\alpha^*)^{-1} = \gamma R_{\alpha^*}^D \mathbf{1}\,. \tag{3.12}$$

When $\bar{K} \neq 1$, we can write $C(\alpha^*) = \alpha^*/(1 - \frac{1}{\bar{K}})$.

Proof. The correct normalization in the definition of the quasi-stationary distribution is to divide the semigroup by $E_x[N_t]$, which is of the order of $e^{\alpha^* t}$. Keeping this in mind, modulo a normalization constant, we multiply both (3.7) and (2.16) by $e^{-\alpha^* t}$ and pass to the limit as $t \to \infty$. Once normalized, the limits are both finite. To calculate them, we take the Laplace transform for $\alpha > \alpha^*$ for each. Via the Tauberian theorem, this will bring in the equivalence between the Yaglom limit (or the normalization we described) and the ratio of the limits, numerator based on (3.7)

$$\lim_{\alpha\downarrow\alpha^*}(\alpha-\alpha^*)R_\alpha\phi(x) = \lim_{\alpha\downarrow\alpha^*}(\alpha-\alpha^*)R^D_\alpha\phi(x) \tag{3.13}$$

$$+\bar{K}\lim_{\alpha\downarrow\alpha^*}\Big[\frac{(\alpha-\alpha^*)}{1-\bar{K}\hat{f}_D(\alpha,\gamma)}\Big]\lim_{\alpha\downarrow\alpha^*}\Big[\hat{f}_D(\alpha,x)\gamma R^D_\alpha\phi\Big] \tag{3.14}$$

and its analogue for the denominator based on (2.24). The resolvent R^D is holomorphic on $\Re\alpha>0$. The first limit is zero. The second limit gives a residue at α^*, which is a constant $c_1(\alpha^*)$. The third limit equals the value at α^*. After simplification, using both (3.11) and (1.9), we proved the theorem. □

Theorem 2. *The distribution (3.12) is a left-side eigenfunction of the semigroup S_t defined in (3.3), with eigenvalue α^* solving (2.25).*

Proof. We recall (3.11). Applied to $\alpha=\alpha^*$ and integrating against γ, this gives $C(\alpha^*)^{-1}=\gamma R^D_{\alpha^*}\mathbf{1}$. We want to apply $\gamma R^D_{\alpha^*}$ (a measure) to the left side of (3.4) and obtain

$$\gamma R^D_{\alpha^*}R_\alpha\phi = (\alpha-\alpha^*)^{-1}\gamma R^D_{\alpha^*}\phi\,,\quad \alpha>\alpha^*\,. \tag{3.15}$$

To see this, first we calculate

(3.16)

$$\gamma R^D_{\alpha^*}\hat{f}_D(\alpha,\cdot) = \gamma R^D_{\alpha^*}\mathbf{1}-\frac{\alpha}{\alpha-\alpha^*}\gamma(R^D_{\alpha^*}\mathbf{1}-R^D_\alpha\mathbf{1}) \qquad \text{resolvent identity}$$

$$= \gamma\frac{1}{\alpha^*}(1-\hat{f}_D(\alpha^*,\cdot))-\frac{\alpha}{\alpha-\alpha^*}\gamma\Big(\frac{1}{\alpha^*}(1-\hat{f}_D(\alpha^*,\cdot))-\frac{1}{\alpha}(1-\hat{f}_D(\alpha,\cdot))\Big) \tag{3.17}$$

$$= C(\alpha^*)^{-1}-\frac{1}{\alpha-\alpha^*}\Big(\alpha C(\alpha^*)^{-1}-(1-\hat{f}_D(\alpha,\gamma))\Big) \tag{3.18}$$

$$= \frac{1}{\alpha-\alpha^*}\Big(-(1-\frac{1}{\bar{K}})+(1-\hat{f}_D(\alpha,\gamma))\Big) \tag{3.19}$$

$$= \frac{1}{\bar{K}(\alpha-\alpha^*)}\Big(1-\bar{K}\hat{f}_D(\alpha,\gamma)\Big)\,. \tag{3.20}$$

This expression is introduced into (3.4) together with applying $\nu=\gamma R^D_{\alpha^*}$ to the left hand side. Using the resolvent identity once more for the first term of (3.4), after simplification, we obtain (3.15). □

4. The heat equation with mass creation

Let $\bar{K}$ be a non-negative constant, $\gamma(dx)$ and the underlying diffusion process $(L, \mathcal{D}(L))$ from (1.4). For any test function ϕ, $\phi(t,\cdot) \in \mathcal{D}_c(L)$ for all $t \geq 0$, we define the boundary condition

$$\bar{K}\langle \phi(t,\cdot), \gamma\rangle = \phi(t,y)\,, \quad y \in (\partial D)_a\,. \tag{4.1}$$

We shall say that $\nu_t(dx) \in C([0,\infty), M_F(D))$ is the weak solution to the heat equation for $(L, \mathcal{D}(L))$ with mass creation $(\gamma(dx), (\partial D)_a, \bar{K})$ and initial value $\nu_0(dx)$ if, for any test function ϕ, $\phi(t,\cdot) \in \mathcal{D}_c(L)$ satisfying the additional boundary condition (4.1), and any $t \geq 0$, the equality holds

$$\langle \phi(t,\cdot), \nu_t\rangle - \langle \phi(0,\cdot), \nu_0\rangle - \int_0^t \langle \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot), \nu_s\rangle\, ds = 0\,. \tag{4.2}$$

Remark. A brief discussion of the soft catalyst case is given in Subsection 5.1.

Theorem 3 is the main result of this section. It solves an essential step in the proof of the hydrodynamic limit from [13].

Theorem 3. *Let $\nu_0 \in M_1(D)$. Then, equation (4.2) has a unique weak solution $\nu_.$ in $C([0,\infty), M_F(D))$, where time continuity is defined in the topology of finite measures. This is a* strong solution *for $t > 0$ in the sense that $\nu_t(dy) = v(t,y)dy$, $t > 0$ with $v \in C^{1,2}((0,\infty)\times D) \cap C((0,\infty)\times \bar{D})$. The solution admits the representation $\langle \nu_t, \phi\rangle = E_{\nu_0}[\langle \zeta_t, \phi\rangle]$, $t \geq 0$, for any $\phi \in \mathcal{D}$. Here $(\zeta_t)_{t\geq 0}$ is the auxiliary measure-valued process $(\zeta_t)_{t\geq 0}$ defined in (2.2), Section 2.*

For convenience, the regularity properties are laid out in the next theorem.

Theorem 4. *For any $0 < t_0 < T$ there exists a constant $C(t_0,T) > 0$ such that*

$$\sup_{t\in[t_0,T], x,y\in D} v^x(t,y) = C(t_0,T) < \infty\,. \tag{4.3}$$

If, in addition, $\nu_0(dy) = v_0(y)dy$, $v_0 \in C(\bar{D})$, then $v \in C([0,\infty)\times\bar{D}) \cap C^{1,2}((0,\infty)\times D)$. The total mass $n_t = \langle \nu_t, 1\rangle$ is positive, strictly increasing and there exists a constant $c(\eta_0)$ depending only on the initial value and $\lambda_* > 0$ such that $0 < n_t < c(\nu_0)e^{\lambda_* t}$, for any $t \geq 0$.

The proof of Theorems 3 and 4 will be done based on the branching process constructed in Section 2.

4.1. **Existence of the solution of the heat equation with mass creation.** Let $\nu_t(dx)$, $t \geq 0$ be the expected value of the unnormalized empirical measure of the process $(Z_t)_{t\geq 0}$. As before, we assume $\nu_0 \in M_1(D)$ (is a probability distribution). For a test function ϕ, we put

$$\langle \phi(t,\cdot), \nu_t \rangle := E_{\nu_0}[\langle \zeta_t, \phi(t,\cdot)\rangle] = E_{\nu_0}[\sum_{j=1}^{N_t} \phi(t, Z_t^j)] \,. \tag{4.4}$$

Based on the estimate on N_t from Proposition 1, $\nu_t^m \in M_F(D)$, being a linear bounded functional on the set of smooth functions vanishing at the boundary.

We first prove the existence and regularity from Theorem 3.

Proposition 6. *The deterministic measure valued process $\nu.$ belongs to $C([0,\infty), M_F(D))$ with the continuity in time in the sense of the convergence of finite measures. For $t > 0$, the solution is Lebesgue absolutely continuous with density satisfying the regularity conditions from Theorem 3.*

Proof. Step 1. Let ϕ such that $\phi(t,\cdot)$ satisfies (1.11) and (4.1) for $t \geq 0$. By construction, we check immediately that $(\langle \zeta_t, \phi(t,\cdot)\rangle)_{t\geq 0}$ is a martingale with respect to the filtration of the process. At the $(\partial D)_r$, the function ϕ has a vanishing normal derivative and no local time is accumulated. The boundary condition represents exactly the fact that the expected value of the mass jump at creation time, i.e. when the boundary $(\partial D)_a$ is hit, is zero. The test function is bounded with smooth bounded derivatives, implying the functional is a proper martingale. Taking the expected value obtains (4.2).

Step 2. Using (3.3) we see that, as a function of time, the deterministic process ν_t, $t \geq 0$, is in the Skorokhod space of right continuous with left limit paths. From (3.7), applied to $\phi(x) \to \phi(t,x)$, we have that $\langle \phi(t,\cdot), \nu_t\rangle = S_t\phi(t,x)$ is the sum of a known continuous part given by S_t^D and a time integral of a rcll function. This shows that $t \to \langle \phi(t,\cdot), \nu_t\rangle$ is continuous, and now re-applying (3.7) we obtain it is in also in C^1 for $t > 0$. To prove the finite measure is Lebesgue absolute continuous, we inspect (3.4) and notice that the inverse Laplace transform in α versus t would not change the fact that on the right hand side of (3.4) the first part is simply the resolvent of $S_\cdot^D$ and the second part is an explicit function of x.

Step 3. The rest of the regularity properties, i.e. the density $v(t,y)$ with $\nu_t(dy) = v(t,y)dy$ is C^2 in x are immediate from both (3.7) and (3.4). □

Theorem 4 refers to ν_t, already well defined as an expected value of the branching process in (4.4). The uniqueness of (4.2) is not needed to study its regularity. However, uniqueness requires the regularity properties, and will be left at the end.

4.2. **Proof of Theorem 4.** *Part 1 - the heat kernel estimate (4.3).* For $t > 0$ we already know that $\nu_t^x(dy) = v^x(t,y)dy$. Re-writing (3.7) we obtain

$$v^x(t,y) = p^D(t,x,y) + \bar{K}\int_0^t \int_D v^{x'}(t-s,y)\gamma(dx')dF_D(s,x)\,. \tag{4.5}$$

Denote $r_\alpha^D(x,y)$ the kernel of the resolvent $R_\alpha^D(x,dy)$. Then (3.4) shows that $R_\alpha(x,dy)$ has a density function, its own kernel $r_\alpha(x,y) = \int_0^\infty e^{-\alpha t}v^x(t,y)dt$ and

$$r_\alpha(x,y) = r_\alpha^D(x,y) + \frac{\bar{K}\hat{f}_D(\alpha,x)}{1-\bar{K}\hat{f}_D(\alpha,\gamma)}\int_D r_\alpha^D(x',y)\gamma(dx')\,, \quad \Re(\alpha) > \alpha^*\,. \tag{4.6}$$

The inverse Laplace transform $g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{t\alpha}\hat{g}(\alpha)dt$, where the integration takes place on the vertical line $\Re(z) = c > \alpha^*$, i.e. $\bar{K}\hat{f}_D(c,\gamma) < 1$. On such a line the factor

$$\frac{\bar{K}\hat{f}_D(z,x)}{1-\bar{K}\hat{f}_D(z,\gamma)} \le \frac{\bar{K}\hat{f}_D(c,x)}{1-\bar{K}\hat{f}_D(c,\gamma)} =: c_1(x)\,, \qquad c_1(x) \quad \text{bounded}\,.$$

We did not bound the denominator directly but wrote the factor in geometric series form.

It follows that as a function of t, the heat kernel $v^x(t,y)$ for the semigroup S_t satisfies the same bounds as $p^D(t,x,y)$. Condition (1.10) implies that there exists $C(t_0,x,y)$ uniformly bounded in (t,x,y) such that

$$p^D(t,x,y) \le C(t_0,x,y)\,, \qquad t \ge t_0\,. \tag{4.7}$$

Then

$$v^x(t,y) \le C_1(t_0,x,y)\,, \qquad C_1(t_0,x,y) = C(t_0,x,y) + c_1(x)\int_D C(t_0,x',y)\gamma(dx')\,.$$

Part 2 - regularity. The continuity and smoothness for continuous initial profile are simple consequences of (4.5). Proposition 3 proves the claims on the total mass n_t^x. □

4.3. **Proof of uniqueness.** When $\nu_0 = \delta_x$, the solution is denoted $v^x(t,y)$ for $t > 0$. We shall use the time reversal in the semigroup, or equivalently the backward equation.

Proposition 7. *Equation (4.2) has a unique solution equal to (4.4).*

Proof. Fix $T > 0$ and $g \in C_0(D)$. Using $\nu_t^x(dy)$ from (4.4) and $v(t,y)dy = \int_D g(y)\nu_t^x(dy)$, which solves (4.2), as shown in Proposition 6, with $\nu_0(dy) = g(y)dy$. Define $\phi(t,x) := v(T-t,x)$, $t \in [0,T]$.

For any sufficiently small $\epsilon > 0$, ϕ is a test function as defined in (4.2)-(5.11) on the time interval $[0, T-\epsilon]$ and satisfies the boundary conditions.

Let $m_t(dx)$ is a weak solution satisfying (4.2). Then

$$\langle \phi(t,\cdot), m_t \rangle = \langle \phi(0,\cdot), m_0 \rangle \,, \qquad 0 \le t \le T - \epsilon \,. \tag{4.8}$$

This implies

$$\begin{aligned}
\langle \phi(T-\epsilon,\cdot), m_{T-\epsilon} \rangle &= \langle \phi(0,\cdot), m_0 \rangle \\
&= \langle v(T,\cdot), m_0 \rangle = \langle \langle g(\cdot), \nu_T \rangle, m_0 \rangle = \int_D \int_D \nu_T^x(dy) g(y) m_0(dx) \\
&= \int_D g(y) \int_D \nu_T^x(dy) m_0(dx) = \int_D g(y) \nu_T^{m_0}(dy) = \langle g, \nu_T^{m_0} \rangle \,.
\end{aligned} \tag{4.9}$$

The left hand side is

$$\begin{aligned}
\langle \phi(T-\epsilon,\cdot), m_{T-\epsilon} \rangle = \langle \int_D g(y)\nu_\epsilon^x(dy), m_{T-\epsilon} \rangle = \langle S_\epsilon g, m_{T-\epsilon} \rangle \\
= \langle S_\epsilon g - g, m_{T-\epsilon} \rangle + \langle g, m_{T-\epsilon} \rangle
\end{aligned}$$

Using the *strong continuity of the semigroup* from Proposition 5, we obtain that the first term converges to zero as $\epsilon \to 0$. The second term approaches $\langle g, m_T \rangle$, which implies that $m_T = \nu_T^{m_0}$. This is true for arbitrary $T > 0$, concluding the proof. $\square$

5. Appendix

In the following, S_t will be a strongly continuous Feller semigroup, i.e. for any $t \ge 0$,

$$\begin{aligned}
&(i) \quad \forall \phi \in C_b(D)\,, \qquad S_t\phi \in C_b(D) \\
&(ii) \quad \forall t, t' \ge 0\,, \quad \forall \phi \in C_b(D)\,, \qquad S_{t+t'}\phi = S_t S_{t'}\phi \quad \text{and} \quad S_0\phi = \phi \\
&(iii) \quad \forall \phi \in C_b(D)\,, \quad t \to S_t\phi \quad \text{is continuous in the supremum norm}\,.
\end{aligned} \tag{5.1}$$

Many results hold by replacing (i) with the weaker condition $(i') \quad S_t 1 \in C_b(D)$.

We shall assume that there exists $\alpha_1 > -\infty$ such that

$$\forall \alpha > \alpha_1 \qquad \sup_{x \in D} \int_0^\infty e^{-\alpha t} S_t 1(x)\, dt < +\infty \,. \tag{5.2}$$

A stronger condition is that there exists $\alpha' > -\infty$ such that $e^{-\alpha' t} S_t$ is a contraction semigroup.

A probability measure $\nu(dx)$ on D is said a *quasi-stationary distribution* (qsd) for the semigroup S_t if

$$\langle \nu, S_t \phi \rangle = \langle \nu, \phi \rangle \langle \nu, S_t 1 \rangle \,, \qquad \forall\, t \geq 0 \,. \tag{5.3}$$

In the context of the process (ζ_t), we define its (marginal) semigroup applied to test functions $F \in C_b(M_F(D))$ of the special form $F(\mu) = \langle \mu, \phi \rangle$, where $\phi \in C_b(D)$

$$S_t \phi(x) = E_x[\langle \zeta_t, \phi \rangle] \,, \qquad \text{with the notation} \quad E_x[F(\zeta_t)] = E[F(\zeta_t) | \zeta_0 = \delta_x] \,. \tag{5.4}$$

Then (5.3) reads explicitly as

$$E_\nu[\sum_{i=1}^{N_t} \phi(Z_t^i)] = E_\nu[N_t] \cdot \langle \phi, \nu \rangle \,, \qquad \forall \quad t \geq 0 \,. \tag{5.5}$$

Equivalently, we can define a qsd by the property that for any two test functions $\phi, \psi \in C_b(D)$

$$\frac{\langle \nu, S_t \phi \rangle}{\langle \nu, S_t \psi \rangle} = \frac{\langle \nu, \phi \rangle}{\langle \nu, \psi \rangle} = \text{constant in } t \geq 0 \,. \tag{5.6}$$

A probability measure $\nu(dx)$ on D is said a *Yaglom limit* for the semigroup S_t if there exists a probability measure ν' such that, for all $\phi \in C_b(D)$

$$\lim_{t \to \infty} \frac{\langle \nu', S_t \phi \rangle}{\langle \nu', S_t 1 \rangle} = \langle \nu, \phi \rangle \,. \tag{5.7}$$

In that case we say ν' is in the *domain of attraction* of ν. If a Yaglom limit has domain of attraction all delta functions, or equivalently, any probability measure ν' on D, it is said a *strong Yaglom limit.*

Theorem 5. *1) $\langle \nu, S_t 1 \rangle$ (expected value of the total number of particles) is exponential.*

2) A qsd ν is a left side eigenfunction of the semigroup.

3) A Yaglom limit is a qsd. A qsd is in its own domain of attraction. A strong Yaglom limit, if it exists, it is unique.

Proof. 1) Using $\phi = S_s \psi$ and $\psi = 1$ in (5.6), we obtain that $t \to \nu S_t 1 = n_t^\nu$ is an exponential function. In case the semigroup is dissipative, the time to extinction is exponentially distributed.

2) The Hille-Yosida theorem shows that the same is true for the generator and resolvent.

3) Let t, t' positive. Then, applying the definition (5.7) with $S_{t'}\phi$ in place of ϕ,

$$\lim_{t\to\infty} \frac{\langle \nu', S_t S_{t'}\phi\rangle}{\langle \nu', S_t 1\rangle} = \langle \nu, S_{t'}\phi\rangle \,. \tag{5.8}$$

$$\frac{\langle \nu', S_t S_{t'}\phi\rangle}{\langle \nu', S_t 1\rangle} = \frac{\langle \nu', S_{t+t'}\phi\rangle}{\langle \nu', S_{t+t'} 1\rangle} \cdot \frac{\langle \nu', S_t S_{t'} 1\rangle}{\langle \nu', S_t 1\rangle} \,. \tag{5.9}$$

Let $t \to \infty$. The first factor converges to $\langle \nu, \phi\rangle$ as $t + t' \to \infty$ and the second factor uses (5.7) with $S_{t'}1$ in place of ϕ, to converge to $\langle \nu, S_{t'}1\rangle$. The equality of the two limits shows that ν is a qsd. □

5.1. **Case of a soft catalyst V.** Given a non-negative bounded $V(\cdot)$ on D, we could consider the the killed process with intensity V, instead of the case of instantaneous killing at the boundary, corresponding to $V(x) = +\infty \mathbf{1}_{(\partial D)_a}(x)$. In this case, the analogue of (4.2) is

$$\langle \phi(t,\cdot), \nu_t\rangle - \langle \phi(0,\cdot), \nu_0\rangle - \int_0^t \Big[\langle \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot), \nu_s\rangle \tag{5.10}$$

$$+ \langle V(\cdot)(\bar{K}\langle \phi(\cdot), \gamma\rangle - \phi(\cdot)), \nu_s\rangle\Big]\, ds = 0\,, \tag{5.11}$$

with (5.11) replacing the boundary condition (4.1).

5.2. **Examples - BM reflected at one, killed at zero.** The next calculations are done in [17], together with a discussion of the extreme case $c \downarrow 0$, a few figures and a discussion of the relation with the BSBM particle system.

We are in $d = 1$ with $D = (0,1)$, $(\partial D)_r = \{1\}$, $(\partial D)_a = \{0\}$, $\gamma = \delta_c$, $c \in (0,1)$ and $L = \frac{1}{2}\frac{d^2}{dy^2}$ with $\nu_0(dx) = v_0(x)dx$. The formal adjoint coincides with L, i.e. $L = L^*$. Let $\nu_t(dy) = v(t,y)dy$ with $v(0+,\cdot) = v_0(\cdot)$ and v has continuous time derivative. We shall show that for any $t > 0$, v is smooth in $(0,c) \cup (c,1)$ and satisfies the boundary conditions

$$v(t,c-) = v(t,c+)\,, \quad v'(t,1) = 0\,, \quad v(t,0) = 0 \tag{5.12}$$

$$(v'(t,c+) - v'(t,c-)) + 2v'(t,0) = 0\,.$$

To see that, and to derive the quasi-invariant measure at the same time, we recall that a quasi-invariant measure is a left-side eigenfunction $g(y)$, with eigenvalue λ of L^*, i.e. when L is applied to functions in $\mathcal{D}_c(L)$ (1.11) satisfying (4.1). In this particular case, we have $2\phi(c) = \phi(0)$ and $\phi'(1) = 0$.

The answer is of course (3.12), and thus $\lambda = \alpha^* > 0$ corresponding to $\bar{K} = 2 > 1$. However, we shall obtain the result by direct computation. Integration by parts gives

$$2\lambda \int_0^1 \phi(y)g(y)dy = \int_0^1 \phi''(y)g(y)dx = \int_0^c \phi''(y)g(y) + \int_c^1 \phi''(y)g(y)dy\,.$$

This implies

$$[\phi'(c)g(c-) - \phi'(0)g(0)] - [\phi(c)g'(c-) - \phi(0)g'(0)]$$

$$+[\phi'(1)g(1) - \phi'(c)g(c+)] - [\phi(1)g'(1) - \phi(c)g'(c+)]$$

$$= \int_0^1 (2\lambda g(y) - g''(y))\phi(y)dy\,.$$

It is important to not consider g smooth at c, as we see from the one-sided limits. Inside the intervals, we obtain $g'' = 2\lambda g$. The boundary conditions derived from the equations above are

$$g(c+) = g(c-) \qquad g(0) = 0 \qquad g'(1) = 0$$

$$(g'(c+) - g'(c-)) + 2g'(0) = 0\,. \tag{5.13}$$

Notice that the boundary conditions are the same as for $y \to v(t, y)$, since that satisfies the *forward equation*. It must be that $g(x) = c_1 e^{\sqrt{2\lambda}y} + c_2 e^{-\sqrt{2\lambda}y}$.

Indeed, the resolvent R_λ^D in (3.12) has density function

$$R_\lambda^D(c, x) = \begin{cases} \frac{2\sqrt{2\lambda}\sinh\sqrt{2\lambda}c}{\cosh\sqrt{2\lambda}} \cosh\sqrt{2\lambda}(1-x) & \text{if } c \le x \le 1 \\ \frac{2\sqrt{2\lambda}\cosh\sqrt{2\lambda}(1-c)}{\cosh\sqrt{2\lambda}} \sinh\sqrt{2\lambda}x & \text{if } 0 \le x < c \end{cases} \tag{5.14}$$

and the normalizing factor from (3.12) is $C(\lambda) = 2\sqrt{2\lambda}$. Naturally, λ, c are connected by the formula (1.9)

$$\frac{1}{K} = \frac{1}{2} = \frac{\cosh\sqrt{2\lambda}(1-c)}{\cosh\sqrt{2\lambda}}\,. \tag{5.15}$$

As c gets smaller, the graph of $R_\lambda^D(c, x)$ forms a sharper "angle" at c.

Another interesting feature that is inherent in (5.15) is that $\sqrt{2\lambda}c$ stabilizes to $\ln 2 \approx 0.69$, i.e. the solution of the equation is such that

$$\lambda \sim \frac{(\ln 2)^2}{2c^2} \qquad c \downarrow 0\,.$$

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